Exceptional Points and the Real Spectral Locus for Complex Airy Operator

CIRM conference on
Mathematical aspects of the physics with non-self-adjoint operators

Sergey Tumanov
based on joint work with Andrey Shkalikov

Lomonosov Moscow State University

Marseille, France, 5 – 9 June 2017

Introduction

We'll talk about spectral properties of the form

$$Ty = -y'' + P(x)y = \lambda y, \quad y(-a) = y(a) = 0,$$

$$0 < a \leq +\infty$$

with conditions for $\mathcal{PT}$ – symmetry: $P(x) = \overline{P(-x)}$.

It's clear that the spectrum is symmetric with respect to real axis.

Introduction

The following problems play an important role in the study of these operators:

- When the spectrum of a $\mathcal{PT}$ – symmetric operator is real.
- When the correspondent operator is similar to self–adjoint?
- Other topics: Resolvent estimates / Basis properties.

Sources of Inspiration

- Bender, Wu, 1969.
- Bessis and Zinn-Justin conjecture (about 1995): $Tw = -w'' + iz^3 w$ in $L_2(-\infty, +\infty)$ has only real spectrum. Proved by Dorey, Dunning and Tateo in 2001.
- Shin genegalized this result to $Tw = -w'' + i(z^3 + az)$, $a \geq 0$ in 2002.

Sources of Inspiration

- Davies, 1998-2000 — wild behavior of resolvent of $Tw = -w'' + c|x|^m w$ in $L_2(\mathbb{R})$ if $c \in \mathbb{C} \setminus \mathbb{R}$. At the same time eigenfunctions form complete set if $\mathrm{Re}\, c > 0$, $m \geq 1$ or $0 < m < 1$ and $|\arg c| < \pi m/2$.
- Krejčiřík, Siegl, 2012 — the completeness of the system of eigenfunctions for $Tw = -w'' + iz^3 w$ in $L_2(\mathbb{R})$.
- Mityagin, Siegl, Viola, 2013 — computation of spectral projectors norms for anharmonic oscillators.
- Eremenko and Gabrielov in 2011 studied the spectral locus of $Tw = -w'' + i(z^3 + az)$ in $L_2(\mathbb{R})$ and especially the real spectral locus $(a, \lambda) \in \mathbb{R}^2$. They found simple analytic disjoint curves Γ_n — the image of a proper analytic embedding of the real line on which each eigenfunction has exactly $2n$ non-real zeros.

Introduction

Consider the operator in $L_2(-a, a)$, P — polynomial with $\mathcal{PT}$–symmetry

$$Ty = -\frac{1}{\varepsilon}y'' + P(x)y = \lambda y, \; y(-a) = y(a) = 0, \; 0 < a \leq +\infty$$

we should distinguish 3 cases:

- small values of the parameter ε
- dynamics of the eigenvalues in the middle
- large values of the parameter [Shkalikov, T, 2016]

Introduction

Let us consider the finit interval $[-a, a] = [-1, 1]$ When ε is small, eigenvalues of T are close to

$$\lambda_n \sim \frac{1}{\varepsilon}\left(\frac{\pi^2 n^2}{4}\right)^2, \quad n \in \mathbb{N},$$

Because of $\mathcal{PT}$–symmetry, in small ε all eigenvalues are real, and the operator is similar to self-adjoint (according Dunford if P is bounded, complex measurable potential, the associated generalized eigenvalue expansion is unconditionally convergent).

And with ε increase it will stay similar to self-adjoint till we get the exceptional point when several eigenvalues collide.

Our aim is to find the exact values of these exceptional points in one nice model case — complex Airy operator.

Complex Airy Operator

Since now let

$$T(\varepsilon)y = -\frac{1}{\varepsilon}y'' + ixy, \quad y(-1) = y(1) = 0.$$

According to the general perturbation theory it's easy to show that when $\varepsilon < 3\pi^2/8 \approx 3.7$ the spectrum is real and consists of simple positive eigenvalues $\lambda_k = \lambda_k(\varepsilon)$, $k \geq 1$.

Moreover, $\lambda_k(\varepsilon) \sim \varepsilon^{-1}(\pi k/2)^2$ when $\varepsilon \to 0$ or $k \to \infty$.

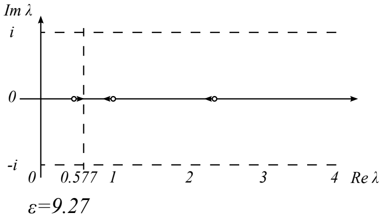
But this estimation for the first exceptional point is rough.

Complex Airy Operator

According to computation, starting from small values of the parameter ε all eigenvalues move from infinity to the left to zero.

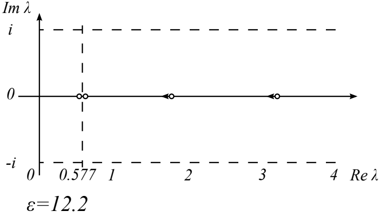
When $\varepsilon \approx 5.1$ the first eigenvalue crosses the nodal point $1/\sqrt{3} \approx 0.58$ continuing its movement to the left.

