On Green functions of second-order elliptic operators on Riemannian manifolds: the critical case

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Mathematical aspects of the physics with non-self-adjoint operators
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Joint work with Debdip Ganguly

The setting

Let P be a second-order linear elliptic operator (not necessarily symmetric) with real coefficients defined on a domain $M \subseteq \mathbb{R}^n$ (or on a smooth noncompact (weighted) Riemannian manifold M of dimension n), where $n \geq 2$. So, in local coordinates P has the form

$$Pu := -\mathrm{div}\left[A(x)\nabla u + u\tilde{b}(x)\right] + b(x) \cdot \nabla u + c(x)u.$$

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Denote by

$$P^\star u := -\mathrm{div}\left[A(x)\nabla u + ub(x)\right] + \tilde{b}(x)\cdot\nabla u + c(x)u$$

the formal adjoint of P.

Green Function

Definition

A function $G_P^M : M \times M \to [-\infty, \infty]$ *is said to be a Green function (fundamental solution) of the operator* P *in* M *if for any* x, $y \in M$

$$P(x, \partial_x) G_P^M(x, y) = \delta_y(x), \qquad P^\star(y, \partial_y) G_P^M(x, y) = \delta_x(y) \text{ in } M,$$

and

$$G_{P^\star}^M(x, y) = G_P^M(y, x) \qquad \forall x, y \in M,$$

where δ_z *denotes the Dirac distribution at* $z \in M$.

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Definition

A positive Green function $G_P^M(x, y)$ *is said to be a positive minimal Green function of* P *in* M *if any other positive Green function* $\hat{G}_P^M(x, y)$ *of* P *in* M *satisfies* $0 < G_P^M(x, y) \leq \hat{G}_P^M(x, y)$ *in* $M \times M$*.*

Criticality theory

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Let P be an elliptic operator on M (as above).

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- If $P \not\geq 0$ in M, then P is *supercritical in* M.

Criticality theory

Remarks

1. *In the symmetric case, $P \geq 0$ iff the quadratic form associated to P is nonnegative on $C_0^\infty(M)$ (i.e. $\int_M P\varphi\,\overline{\varphi}\,\mathrm{d}x \geq 0 \quad \forall\,\varphi \in C_0^\infty(M)$).*

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4. *If* P *is critical in* M*, then the equation* $Pu = 0$ *admits a unique positive (super)solution* ψ *in* M*, called the* (Agmon) ground state of P in M.

Examples

Example

The Laplacian $P := -\Delta$ is subcritical in $\mathbb{R}^n$ iff $n \geq 3$. The corresponding positive minimal Green function is given by

$$G_{-\Delta}^{\mathbb{R}^n}(x, y) = C_n |x - y|^{2-n},$$

while for $n = 1, 2$ the ground state is given by $\psi(x) = 1$.

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Let $M \Subset \mathbb{R}^n$ and P be a uniformly elliptic operator with up to the boundary smooth enough coefficients.
Let λ_0 be the principal eigenvalue of P in M.
Then $P - \lambda$ is subcritical if $\lambda < \lambda_0$, critical if $\lambda = \lambda_0$, and supercritical if $\lambda > \lambda_0$.

Examples

Example

Let $M := \mathbb{R}^n \setminus \{0\}$, $n \geq 3$. Consider the n-dimensional Hardy inequality

$$\int_M |\nabla\varphi|^2 \,\mathrm{d}x \geq \left(\frac{n-2}{2}\right)^2 \int_M \frac{|\varphi(x)|^2}{|x|^2} \,\mathrm{d}x \qquad \forall\varphi \in C_0^\infty(M),$$

