Eigenvalue asymptotics for the damped wave equation on metric graphs

Jiří Lipovský [1]

University of Hradec Králové, Faculty of Science
jiri.lipovsky@uhk.cz

joint work with P. Freitas

Marseille, June 6th, 2017

[1] Support of project 15-14180Y "Spectral and resonance properties of quantum models" of the Czech Science Foundation is acknowledged.

Description of the model

- metric graph with N finite edges
- on each edge damped wave equation

$$\partial_{tt} w_j(t,x) + 2a_j(x)\partial_t w_j(t,x) = \partial_{xx} w_j(t,x) + b_j(x) w_j(t,x)$$

- the ansatz $w_j(t,x) = \mathrm{e}^{\lambda t} u_j(x)$ allows us to rewrite equation it as

$$\partial_{xx} u_j(x) - (\lambda^2 + 2\lambda a_j(x) - b_j(x)) u_j(x) = 0$$

- we define $\tilde{\lambda}_j(\lambda) := \sqrt{\lambda^2 + 2\lambda a_j(x) - b_j(x)}$ and for a_j, b_j constant on each edge we obtain solutions $\mathrm{e}^{\pm\tilde{\lambda}_j x}$
- coupling conditions at the vertices

$$(U - I)\Psi + i(U + I)\Psi' = 0$$

- substituting for Ψ and Ψ' one obtains the secular equation

- alternatively, eigenvalues of the operator

$$H = \begin{pmatrix} 0 & I \\ I\frac{\mathrm{d}^2}{\mathrm{d}x^2} + B & -2A \end{pmatrix},$$

  with diagonal $N \times N$ matrices A and B
- the domain of the operator consists of functions $(\psi_1(x), \psi_2(x))^{\mathrm{T}}$ with components of both ψ_1 and ψ_2 in $W^{2,2}(e_j)$ and satisfying the coupling conditions

$$(U - I)\Psi + i(U + I)\Psi' = 0\,,$$

  at the vertices

Location of eigenvalues and high frequency abscissas

Theorem

If λ is an eigenvalue of H with nontrivial imaginary part $Im(\lambda) \neq 0$, then its real part satisfies

$$Re(\lambda) = -\frac{\sum_{j=1}^{N} \int_0^{l_j} a_j(x)|u_j(x)|^2 \,\mathrm{d}x}{\sum_{j=1}^{N} \|u_j(x)\|_2^2},$$

where $u_j(x)$ denotes the corresponding wavefunction components.

Corollary

Let us consider a damped wave equation on the graph Γ with damping functions on the edges $a_j(x)$ and potentials $b_j(x)$. Denote the average of the damping function on each edge by $\bar{a}_j$. Then all high frequency abscissas lie in the interval $[-\max_j \bar{a}_j, -\min_j \bar{a}_j]$.

Location of high frequency abscissas

- we say that ω_0 is a *high frequency abscissa* of the operator H if there exists a sequence of eigenvalues of H, say $\{\lambda_n\}_{n=1}^{\infty}$, such that

$$\lim_{n\to\infty} \mathrm{Im}\,\lambda_n = \pm\infty \text{ and } \lim_{n\to\infty} \mathrm{Re}\,\lambda_n = \omega_0 = \mathrm{Re}\, c_0^{(s)}.$$

Theorem

Let Γ be a graph with standard coupling which contains a cycle of N edges of length one with averages of damping coefficients on all of the edges of the cycle equal to a. Then there is a high frequency abscissa at $-a$.

Pseudo-orbit expansion

- pseudo-orbit expansion used earlier for quantum graphs (Band, R., Harrison, J. M., and Joyner, C. H. Finite pseudo orbit expansion for spectral quantities of quantum graphs. *J. Phys. A: Math. Theor.*, 2012. vol. 45, p. 325204)
- graph Γ replaced by a directed graph Γ_2
- a *periodic orbit* is a closed trajectory on the graph Γ_2
- an *irreducible pseudo-orbit* $\bar{\gamma}$ is a collection of periodic orbits where none of the directed bonds is contained more than once
- let $m_{\bar{\gamma}}$ denote the number of periodic orbits in $\bar{\gamma}$, $L_{\bar{\gamma}} = \sum_{e\in\bar{\gamma}} \tilde{\lambda}_e \ell_e$ where the sum is over all directed bonds in $\bar{\gamma}$
- the coefficients $A_{\bar{\gamma}} = \Pi_{\gamma_j\in\bar{\gamma}} A_{\gamma_j}$ with A_{γ_j} given as multiplication of entries of $S(\lambda)$ along the trajectory γ_j.

