A bound on the pseudospectrum of the harmonic oscillator with imaginary potential

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Joint work with Patrick Dondl (Freiburg University), and Patrick Dorey (Durham University)

Outline

1. Non-selfadjoint Operators and Pseudospectra

2. Schrödinger Operators with Growing Potential

3. Our Results

4. Further Results and Open Questions

Non-selfadjoint Operators and Pseudospectra

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In addition, if H has compact resolvent, the eigenfunctions of H form a basis.

Non-Selfadjoint Operators

If H is *not* selfadjoint, *none* of the above results is true in general!

$\rightsquigarrow$ Spectrum contains very little information about H!

A Non-Selfadjoint Example

Consider $H : L^2(\mathbb{R}) \supset \mathcal{D}(L) \to L^2(\mathbb{R})$, where

$$L = -\frac{d^2}{dx^2} + ix^3 + cx^2, \qquad c \in \mathbb{R}$$

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$$L = -\frac{d^2}{dx^2} + ix^3 + cx^2, \qquad c \in \mathbb{R}$$

$$\mathcal{D}(L) = \{\psi \in L^2(\mathbb{R}) : L\psi \in L^2(\mathbb{R})\}$$

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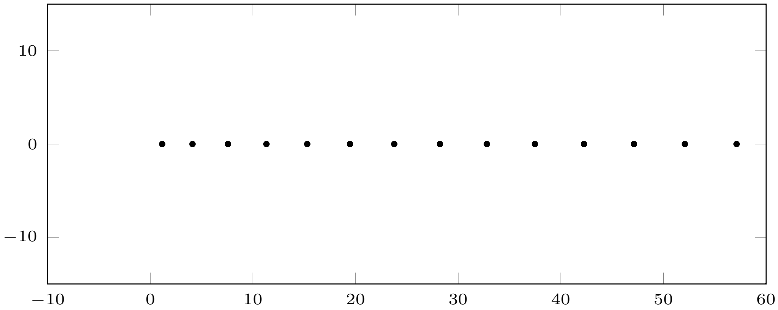

Figure: The spectrum of L

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The spectrum of L consists of discrete eigenvalues which are real, positive and algebraically simple.

Theorem ([Novak (2015)])

The eigenfunctions of L form a complete set in $L^2(\mathbb{R})$.

The Pseudospectrum

This motivates the definition of a *finer* indicator:

Definition

For $\varepsilon > 0$ the set

$$\sigma_\varepsilon(H) := \sigma(H) \cup \left\{ z \in \mathbb{C} : \|(z - H)^{-1}\| > \tfrac{1}{\varepsilon} \right\}$$

is called the ε-*pseudospectrum*.

Properties of the Pseudospectrum

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$\rightsquigarrow$ Pseudospectrum contains information about stability of eigenvalues.

Schrödinger Operators with Growing Potential

The Operator of Interest

In [Dondl, Dorey, R. (2016)] we are interested in the operator H defined as the closure of

$$H = -\Delta + V \quad \text{on} \quad C_0^\infty(\mathbb{R}^n)$$

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(I) There exist $a, b > 0$ such that $|\nabla V|^2 \leq a + b|V|^2$

(II) There exist $c, d > 0$ such that $\operatorname{Re} V(x) \geq c|x|^2 - d$.

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Theorem (Edmunds, Evans (1987))

$H+d$ is m-accretive and thus $-(H+d)$ generates a one-parameter semigroup of contractions.

On the Other Hand...

Theorem ([Novak (2015)], [Krejčiřík et. al. (2014)])

The operator $L_+ := -\frac{d^2}{dx^2} + ix^3 + x^2$ *on* $L^2(\mathbb{R})$ *has the following properties:*

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- *The eigenfunctions of* L_+ *do not form a (Schauder) basis in* $L^2(\mathbb{R})$.
- $-iL_+$ *does not generate a bounded semigroup.*
- *For any* $\delta > 0$ *there exist* $A, B > 0$ *such that for all* $\varepsilon > 0$

$$\sigma_\varepsilon(L_+) \supset \left\{ z \in \mathbb{C} : |z| > A, |\arg(z)| < \arctan(\operatorname{Re} z) - \delta, |z| \geq B \left(\log \tfrac{1}{\varepsilon}\right)^{\frac{6}{5}} \right\}$$