When $\varepsilon = \varepsilon_{1,turn} \approx 9.3$ the first eigenvalue reaches the point $\lambda_{1,turn} \approx 0.45$, stops at this point and with further increase of $\varepsilon > \varepsilon_{1,turn}$ moves back to the right, at the same time all other eigenvalues continue to move to the left.



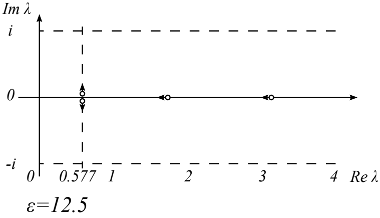
Complex Airy Operator

Afterwards the first and the second eigenvalues move towards each other and collide at the nodal point $1/\sqrt{3}$ when $\varepsilon_1 \approx 12.3$.



Complex Airy Operator

After the collision the first and the second eigenvalues become the pair of complex conjugate eigenvalues, escaping from the real axis at right angle, getting closer and closer to segments $[1/\sqrt{3}, \pm i]$ moving to $\pm i$.

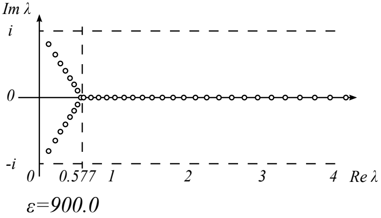


Complex Airy Operator

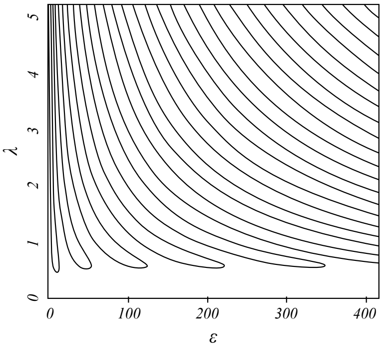
With further ε increase the third eigenvalue crosses the nodal point, moves to the left to $\lambda_{2,turn} < 1/\sqrt{3}$, when it stopes and then moves to the right towards the fourth eigenvalue until collision at the nodal point $1/\sqrt{3}$.

The same picture is observed for all pairs λ_{2k-1} and λ_{2k}, $k \in \mathbb{N}$.

In large ε eigenvalues concentrate near segments $[i, 1/\sqrt{3}]$, $[-i, 1/\sqrt{3}]$ and ray $[1/\sqrt{3}, +\infty)$, forming together so called **spectral tie** [Shkalikov, 1997].



Real Spectral Locus for Airy Operator



Theorem

The real spectral locus for Airy operator consists of pairwise disjoint regular analytic Jordan curves with ends at infinity. Each curve corresponds to two eigenvalues: odd and even.

Main Result

Consider the Airy equation

$$y''(\xi) = \xi \cdot y(\xi)$$

and standard solutions $\mathrm{Ai}\,(\xi)$, $\mathrm{Bi}\,(\xi)$. We introduce two additional ones:

$$\mu_0(\xi) = -\sqrt{3}\mathrm{Ai}\,(\xi) + \mathrm{Bi}\,(\xi), \quad \mu_1(\xi) = \sqrt{3}\mathrm{Ai}\,(\xi) + \mathrm{Bi}\,(\xi)$$

with following properties: $\mu_0(0) = 0$, $\mu_1'(0) = 0$.

Lemma

Zeros of μ_0 and μ_1 lie on the rays $\arg z = \pi/3 + 2\pi k/3$, $k = -1, 0, 1$ symmetrically relative to the origin. Let $\alpha_0 = 0$, α_k, $(k = 1, 2, \dots)$ — the absolute values for μ_0 zeros, β_l, $(l = 1, 2, \dots)$ — the absolute values for μ_1 zeros.

The zeros of both functions alternate:

$$\alpha_0 < \beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \beta_3 < \dots$$

Main Result

Theorem

As before, let $\beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \beta_3 < \dots$ *be the absolute values of* μ_1 *and* μ_0. *We assign*

$$\delta_k = \left(\beta_k \frac{\sqrt{3}}{2}\right)^3, \quad \varepsilon_k = \left(\alpha_k \frac{\sqrt{3}}{2}\right)^3, \quad k \in \mathbb{N}.$$

It's obvious that, $0 < \delta_1 < \varepsilon_1 < \delta_2 < \varepsilon_2 < \dots$

- *When* $\varepsilon \in (0, \varepsilon_1]$ *all eigenvalues are real,* $T(\varepsilon)$ *is similar to self-adjoint operator.*
- *When* $\varepsilon \in (0, \delta_1) \cup \bigcup_{k=1}^{\infty} (\varepsilon_k, \delta_{k+1})$ *the interval* $(0, 1/\sqrt{3})$ *doesn't contain eigenvalues.*
- *When* $\varepsilon \in \bigcup_{k=1}^{\infty} (\delta_k, \varepsilon_k)$ *the interval* $(0, 1/\sqrt{3})$ *contains the single eigenvalue* λ_{2k-1}.