Theorem

The secular equation for the damped wave equation on a metric graph is given by

$$\sum_{\bar{\gamma}} (-1)^{m_{\bar{\gamma}}} A_{\bar{\gamma}}(\lambda) \exp(-L_{\bar{\gamma}}(\lambda)) = 0$$

where the sum goes through all irreducible pseudo-orbits $\bar{\gamma}$*.*

Number of distinct high frequency abscissas

Theorem

Let Γ be an equilateral graph with N edges of the length 1. Let us assume a damped wave equation on Γ with damping and potential functions constant on each edge $a_j(x) \equiv a_j$, $b_j(x) \equiv b_j$ and with general coupling for a given unitary matrix U. Then there exist numbers $n_0 \in \mathbb{N}$, $c_0^{(s)} \in \mathbb{C}$, $s = 1, \dots, 2N$ and $c_1 \in \mathbb{R}$ such that for every $n \geq n_0$ all eigenvalues of H are within the following set

$$\{\lambda, |\mathrm{Im}\,\lambda| \leq 2\pi n_0\} \cup \bigcup_{s=1}^{2N} \bigcup_{n=n_0}^{\infty} B\left(2\pi n i + c_0^{(s)}, \frac{c_1}{n}\right),$$

where $B(x_0, r)$ denotes the circle in the complex plane with center x_0 and radius r.

Theorem

Let Γ be a graph with N edges all of which have lengths equal to 1, (general) Robin coupling at the boundary and standard coupling otherwise. Let us suppose that the graph is bipartite, i.e. it does not have any closed cycle of odd length (there is no such a sequence of odd number of edges in which the m-th and $(m+1)$-th edges, including both the last and first one, have a joint vertex). Then for any damping functions bounded and $\mathcal{C}^2$ at each edge there are at most N high frequency absicssas.

- in the secular equation only the terms with $\mathrm{e}^{2c_0^{(s)}}$
- polynomial equation in $\mathrm{e}^{2c_0^{(s)}}$ of N-th order – at most N roots

Theorem

Let Γ be a tree graph with N edges all with unit length, Robin coupling at the boundary and standard coupling otherwise. Let us suppose that all vertices have odd degree. Then there always exists such a damping for which the number of high frequency abscissas is greater than or equal to N.

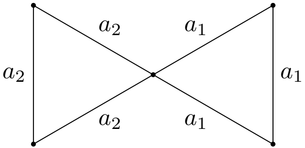
- the first term of the n expansion of the secular equation can be written as (for simplicity we omit n to the corresponding power)

$$C_N \mathrm{e}^{2a_1+2a_2+\cdots+2a_N} y^N + C_{N-1}\mathrm{e}^{2a_1+2a_2+\cdots+2a_{N-1}} [1+ \\ +\mathcal{O}\left(\mathrm{e}^{-2(a_{N-1}-a_N)}\right)\Big] y^{N-1} + \cdots + \\ + C_2 \mathrm{e}^{2a_1+2a_2}\left[1+\mathcal{O}\left(\mathrm{e}^{-2(a_2-a_3)}\right)\right] y^2 + \\ + C_1 \mathrm{e}^{2a_1}\left[1+\mathcal{O}\left(\mathrm{e}^{-2(a_1-a_2)}\right)\right] y + C_0 = 0\,,$$

- we use $0 \ll a_N \ll a_{N-1} \ll \cdots \ll a_1$
- for y being close to e^{-2a_1} the last two terms are dominant
- hence for the roots of the previous polynomial equation of the N-th order we get

$$y_j = -\frac{C_{j-1}}{C_j}\,\mathrm{e}^{-2a_j}\left[1 + \mathcal{O}\left(\mathrm{e}^{-2(a_j - a_{j+1})}\right)\right]\,,$$