Then $P = -\Delta - \left(\frac{n-2}{2}\right)^2 |x|^{-2}$ is *critical* in M with ground state

$$\psi(x) := |x|^{(2-n)/2}.$$

The positive minimal Green function

Suppose that $P \geq 0$ in M, then the generalized maximum principle holds in any compact subdomain $\tilde{M} \Subset M$, and the Dirichlet problem

$$Pu_f = f \text{ in } \tilde{M}, \qquad u_f = 0 \text{ on } \partial\tilde{M},$$

is uniquely solvable in $\tilde{M}$; the solution is given by the Dirichlet Green function $G_P^{\tilde{M}}(x, y)$ of P in $\tilde{M}$:

$$u_f(x) = \int_{\tilde{M}} G_P^{\tilde{M}}(x, y) f(y) \, \mathrm{d}y.$$

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Let $\{M_j\}_{j=1}^{\infty}$ be a (compact) exhaustion of M, i.e. a sequence of smooth, relatively compact domains in M such that $M_1 \neq \emptyset$, $M_j \Subset M_{j+1}$ and $\cup_{j=1}^{\infty} M_j = M$.

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Let $G_P^{M_j}(x,y)$ be the Dirichlet Green function of P in M_j.

The positive minimal Green function

By the generalized maximum principle, $\{G_P^{M_j}(x,y)\}_{j=1}^{\infty}$ is an increasing sequence of positive functions which (by the Harnack principle) converges locally uniformly in $M \times M \setminus \{(x,x) \mid x \in M\}$, and

$$\lim_{j\to\infty} G_P^{M_j}(x,y) = \begin{cases} G_P^M(x,y) & \text{if } P \text{ is subcritical in } M, \\ \infty & \text{if } P \text{ is critical in } M. \end{cases}$$

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The existence of a fundamental solution for differential operators with constant coefficients has been proved by L. Ehrenpreis (1954) and B. Malgrange (1955), and for elliptic operators with analytic coefficients by F. John (1955) using the unique continuation property.

Li-Tam's Green Function

Theorem (Peter Li & Luen-Fai Tam, AJM, 1987)

Let M be a complete noncompact Riemannian manifold. Then for the Laplace-Beltrami operator there exists a symmetric Green function $G^M_{-\Delta}(x, y)$. In particular, $G^M_{-\Delta}(x, y)$ satisfies equation

$$-\Delta_x \left(G^M_{-\Delta}(x, y) \right) = \delta_y(x) \qquad \forall y \in M.$$

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$$-\Delta_x \left(G_{-\Delta}^{M}(x,y) \right) = \delta_y(x) \qquad \forall y \in M.$$

The proof relies on the unique continuation property and the completeness of M, and hinges on a construction of a converging sequence of the form

$$\left\{ G_{-\Delta}^{M_j}(x,y) - a_j \right\}_{j=1}^{\infty},$$

where $\{a_j\}$ is an appropriate sequence (in the critical case $\lim a_j = \infty$).

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We call a Green function that is obtained by such a construction a Li-Tam (LT) Green function.

The main theorem

Theorem

Let P be a critical operator on a noncompact Riemannian manifold M of dimension $n \geq 2$. Denote by Φ and $\Phi^{\star}$ the ground states of P and $P^{\star}$.

1. *P admits a LT Green function $G_P^M(x, y)$ in M.*

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1. P *admits a LT Green function* $G_P^M(x,y)$ *in* M*.*
2. *Any LT Green function* $G_P^M(x,y)$ *satisfies the following boundedness property: For any* $y \in M$ *and any neighborhood* U_y *of* y *there exists* $C > 0$ *depending on* U_y *such that*

$$G_P^M(x,y) \leq C\Phi(x) \quad \text{and} \quad G_{P^\star}^M(x,y) \leq C\Phi^\star(x) \qquad \forall x \in M \setminus U_y.$$

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3. *For any Green function $\hat{G}_P^M$ and $y \in M$, we have*
$$\liminf_{x \to \bar{\infty}} \frac{\hat{G}_P^M(x,y)}{\Phi(x)} = -\infty.$$

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4. *For any $z \in M$ there exists an LT Green function $\hat{G}_P^M(x,y)$ such that in some neighborhood U_z of z we have*
$$\hat{G}_P^M(x,z) < 0 \qquad \forall x \in M \setminus U_z.$$

Uniqueness

Theorem

Let P be a critical operator in M, and let $\tilde{G}_P^M$, and G_P^M be two LT Green functions. Then there exists $C \in \mathbb{R}$ such that

$$\hat{G}_P^M(x,y) = G_P^M(x,y) + C\Phi(x)\Phi^\star(y) \qquad \forall x, y \in M.$$

In particular, if $\tilde{G}_P^M(x_0,y_0) = G_P^M(x_0,y_0)$ for some $x_0, y_0 \in M$, then $\tilde{G}_P^M = G_P^M$.