Main Result

Theorem (continuation)

- *All critical points of the spectral locus coincide with the set*

$$\mathcal{M} = \left\{(\varepsilon_k, \frac{1}{\sqrt{3}})\right\},$$

the multiplicity of each one is 2. With increasing of ε when the value passes through $\varepsilon = \varepsilon_k$ the pair of eigenvalues λ_{2k-1} and λ_{2k} collides in $\tilde{\lambda} = 1/\sqrt{3}$, scattering then at right angle to the real axis. Subsequently eigenvalues do not return back to the real axis.
If $\varepsilon \notin \{\varepsilon_k\}$ then $T(\varepsilon)$ is similar to normal operator.

- *When $\varepsilon = \delta_k$ the eigenvalue λ_{2k-1} passes through $\tilde{\lambda} = 1/\sqrt{3}$. This value is not exceptional.*
- *When $\varepsilon = \delta_k$ the eigenfunction for $\lambda_{2k-1} = 1/\sqrt{3}$ can be explicitly specified:*

$$y(z) = \mu_1\Big(\delta_k^{1/3}(\frac{1}{\sqrt{3}} - iz)\Big).$$

Main Result

Theorem (continuation)

- *When* $\varepsilon = \varepsilon_k$ *the eigenfunction for* $\lambda_{2k-1} = \lambda_{2k} = 1/\sqrt{3}$ *can be explicitly specified:*

$$y(z) = \mu_0\Big(\varepsilon^{1/3}(\frac{1}{\sqrt{3}} - iz)\Big).$$

- *The asymptotic formulas for* $k \to \infty$*:*

$$\varepsilon_k = \frac{\sqrt{3}}{4}\left(\frac{3}{2}\right)^3\left(\pi k - \frac{\pi}{12} + O(\frac{1}{k})\right)^2,$$

$$\delta_k = \frac{\sqrt{3}}{4}\left(\frac{3}{2}\right)^3\left(\pi k - \frac{5\pi}{12} + O(\frac{1}{k})\right)^2.$$

Main Result

Theorem (continuation)

The turning points can be estimated as follows:

$$\lambda_{2k-1,turn} < \cot \arg z_k < 1/\sqrt{3},$$

where z_k *— the complex zeros of* Bi *, lying in the first quadrant.*

Ideas of Proof

Changing the variable to $\xi = \varepsilon^{1/3}(\lambda - ix)$ we come to the study of zeros for solutions of Airy equation.

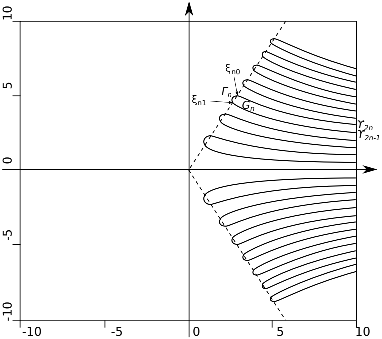
$$y'' = \xi \cdot y.$$

When $a \in \mathbb{C}$ the family $V_a(\xi) = a\mathrm{Ai}\,(\xi) + \mathrm{Bi}\,(\xi)$ covers all solutions up to constant factor, except Ai.

If a is real, the complex (non-real) zeros of $V_a(\xi)$ provide alternative parametrization of real spectral locus: for each pair (ε, λ) the pair of complex conjugate zeros of some real solution of the Airy equation corresponds uniquely.

Backward: $\varepsilon = |\mathrm{Im}\,\xi_0|^3$, $\lambda = \mathrm{Re}\,\xi_0/|\mathrm{Im}\,\xi_0|$ for each zero ξ_0 of $V_a(\xi)$, $a \in \mathbb{R}$.

Ideas of Proof



The equation $V_a(\xi) = 0$ determines countable number of implicit functions $\xi_n = \xi_n(a)$, $n \in \mathbb{Z}$, analytically continued to a neighborhood of the real axis. The ξ_n-images of the real axis — pairwise disjoint Jordan analytic curves Γ_n.

ξ_{n1} corresponds to the movement of λ_{2n-1} through the node, ξ_{n0} corresponds to collision of λ_{2n-1} and λ_{2n}.

Bibliography

N. Dunford, *A survey of the theory of spectral operators*, Bull. Amer. Math. Soc., **64** No. 5 (1958), 217-274.

A. Eremenko, A. Gabrielov. *Singular perturbation of polynomial potentials with applications to $\mathcal{PT}$–symmetric families*, Mosc. Math. J., **11** No. 3 (2011), 473-503.

A. Gil, J. Segura. *On the complex zeros of Airy and Bessel functions and those of their derivatives*, Anal. Appl. **12** (2014), 537-561.

A. A. Shkalikov, *The limit behavior of the spectrum for large parameter values in a model problem* Mat. Zametki, **62**, 950–953 (1997); English translation: Math. Notes, **62**, 796-799 (1997).

Thank You!!!