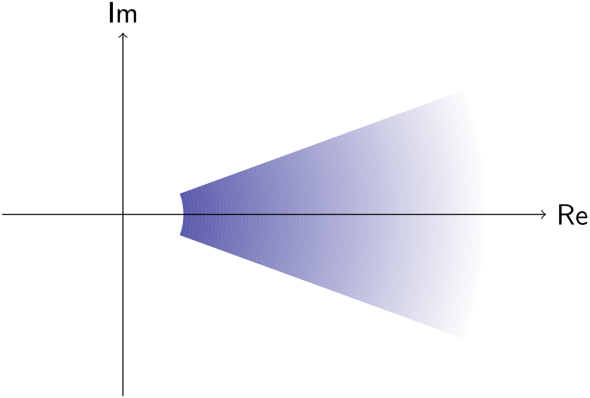
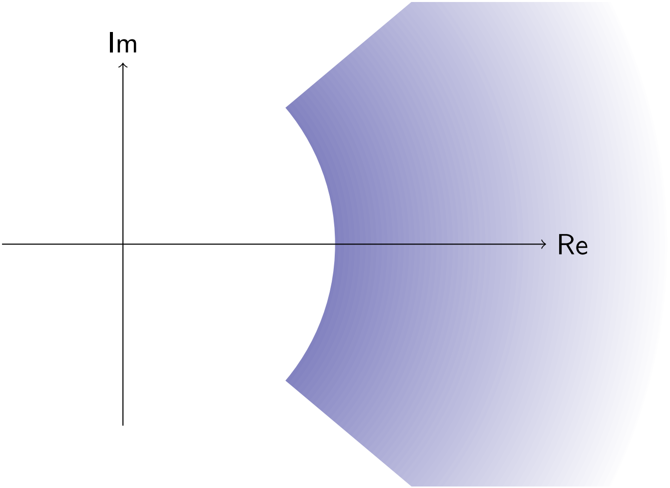


FIGURE: The pseudospectrum of the harmonic oscillator with imaginary cubic potential contains a set of the above shape.



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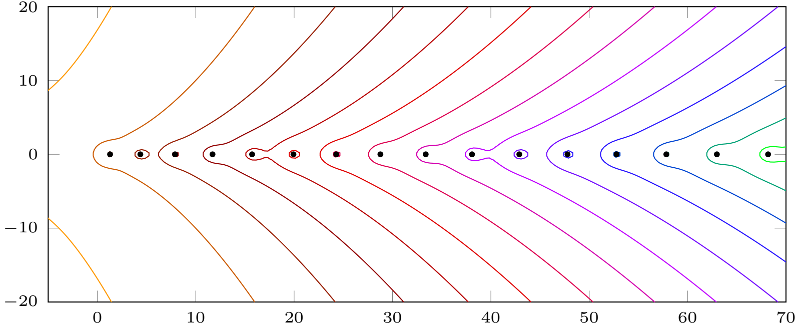

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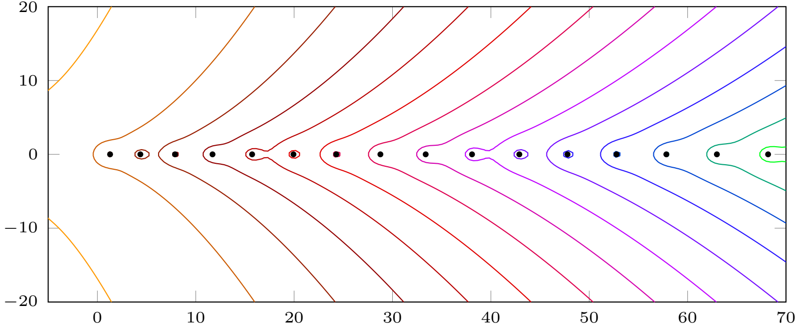

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$\rightsquigarrow$ Complementary inclusion result?

Our Results

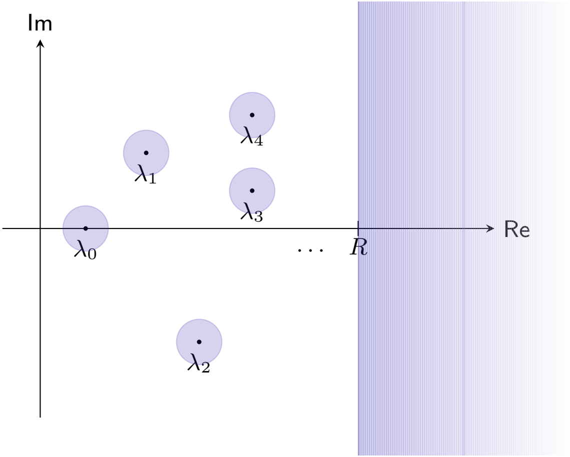
Our Result

Theorem

For every $\delta > 0, R > 0$ *there exists an* $\varepsilon > 0$ *such that*

$$\sigma_\varepsilon(-\Delta + V) \subset \{z : Re(z) \geq R\} \cup \bigcup_{\lambda \in \sigma(-\Delta+V)} \{z : |z - \lambda| < \delta\}. \tag{1}$$