Key Lemma

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Suppose that $P(1) = 0$ *in* M *(in particular,* $P \geq 0$ *in* M*). Fix* $p \in M$*. Then the sequence of Green functions* $\{G_P^{M_j}(\cdot, p)\}_{j=j_0}^{\infty}$ *has locally uniform bounded oscillation in* $M \setminus \{p\}$*.*

Outline of the proof of the main Theorem

1. **Reduction:** Use a modified ground state transform to define the critical operator

$$L(u) := \Phi^\star P(\Phi u),$$

where Φ and $\Phi^\star$ denote the ground state of the operator P and $P^\star$, respectively. So, $L(1) = L^\star(1) = 0$, with ground states $\widetilde{\Phi} = \widetilde{\Phi^\star} = 1$.

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2. **Claim:** Fix $p \in M_1$, then, up to a subsequence,
$$J_L^{M_j}(x,p) := G_L^{M_j}(x,p) - \alpha_j^{(p)} \underset{j\to\infty}{\longrightarrow} G_L^M(x,p)$$
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4. For any fixed $y \in M$, the sequence $\left\{G_L^{M_j}(x,y) - \alpha_j^{(p)}\right\}$ converges for all $x \neq y$ to a function $G_L^M(x,y)$.
5. $G_P^M(x,y) := \Phi(x) G_L^M(x,y) \Phi^\star(y)$ is a Green function of P in M.

The behavior of the LT Green function at ∞

Problem

Let G_P^M *be a LT Green function of a critical operator* P *in* M *with a ground state* Φ*. Does the following assertion hold true?*

$$\lim_{x \to \bar{\infty}} \frac{G_P^M(x,y)}{\Phi(x)} = -\infty.$$

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Theorem

Let $G_P^M(x,y)$ be a LT Green function of a symmetric (or even quasi-symmetric) critical operator P in M. Suppose that for $0 \lneqq W \in C_0^\infty(M)$, the Martin boundary of $P + W$ in M is a singleton. Then $\lim_{x\to\bar{\infty}} \frac{G_P^M(x,y)}{\Phi(x)} = -\infty$.

Examples

Example

$$G_{-\Delta}^{\mathbb{R}^1}(x,y) = -\frac{1}{2}|x-y| + C, \qquad G_{-\Delta}^{\mathbb{R}^2}(x,y) = -\frac{1}{2\pi}\log|x-y| + C.$$

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Example

Let $M = \mathbb{R}^n \setminus \{0\}$, where $n \geq 3$, and consider the critical Hardy operator

$$P := -\Delta - \frac{(n-2)^2}{4}\frac{1}{|x|^2}\,.$$

with the ground state $v(x) = |x|^{(2-n)/2}$.

For $\zeta = 0$ or $\zeta = \infty$ the limit $\lim_{x\to\zeta} \frac{G_P^M(x,x_0)}{|x|^{(2-n)/2}}$ exists. Moreover, the limit is equal to $-\infty$ at least at one of these points. We do not know whether the limit is equal to $-\infty$ at *both* ends.

Note that $\lim_{x\to\infty} G_P^M(x,x_0) = 0$ but $\lim_{x\to 0} G_P^M(x,x_0) = -\infty$.

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Suppose that $P(1) = 0$ in M (in particular, $P \geq 0$ in M). Fix $p \in M$. Then the sequence of Green functions $\{G_P^{M_j}(\cdot, p)\}_{j=j_0}^{\infty}$ has locally uniform bounded oscillation in $M \setminus \{p\}$.

Proof of the key lemma

Consider 'annuli' of the form $A_p(k) := M_k \setminus B(p, \frac{1}{k})$, $k \geq 1$.

