New spectral bounds for damped systems

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1. 2nd order Cauchy problems

[Jacob/Trunk '07/09, K. Veselić '06]

In a Hilbert space $\mathcal{H}$

$$\ddot{z}(t) + D\dot{z}(t) + A_0 z(t) = 0, \quad z(0) = z_0, \; \dot{z}(0) = z_1,$$

where

A_0 unbounded self-adjoint, $A_0 \geq a_0^2 > 0$,

D accretive with $\mathcal{D}(A_0) \subset \mathcal{D}(D)$, $A_0^{-1/2} D A_0^{-1/2}$ bounded.

$\uparrow$ i.e. $\operatorname{Re}(Dx, x) \geq 0$, $x \in \mathcal{D}(D)$

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Linearize in time: $x = (z, \dot{z})^{\mathrm{t}}$, $x_0 = (z_0, z_1)^{\mathrm{t}}$ $\rightsquigarrow$

$$\dot{x}(t) = \mathcal{A}\, x(t), \ x(0) = x_0, \quad \mathcal{A} = \begin{pmatrix} 0 & I \\ -A_0 & -D \end{pmatrix},$$

in $\mathcal{H}_{1/2} \oplus \mathcal{H}$ where $\mathcal{H}_{1/2} = \mathcal{D}(A_0^{1/2})$ with $\|\cdot\|_{1/2} = \|A_0^{1/2} \cdot \|$ on

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{H}_{1/2} \oplus \mathcal{H}_{1/2} : A_0 x_1 + D x_2 \in \mathcal{H} \right\} \supset \mathcal{D}(A_0) \oplus \mathcal{D}(A_0).$$

Note: here $I : \mathcal{H} \to \mathcal{H}_{1/2}$ unbounded and $\mathcal{A}$ **no operator matrix (OM)**!

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Question: Spectral bounds for $\mathcal{A}$ in terms of damping operator D ???

Ex. *Wave equation with strong (viscoelastic and frictional) damping*:

$$\ddot{z}(t) + \big(\underbrace{-d\Delta + V}_{=D}\big)\dot{z}(t) + \big(\underbrace{-\Delta + b}_{=A_0}\big)z(t) = 0, \quad z(0) = z_0,\ \dot{z}(0) = z_1,$$

in $\mathcal{H} = L^2(\Omega, \mathbb{C})$ with $\Omega = \mathbb{R}^n$ or $\Omega \subset \mathbb{R}^n$ bounded with C^2-boundary where

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$d \geq 0$ (viscoelastic damping constant),

$$b \in L^\infty(\Omega, \mathbb{R}), \qquad \operatorname{ess\,inf} b \begin{cases} > 0 & \text{if } \Omega = \mathbb{R}^n, \\ \geq 0 & \text{if } \Omega \text{ bounded}, \end{cases}$$

$$\operatorname{ess\,inf} \operatorname{Re} V \geq 0, \ \ V \in \begin{cases} L^{n/2}_{\mathrm{w}}(\Omega, \mathbb{C}) + L^\infty(\Omega, \mathbb{C}) & \text{if } n \geq 5^*, \\ L^{2+\epsilon}(\Omega, \mathbb{C}) + L^\infty(\Omega, \mathbb{C}) & \text{if } n = 4 \text{ with } \epsilon > 0, \\ L^2(\Omega, \mathbb{C}) + L^\infty(\Omega, \mathbb{C}) & \text{if } n \leq 3, \end{cases}$$

(frictional damping).

* sufficient: $V \in L^p_{\mathrm{w}}(\Omega, \mathbb{C}) + L^\infty(\Omega, \mathbb{C})$ with $p \geq \frac{n}{2}$.

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Here

$$\mathcal{H}_{1/2} = H_0^1(\Omega, \mathbb{C}), \quad \mathcal{H}_{-1/2} = H^{-1}(\Omega, \mathbb{C}).$$

2. Numerical range and quadratic numerical range

Numerical range (NR): of a linear operator $\mathcal{A}$ in a Hilbert space $\mathcal{H}$

$$W(\mathcal{A}) := \big\{(\mathcal{A}x, x) : x \in \mathcal{D}(\mathcal{A}), \|x\| = 1\big\} \subset \mathbb{C}$$

convex, with spectral inclusion property

$$\sigma_{\mathrm{p}}(\mathcal{A}) \subset W(\mathcal{A}), \quad \sigma_{\mathrm{app}}(\mathcal{A}) \subset \overline{W(\mathcal{A})}.$$