In particular, the unbounded part of the pseudospectrum is contained in a half plane which moves towards $+\infty$ *as* ε *decreases.*



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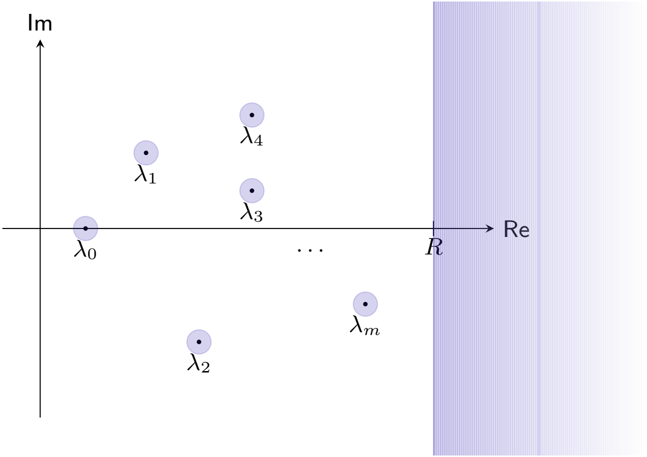

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Idea of Proof
(inspired by [Boulton (2002)])
(only for $d = 0$)

Use the standard estimate

Theorem (Hille-Yosida)

Let $-H$ be the generator of a one-parameter semigroup with $\|e^{-tH}\| \leq Me^{-\mu t}$ for all $t \geq 0$. Then

$$\|(z-H)^{-1}\| \leq \frac{M}{\mu - \operatorname{Re} z} \qquad \forall z : \operatorname{Re} z < \mu. \tag{2}$$

and show that $\mu > 0$ is possible.

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Theorem ([Davies (1980)])

If T_t is a one-parameter semigroup on a Banach space then

$$a := \lim_{t\to\infty} t^{-1} \log \|T_t\| \tag{3}$$

exists with $-\infty \leq a < \infty$. Moreover

$$r(T_t) := \max\{|\lambda| : \lambda \in \sigma(T_t)\} = e^{at} \quad \forall t > 0.$$

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- $\rightsquigarrow$ Can determine a if we know the spectral radius of e^{-tH}.
- $\rightsquigarrow$ If a turns out to be negative, can choose $0 < \mu < -a$
- $\rightsquigarrow$ Obtain a bound on $\|(H-z)^{-1}\|$ for $\mathrm{Re}(z) < \mu$

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$\Leftrightarrow$ $$a = -\mathrm{Re}\,\lambda_0$$

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  - $\rightsquigarrow$ Obtain bound on the resolvent for $\operatorname{Re} z < \lambda_{m+1}$.

Further Results and Open Questions

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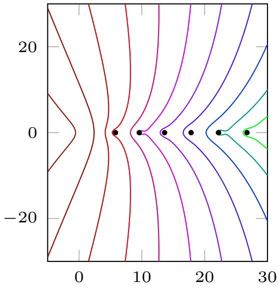

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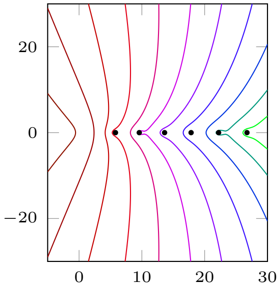
  $\rightsquigarrow$ No analogue of our theorem possible.



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Indeed, using the methods of [Krejčiřík, Siegl, Tater, Viola (2015)], we have the following theorem:

Theorem

For every $C, R, M > 0$ *there exists* $z \in \mathbb{C}$ *such that* $\operatorname{Re} z < -R$, $|z| > M$ *and*

$$\|(L_- - z)^{-1}\| \geq C. \tag{4}$$

In particular, L_- *does not generate a one-parameter semigroup.*

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Theorem

For the pseudospectrum of L_0 the inclusion (1) holds and in addition there exists a $C > 0$ such that for every $\delta > 0$ there is an $\varepsilon > 0$ such that

$$\sigma_\varepsilon(L_0) \subset \left\{ z : \operatorname{Re} z \geq C \left(\log \frac{1}{\varepsilon} \right)^{6/5} \right\} \cup \bigcup_{\lambda \in \sigma(L_0)} \{z : |z - \lambda| < \delta\}. \quad (5)$$

In particular, apart from disks around the eigenvalues, the ε-pseudospectrum is contained in the half plane $\left\{ \operatorname{Re} z \geq C \left(\log \frac{1}{\varepsilon} \right)^{6/5} \right\}$.

(based on [Henry (2014)].)

Thank You!