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For $j > k$, denote by $\omega_j(k)$ the oscillation of $G_P^{M_j}(x,p)$ on $A_p(k)$:

$$\omega_j(k) = \sup_{x \in A_p(k)} \{G_P^{M_j}(x,p)\} - \inf_{x \in A_p(k)} \{G_P^{M_j}(x,p)\}.$$

It suffices to prove that for $\forall k \geq 1$, the sequence $\{\omega_j(k)\}_{j>k}$ is bounded.

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For $j > k$, denote by $\omega_j(k)$ the oscillation of $G_P^{M_j}(x,p)$ on $A_p(k)$:

$$\omega_j(k) = \sup_{x \in A_p(k)} \{G_P^{M_j}(x,p)\} - \inf_{x \in A_p(k)} \{G_P^{M_j}(x,p)\}.$$

It suffices to prove that for $\forall k \geq 1$, the sequence $\{\omega_j(k)\}_{j>k}$ is bounded.

Fix k. Suppose that there exists a subsequence of $\omega_j := \omega_j(k)$ such that $\omega_j \to \infty$.

Proof of the key lemma

Consider 'annuli' of the form $A_p(k) := M_k \setminus B(p, \frac{1}{k})$, $k \geq 1$.

For $j > k$, denote by $\omega_j(k)$ the oscillation of $G_P^{M_j}(x,p)$ on $A_p(k)$:

$$\omega_j(k) = \sup_{x \in A_p(k)} \{G_P^{M_j}(x,p)\} - \inf_{x \in A_p(k)} \{G_P^{M_j}(x,p)\}.$$

It suffices to prove that for $\forall k \geq 1$, the sequence $\{\omega_j(k)\}_{j>k}$ is bounded.

Fix k. Suppose that there exists a subsequence of $\omega_j := \omega_j(k)$ such that $\omega_j \to \infty$.

Define for $j > k$ functions h_j by

$$h_j(x) := \omega_j^{-1} G_P^{M_j}(x,p) - \omega_j^{-1} \inf_{z \in M_k} \{G_P^{M_j}(z,p)\}.$$

Clearly, $Ph_j = 0$ in $M_j \setminus \{p\}$, and $\mathrm{Osc}(h_j) = 1$ in $A_p(k)$.

Proof of the key lemma (continued)

Note that for a fixed j

$$h_j(x) := \omega_j^{-1} G_P^{M_j}(x,p) - \omega_j^{-1} \inf_{z\in M_k}\{G_P^{M_j}(z,p)\} \underset{x\to p}{\sim} \omega_j^{-1} G_P^{M_1}(x,p).$$

Proof of the key lemma (continued)

Note that for a fixed j

$$h_j(x) := \omega_j^{-1} G_P^{M_j}(x,p) - \omega_j^{-1} \inf_{z\in M_k}\{G_P^{M_j}(z,p)\} \underset{x\to p}{\sim} \omega_j^{-1} G_P^{M_1}(x,p).$$

Furthermore, by the WMP on the domain M_k

$$\inf\{h_j(x) : x \in M_k\} = \inf\{h_j(x) : x \in \partial M_k\} = 0.$$

Proof of the key lemma (continued)

Note that for a fixed j

$$h_j(x) := \omega_j^{-1} G_P^{M_j}(x,p) - \omega_j^{-1} \inf_{z\in M_k} \{G_P^{M_j}(z,p)\} \underset{x\to p}{\sim} \omega_j^{-1} G_P^{M_1}(x,p).$$

Furthermore, by the WMP on the domain M_k

$$\inf\{h_j(x) : x\in M_k\} = \inf\{h_j(x) : x \in \partial M_k\} = 0.$$

On the other hand, it follows that

$$\omega_j^{-1} G_P^{M_k}(x,p) \leq h_j(x) \leq \omega_j^{-1} G_P^{M_k}(x,p) + 1 \qquad \forall x\in M_k,$$

Proof of the key lemma (continued)

Note that for a fixed j

$$h_j(x) := \omega_j^{-1} G_P^{M_j}(x,p) - \omega_j^{-1} \inf_{z \in M_k} \{ G_P^{M_j}(z,p) \} \underset{x \to p}{\sim} \omega_j^{-1} G_P^{M_1}(x,p).$$