Recall: $\lambda \in \sigma_{\mathrm{app}}(\mathcal{A}) :\Longleftrightarrow$

$$\exists\ (\mathbf{x}_n)_1^\infty \subset \mathcal{D}(\mathcal{A}),\ \|\mathbf{x}_n\| = 1 : \quad (\mathcal{A} - \lambda)\mathbf{x}_n \to 0,\ n \to \infty.$$

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Quadratic numerical range (QNR): of linear operator matrix

$$\mathcal{A}_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{on} \quad \mathcal{D}(\mathcal{A}_0) = \mathcal{D}_1 \oplus \mathcal{D}_2$$

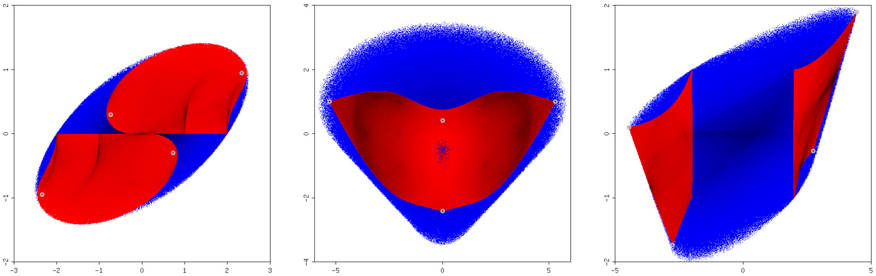
in product $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of Hilbert spaces: [Langer/T.'98, T. '08/09]

$$W^2(\mathcal{A}_0) := \bigcup_{\substack{x_1 \in \mathcal{D}_1, x_2 \in \mathcal{D}_2 \\ \|x_1\| = \|x_2\| = 1}} \sigma_{\mathrm{p}}\big(\mathcal{A}_{0;x_1,x_2}\big)$$

where

$$\mathcal{A}_{0;x_1,x_2} := \begin{pmatrix} (Ax_1, x_1) & (Bx_2, x_1) \\ (Cx_1, x_2) & (Dx_2, x_2) \end{pmatrix} \in M_2(\mathbb{C}).$$

A small QNR / NR gallery . . .



. . . for 4×4 matrices divided in 2×2 blocks

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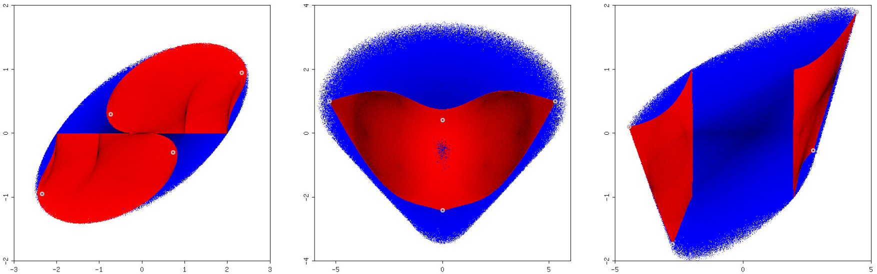

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Rem.

- $W^2(\mathcal{A}_0) \subset W(\mathcal{A}_0)$,
- $W^2(\mathcal{A}_0)$ not convex, at most two components,
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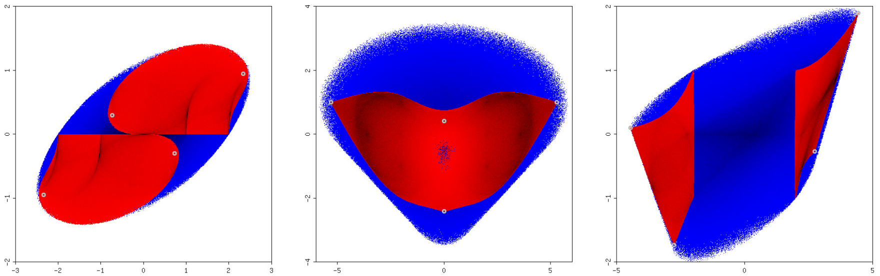

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- $\sigma_{\mathrm{app}}(\mathcal{A}_0) \subset \overline{W^2(\mathcal{A}_0)}$ *so far* for diag. dom. & off-diag. dom. OM

But not yet for top or bottom dominant and coupled domain!

