Spectral enclosures for non-self-adjoint waveguides with Robin boundary conditions

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in collaboration with

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Outline

Uniformly regular domains

[1] R. Freeman, *Pacific J. Math.* **12** (1962), 121–135.

Uniformly regular domains

$\Omega \subset \mathbb{R}^d$ – C^∞-smooth uniformly regular domain[1]; $\Omega^{\rm c} := \mathbb{R}^d \setminus \overline{\Omega}$, $\Sigma := \partial\Omega$.

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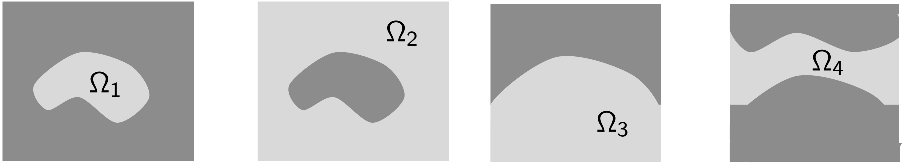

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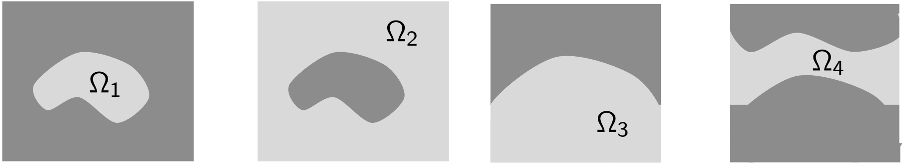

The class includes

(i) Bounded and exterior C^∞-smooth domains.

(ii) Hypographs of C^∞-functions with bounded derivatives.

(iii) Bent waveguides and layers without increasing oscillations at ∞.

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Neumann trace of $u \in H^2(\Omega)$ and its jump for $w \in H^2(\Omega) \oplus H^2(\Omega^{\mathrm{c}})$

$\partial_\nu u|_\Sigma$ " $=$ " $\nu(x) \cdot (\nabla u)|_\Sigma$ with ν pointing outwards of Ω.

$[\partial_\nu w]|_\Sigma = \partial_\nu w_+|_\Sigma - \partial_\nu w_-|_\Sigma$ with $w_+ = w|_\Omega$ and $w_- = w|_{\Omega^{\mathrm{c}}}$.

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The Robin Laplacian in $L^2(\Omega)$

$\mathsf{H}^\Omega_\alpha u = -\Delta u, \quad \operatorname{dom} \mathsf{H}^\Omega_\alpha = \{u \in H^2(\Omega) : \partial_\nu u|_\Sigma = \alpha u|_\Sigma\}.$

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Schrödinger operator in $L^2(\mathbb{R}^d)$ with δ-interaction on Σ

$\mathsf{H}^\Sigma_\alpha w = (-\Delta w_+) \oplus (-\Delta w_-),$
$\mathrm{dom} \mathsf{H}^\Sigma_\alpha = \{w_+ \oplus w_- \in H^2(\Omega) \oplus H^2(\Omega^c) \colon w_+|_\Sigma = w_-|_\Sigma,\ [\partial_\nu w]_\Sigma = \alpha w|_\Sigma\}$

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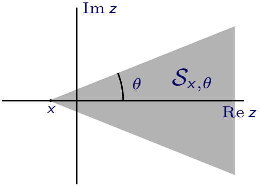
- $\mathcal{S}_{x,\theta} := \{z \in \mathbb{C} \colon |\mathrm{Im}\, z| \le \tan\theta\, \mathrm{Re}\,(z - x)\}$ with $x \in \mathbb{R}$, $\theta \in (0, \frac{\pi}{2})$
- for m-sectorial H, $\sigma(\mathsf{H}) \subset \mathcal{S}_{x,\theta}$ with proper x, θ



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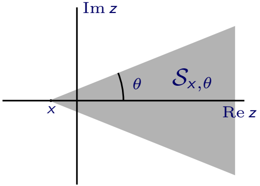

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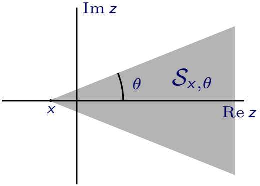

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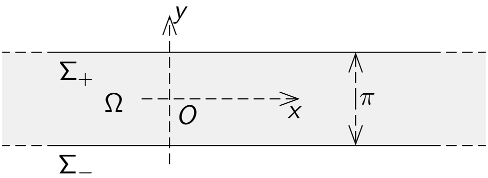
Local and global absence of non-real spectrum for some complex α.

$\mathcal{PT}$-symmetric waveguide as a special case

[2] D. Borisov and D. Krejčiřík, *IEOT* **62** (2008), 489–515.