Furthermore, by the WMP on the domain M_k

$$\inf\{h_j(x) : x \in M_k\} = \inf\{h_j(x) : x \in \partial M_k\} = 0.$$

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$$\omega_j^{-1} G_P^{M_k}(x,p) \leq h_j(x) \leq \omega_j^{-1} G_P^{M_k}(x,p) + 1 \qquad \forall x \in M_k,$$

$\{h_j\}$ converges in $M_k \setminus \{p\}$ to a function h satisfying $Ph = 0$ in $M_k \setminus \{p\}$ and $0 \leq h \leq 1$. By a removable singularity theorem, h can be extended to $\bar{h}$ which satisfies $P\bar{h} = 0$ in M_k

Proof of the key lemma (continued)

Note that for a fixed j

$$h_j(x) := \omega_j^{-1} G_P^{M_j}(x,p) - \omega_j^{-1} \inf_{z\in M_k} \{G_P^{M_j}(z,p)\} \underset{x\to p}{\sim} \omega_j^{-1} G_P^{M_1}(x,p).$$

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Since $\bar{h}$ has maximum at p, SMP implies $\bar{h} = \mathrm{constant} = C$ in M_k.

Proof of the key lemma (continued)

Note that for a fixed j

$$h_j(x) := \omega_j^{-1} G_P^{M_j}(x,p) - \omega_j^{-1} \inf_{z\in M_k}\{G_P^{M_j}(z,p)\} \underset{x\to p}{\sim} \omega_j^{-1} G_P^{M_1}(x,p).$$

Furthermore, by the WMP on the domain M_k

$$\inf\{h_j(x) : x \in M_k\} = \inf\{h_j(x) : x \in \partial M_k\} = 0.$$

On the other hand, it follows that

$$\omega_j^{-1} G_P^{M_k}(x,p) \le h_j(x) \le \omega_j^{-1} G_P^{M_k}(x,p) + 1 \qquad \forall x \in M_k,$$

$\{h_j\}$ converges in $M_k \setminus \{p\}$ to a function h satisfying $Ph = 0$ in $M_k \setminus \{p\}$ and $0 \le h \le 1$. By a removable singularity theorem, h can be extended to $\bar{h}$ which satisfies $P\bar{h} = 0$ in M_k

Since $\bar{h}$ has maximum at p, SMP implies $\bar{h} = \mathrm{constant} = C$ in M_k. Further, WMP implies that $f_j := C + 1 - h_j \ge 0$ in M_j. Hence, it converges to a positive solution f in M.

Proof of the key lemma (continued)

Note that for a fixed j

$$h_j(x) := \omega_j^{-1} G_P^{M_j}(x,p) - \omega_j^{-1} \inf_{z\in M_k}\{G_P^{M_j}(z,p)\} \underset{x\to p}{\sim} \omega_j^{-1} G_P^{M_1}(x,p).$$

Furthermore, by the WMP on the domain M_k

$$\inf\{h_j(x) : x\in M_k\} = \inf\{h_j(x) : x\in \partial M_k\} = 0.$$

On the other hand, it follows that

$$\omega_j^{-1} G_P^{M_k}(x,p) \leq h_j(x) \leq \omega_j^{-1} G_P^{M_k}(x,p) + 1 \qquad \forall x\in M_k,$$

$\{h_j\}$ converges in $M_k \setminus \{p\}$ to a function h satisfying $Ph=0$ in $M_k\setminus\{p\}$ and $0\leq h\leq 1$. By a removable singularity theorem, h can be extended to $\bar{h}$ which satisfies $P\bar{h}=0$ in M_k

Since $\bar{h}$ has maximum at p, SMP implies $\bar{h} = \mathrm{constant} = C$ in M_k.

Further, WMP implies that $f_j := C+1-h_j \geq 0$ in M_j. Hence, it converges to a positive solution f in M.

Note that $f=1$ on M_k, hence $\mathrm{Osc}(f)=0$ in $A_p(k)$. This contradicts that

$$\mathrm{Osc}(f) = \mathrm{Osc}(h_j) = 1 \text{ in } A_p(k). \quad \square$$

Thank you for your attention!