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- $\mathcal{A}$ closed with $\mathcal{A} = \overline{\mathcal{A}|_{\mathcal{D}(A_0)\oplus\mathcal{D}(A_0)}}$ $\rightsquigarrow$

$$\mathcal{A}|_{\mathcal{D}(A_0)\oplus\mathcal{D}(A_0)} = \begin{pmatrix} 0 & I \\ -A_0 & -D \end{pmatrix}$$ operator matrix,

but neither diag. dom. nor off-diag. dom. !!!

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- $0 \in \rho(\mathcal{A})$ and $\mathrm{Re}\, W(\mathcal{A}) \leq 0$ with

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  in particular, if $D = D^*$, then $(-\sup W(D), 0] \subset W(\mathcal{A})$.

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Def. $\beta := \inf_{z\in H_{1/2}, z\neq 0} \frac{\operatorname{Re}\langle Dz,z\rangle}{\|z\|^2} \geq 0,$

$\mu := \inf_{z\in H_{1/2}, z\neq 0} \frac{\operatorname{Re}\langle Dz,z\rangle}{\|z\|\|z\|_{1/2}} \geq 0, \quad \Longrightarrow \quad \beta \geq a_0\mu \geq a_0^2\delta, \quad \mu^2 \geq \beta\delta.$

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(↑ $\|z\|_{1/2} \geq a_0\|z\|$)

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($\uparrow$ $\|z\|_{1/2} \geq a_0\|z\|$)

Note: $\beta > 0 \Leftrightarrow D$ uniformly accretive; $\quad \delta > 0 \implies \mu > 0 \implies \beta > 0$.

Theorem. *For* $\mathcal{A} = \begin{pmatrix} 0 & I \\ -A_0 & -D \end{pmatrix} = \overline{\mathcal{A}|_{\mathcal{D}(A_0)\oplus\mathcal{D}(A_0)}} =: \overline{\mathcal{A}_0}$ *as above,*

$$\sigma_{\rm p}(\mathcal{A}) \subset \overline{\sigma_{\rm p}(\mathcal{A}_0)} \subset \overline{W^2(\mathcal{A}_0)} \subset \overline{W(\mathcal{A})},$$

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(I) ***D*** **uniformly accretive:** $\delta > 0$, and hence $\beta > 0$:

- **not** reflected in **NR** $W(\mathcal{A})$;
- but reflected in **QNR** $\rightsquigarrow$ for $\lambda \in \sigma(\mathcal{A})$, $-\beta < \mathrm{Re}\lambda \le 0$,

$$|\mathrm{Im}\lambda| \le \sqrt{\frac{\beta}{\delta}\frac{|\mathrm{Re}\lambda|}{\beta - |\mathrm{Re}\lambda|} - |\mathrm{Re}\lambda|^2},$$

and, if $\beta\delta > 4$, additionally,

$$\mathrm{Re}\lambda \notin \Big[-\frac{\beta}{2} - \sqrt{\Big(\frac{\beta}{2}\Big)^2 - \frac{\beta}{\delta}}, -\frac{\beta}{2} + \sqrt{\Big(\frac{\beta}{2}\Big)^2 - \frac{\beta}{\delta}} \Big],$$

i.e. $\mathcal{A}$ has a spectral free strip around $\mathrm{Re}\lambda = -\dfrac{\beta}{2}$!

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$$|\mathrm{Im}\lambda| \le k|\mathrm{Re}\lambda| + 2\sqrt{\frac{1}{\delta}|\mathrm{Re}\lambda|};$$

  no sectorial enclosure possible since $0 \in \rho(\mathcal{A})$;

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where $0 < \lambda_1 < \lambda_2$ are explicitly known,

$$k_\mu^2 = \frac{2}{\mu^2} + \frac{k^2-1}{2} + \sqrt{\left(\frac{2}{\mu^2} + \frac{k^2-1}{2}\right)^2 + k^2} \in \left[k^2, k^2 + \frac{4}{\mu^2}\right],$$

and $h(t)$ is the largest non-negative zero of $(x^2+t^2)(x-kt) = \frac{2}{\delta}tx$.