$\mathcal{PT}$-symmetric waveguide as a special case

$\Omega = \mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$, $\Sigma = \Sigma_+ \cup \Sigma_-$ where $\Sigma_\pm = \mathbb{R} \times \{\pm\frac{\pi}{2}\}$



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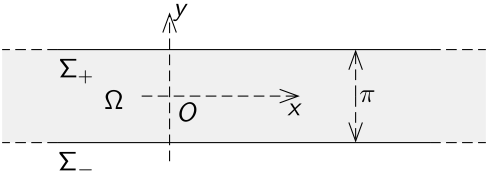

$\mathcal{PT}$-symmetry

$\alpha_\pm := \alpha|_{\Sigma_\pm}$ and $\alpha_+ = \overline{\alpha_-}$.

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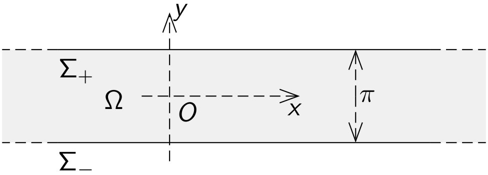

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Hamiltonian of $\mathcal{PT}$-symmetric waveguide[2]

(i) J-selfadjoint with $(\mathsf{J}u)(x,y) = u(x,-y)$; $(\mathsf{H}^\Omega_\alpha)^* = \mathsf{J}\mathsf{H}^\Omega_\alpha\mathsf{J}$.

(ii) $\sigma(\mathsf{H}^\Omega_\alpha) \subset \mathbb{R}$ if $\alpha_+ = \mathrm{i}\beta$ where $\beta \in C_0(\mathbb{R}) \cap W^1_\infty(\mathbb{R})$ is real and odd.

(iii) If and only if condition for $\sigma(\mathsf{H}^\Omega_\alpha) \subset \mathbb{R}$ can hardly be found.

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Universal spectral enclosures I

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Universal spectral enclosures I

Truncated parabolic region for $x_0 < x_1 < 0$ and $k > 0$

$$\mathcal{P}_{x_0,x_1,k} = \{z \in \mathbb{C} \colon \mathrm{Re}\, z \geq x_1, |\mathrm{Im}\, z| \leq k(\mathrm{Re}\, z - x_0)^{1/2}\}.$$

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Theorem (Behrndt-Langer-VL-Rohleder)

$\sigma(\mathsf{H}_\alpha^\Omega) \subset \mathcal{P}_{x_0,x_1,k}$ *and* $\sigma(\mathsf{H}_\alpha^\Sigma) \subset \mathcal{P}_{y_0,y_1,\ell}$.

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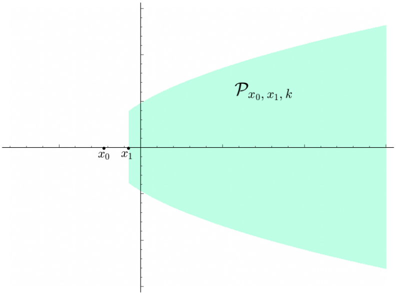

- x_0, x_1, k, y_0, y_1, ℓ depend on α and Σ.
- such a result was known for compact $\partial\Omega$[3]
- p-subordinate additive perturb.: (Markus-Matsaev-81, Wyss-10)
- varying Robin coefficient is a non-additive perturbation.

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Universal spectral enclosures II

Neighbourhood of $\mathbb{R}_+$

$\mathcal{D}(R) = \{z \in \mathbb{C} \colon \mathrm{dist}\,(z, \mathbb{R}_+) \leq R\}$ for $R > 0$.

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$$\sigma(\mathsf{H}_\alpha^\Omega) \subset \mathcal{S}_{x,\theta} \cup \mathcal{D}(a\|\alpha\|_\infty^{1/\beta}) \quad \textit{and} \quad \sigma(\mathsf{H}_\alpha^\Sigma) \subset \mathcal{S}_{x,\theta} \cup \mathcal{D}(b\|\alpha\|_\infty^{1/\beta})$$

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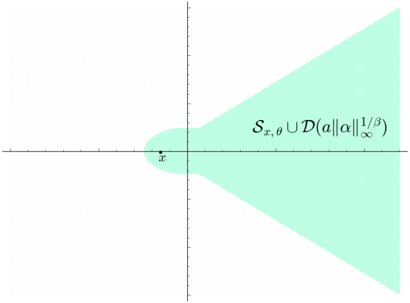

- $\beta = \frac{1}{2}$?
- Note that θ is arbitrary.
- for relatively compact additive perturbations Gokhberg-Krein-69, Cuenin-Tretter-16

Spectral enclosures for H_α^Σ with compact Σ

[4] A. A. Abramov, A. Aslanyan, and E. B. Davies, *J. Phys. A* **34** (2001), 57–72.
[5] R. L. Frank, *Bull. Lond. Math. Soc.* **43** (2011), 745–750.
[6] L. Fanelli, D. Krejčiřík and L. Vega, `arXiv:1506.01617`.

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$$\sigma(\mathsf{H}_\alpha^\Sigma) \setminus \mathbb{R}_+ \subset \left\{ z \in \mathbb{C} \colon |z| \leq 2\exp\left(-\frac{1}{c\|\alpha\|_\infty} \right) \right\}, \qquad \text{for } \|\alpha\|_\infty < \alpha_*.$$

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Preceeding related results for complex regular potentials[4,5,6].