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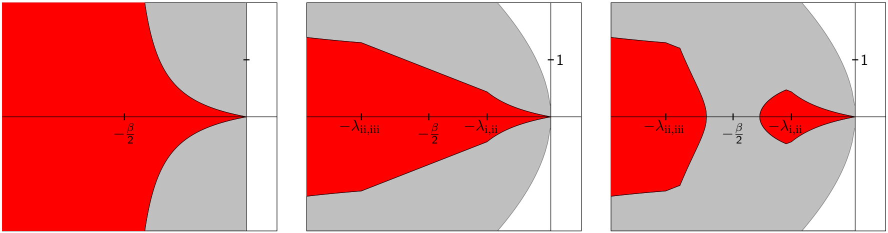
(III) ***D* = *D**:** i.e. $k = 0$, and e.g. $\delta > 0$:

$$\textbf{QNR} \rightsquigarrow \begin{cases} \sigma(\mathcal{A}) \subset (-\infty, 0) & \text{if } \mu \geq 2, \\ \sigma(\mathcal{A}) \setminus \mathbb{R} \text{ bounded} & \text{if } \mu < 2, \end{cases}$$

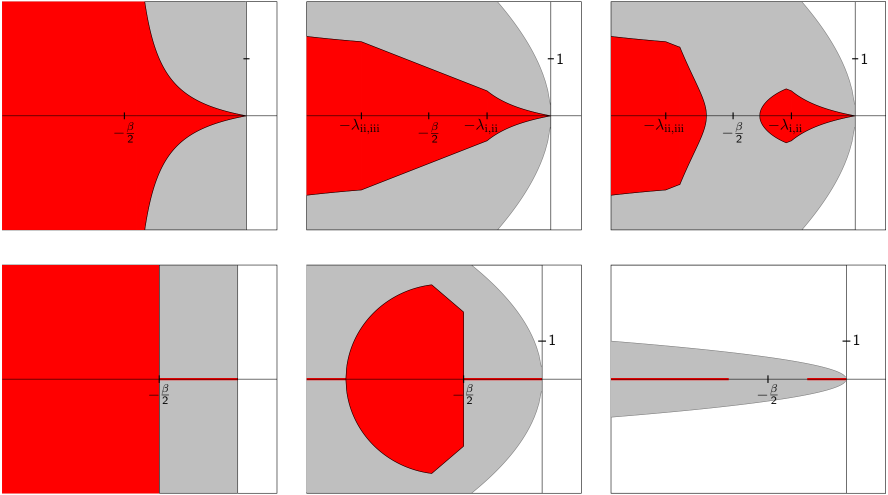
while for **NR** always $\mathrm{Im}\, W(\mathcal{A})$ unbounded!

D uniformly accretive and sectorial (top)



$\beta > 0$, $\mu = \delta = 0$ $\qquad$ $\beta > 0$, μ, $\delta > 0$ $\qquad$ $\beta > 0$, μ, $\delta > 0$, $\beta\delta > 4$

D uniformly accretive and sectorial (top) / self-adjoint (bottom)



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Appl. *Small transverse oscillations of ideal incompressible fluid in a pipe*:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial r^2}\left[E\frac{\partial^2 u}{\partial r^2} + C\frac{\partial^3 u}{\partial r^2 \partial t}\right] + K\frac{\partial^2 u}{\partial t \partial r} = 0, \quad r \in (0,1),\ \ t > 0,$$

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$$A_0 = E\frac{\mathrm{d}^4}{\mathrm{d}r^4}, \quad D = C\frac{\mathrm{d}^4}{\mathrm{d}r^4} + K\frac{\mathrm{d}}{\mathrm{d}r},$$

same strength!

hence $\beta = C\pi^4$, $\delta = \frac{C}{E}$, $\mu = \frac{C}{\sqrt{E}}\pi^2$, $k = \frac{K}{C\pi^3}$.

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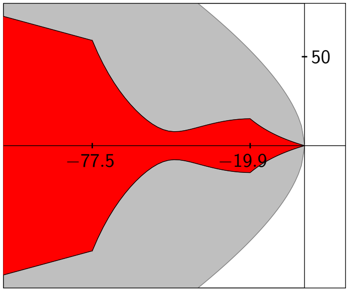
$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial r^2}\left[E\frac{\partial^2 u}{\partial r^2} + C\frac{\partial^3 u}{\partial r^2 \partial t}\right] + K\frac{\partial^2 u}{\partial t \partial r} = 0, \quad r \in (0,1), \ t > 0,$$

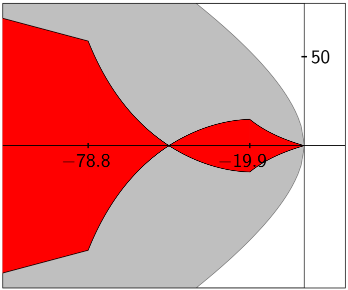
[Shkalikov '96]; here

$$A_0 = E\frac{\mathrm{d}^4}{\mathrm{d}r^4}, \quad D = C\frac{\mathrm{d}^4}{\mathrm{d}r^4} + K\frac{\mathrm{d}}{\mathrm{d}r},$$ *same strength*!

hence $\beta = C\pi^4$, $\delta = \dfrac{C}{E}$, $\mu = \dfrac{C}{\sqrt{E}}\pi^2$, $k = \dfrac{K}{C\pi^3}$. For $E = 25$, $K = 14$:



$C = 1$



$C = 10/\pi^2 \sim 1.014...$

Ex. *Wave equation with strong (viscoelastic and frictional) damping*:

$$\ddot{z}(t) + \underbrace{\left(-d\Delta + V\right)}_{=D}\dot{z}(t) + \underbrace{\left(-\Delta\right)}_{=A_0} z(t) = 0, \quad z(0) = z_0,\ \dot{z}(0) = z_1,$$

in $H = L^2(\Omega, \mathbb{C})$ with $\Omega \subset \mathbb{R}^n$ bounded with C^2-boundary where

$$d > 0, \quad \mathrm{ess\ inf}\,\mathrm{Re}\,V =: V_{\mathrm{Re,inf}} \geq 0 \qquad (b \equiv 0 \text{ for simplicity}).$$

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Here

$$a_0^2 = \min \sigma(-\Delta) = \lambda_1(\Omega) > 0,$$

$$\beta = \inf_{z \in H_0^1(\Omega,\mathbb{C})} \frac{d\langle -\Delta z, z\rangle + \langle \mathrm{Re}\, V z, z\rangle}{\|z\|^2} \geq d\lambda_1(\Omega) + V_{\mathrm{Re,inf}} =: \beta_-,$$

$$\delta = \inf_{z \in H_0^1(\Omega,\mathbb{C})} \frac{d\langle -\Delta z, z\rangle + \langle \mathrm{Re}\, V z, z\rangle}{\langle -\Delta z, z\rangle} \geq d =: \delta_-.$$

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Then:

$W(\mathcal{A})$ is *entire* closed left half-plane, **but:**

$W^2(\mathcal{A})$ gives *non-trivial* enclosure in strip $\{z \in \mathbb{C} : -\beta < \mathrm{Re}\, z \leq 0\}$,

and spectral free strip if $\beta\delta > 4 \rightsquigarrow d_{\mathrm{crit}} := -\frac{1}{2}\frac{V_{\mathrm{Re,inf}}}{\lambda_1(\Omega)} + \sqrt{\left(\frac{1}{2}\frac{V_{\mathrm{Re,inf}}}{\lambda_1(\Omega)}\right)^2 + \frac{4}{\lambda_1(\Omega)}}$

Ex. *Wave equation with strong (viscoelastic and frictional) damping*:

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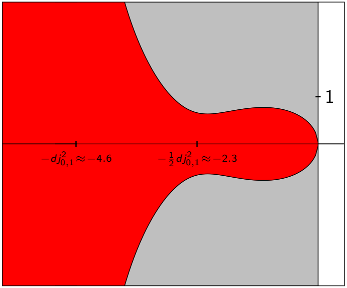
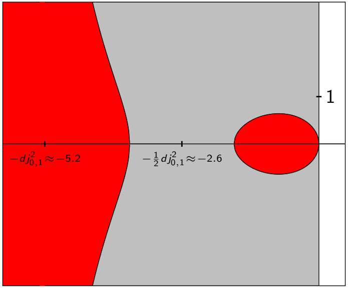
E.g. for $\Omega = S_1(0) \subset \mathbb{R}^2 \rightsquigarrow \lambda_1(\Omega) = j_{0,1}^2$, $V_{\mathrm{Re,inf}} = 0$:



$d = 0.8 \leq d_{\mathrm{crit}}$



$d = 0.9 > d_{\mathrm{crit}}$

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Thank you very much for your attention !