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Enclosures for $\mathcal{PT}$-symmetric waveguides

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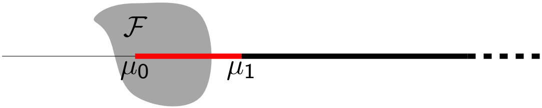
Fix a compact set $\mathcal{F} \subset \mathbb{C}$ with $\mathcal{F} \cap \mathbb{R} \subset (-\infty, \mu_1)$



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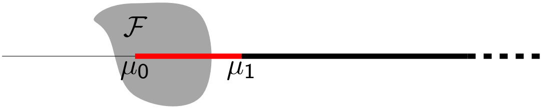

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There exists $\gamma = \gamma(\kappa, \mathcal{F}) > 0$ *such that* $\sigma(\mathsf{H}^\Omega_\alpha) \cap \mathcal{F} \subset \mathbb{R}$ *for* $\alpha = \alpha_+ \oplus \alpha_-$ *with* $\alpha_+ = \overline{\alpha_-}$ *and* $\|\alpha - \beta\|_\infty < \gamma$.

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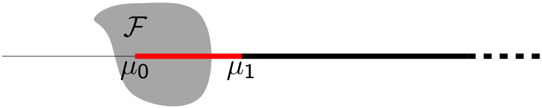

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Non-real spectrum of H^Ω_α does not appear near low-lying real spectrum.

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$$\sigma(\mathsf{H}^\Omega_\alpha) = \{\lambda_0, \lambda_1, \overline{\lambda_1}\} + \mathbb{R}_+, \qquad \lambda_0 \in \mathbb{R}, \ \mathrm{Im}\, \lambda_1 > 0.$$

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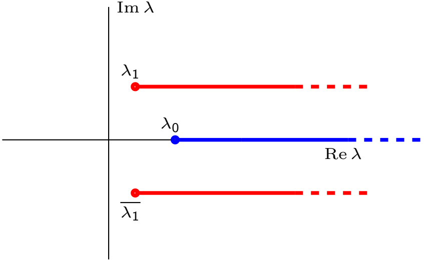

$\lambda_0(z), \lambda_1(z) \to 1$ as $z \to \mathrm{i}$.

Necessity of the assumption $\kappa \neq 1$

For $\beta = \mathrm{i} \oplus (-\mathrm{i})$, $(\kappa = 1)$, holds $\sigma(\mathsf{H}^{\Omega}_{\beta}) = [1, \infty)$.

Example

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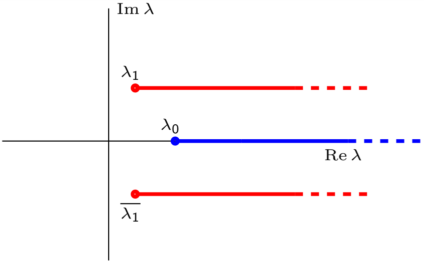

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There exists arbitrarily small perturbation of $\beta = \mathrm{i} \oplus (-\mathrm{i})$ which creates non-real spectrum in any compact domain in $\mathbb{C}$ intersecting $[1, \infty)$.

Outline

Parabolic regions arise as intersection of sectors

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Consider the BVP for $\psi \in L^2(\Sigma)$ and $z \in \mathbb{C} \setminus \mathbb{R}_+$

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$\sigma(\mathsf{H}^\Omega_\alpha) \subset \mathcal{S}_{\eta,\theta(\eta)}$ for $\eta < \eta_*$ and using decay of $\|\Lambda_\Omega(\eta)\|$ as $\eta \to -\infty$

$$\mathcal{P}_{x_0,x_1,k} = \bigcap_{\eta<\eta_*} \mathcal{S}_{\eta,\theta(\eta)}.$$

Galkowski-Smith bounds and spectral enclosures for H_α^Σ

[7] J. Galkowski and H. Smith, *Int. Math. Res Notices* **16** (2015), 7473–7509.

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Assume that $\Sigma(= \partial\Omega)$ is compact. Recall $\Omega^c := \mathbb{R}^d \setminus \overline{\Omega}$.

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We combine BS-principle and sharp upper bounds[7] on $\|\Lambda_\Sigma(z)\|$.

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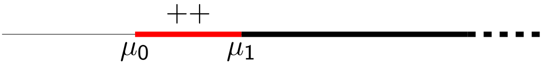

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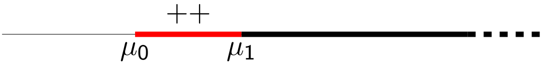

Claims on H_α^Ω for more general α are obtained via stability of σ_{++} under small perturbations (Azizov, Behrndt, Jonas, Philipp, Trunk,...).

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Non-real spectrum can be excluded in a compact set $\mathcal{F} \subset \mathbb{C}$ satisfying certain geometric condition.

References

References

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