Example – two loops with different damping coefficients



- secular equation

$$\sinh \frac{3}{2}\tilde{\lambda}_1 \sinh \frac{3}{2}\tilde{\lambda}_2$$
$$\left[(\tilde{\lambda}_1 + \tilde{\lambda}_2) \sinh \frac{3(\tilde{\lambda}_1 + \tilde{\lambda}_2)}{2} + (\tilde{\lambda}_1 - \tilde{\lambda}_2) \sinh \frac{3(\tilde{\lambda}_1 - \tilde{\lambda}_2)}{2}\right] = 0\,.$$

- using the asymptotic expansion $\lambda_n = 2\pi i n + c_0^{(s)} + \mathcal{O}\left(\frac{1}{n}\right)$ one obtains

$$4\pi i n \left(\mathrm{e}^{6\pi i n + \frac{3}{2}(a_1 + a_2 + 2c_0)} - \mathrm{e}^{-6\pi i n - \frac{3}{2}(a_1 + a_2 + 2c_0)}\right) + \mathcal{O}(1) = 0$$

- and hence

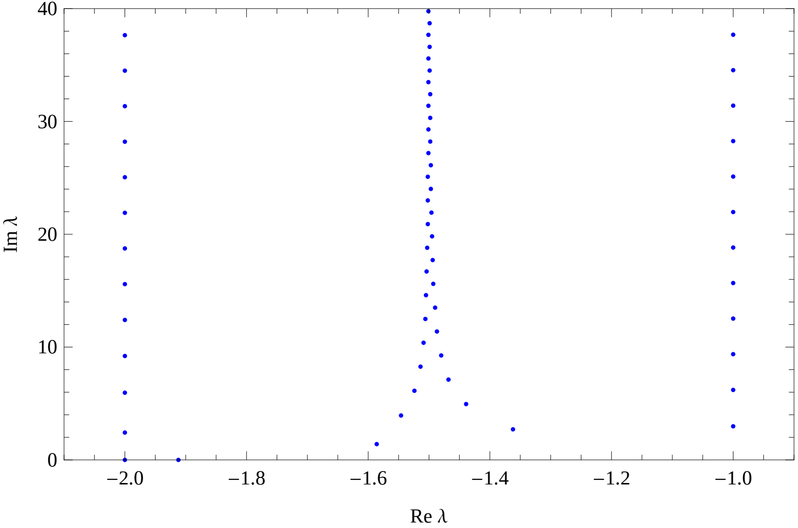
$$c_0^{(s)} = -\frac{a_1 + a_2}{2} + \frac{s\pi i}{6}\,, \quad s \in \{0, \dots, 5\}$$

- from the equation $\sinh\left(\frac{3}{2}\tilde{\lambda}_j(\lambda_n)\right) = 0$ we have

$$3a_j + 3c_0^{(s)} + \mathcal{O}\left(\frac{1}{n}\right) = 2\pi is$$

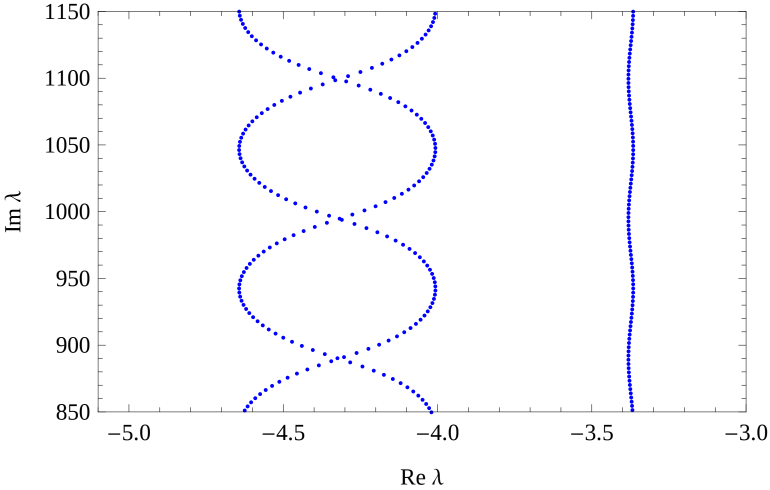
$$c_0^{(s)} = -a_j + \frac{2\pi is}{3}\,, \quad s \in \{0, 1, 2\}$$

- three sequences of eigenvalues $-a_1$, $-a_2$, $-\frac{a_1+a_2}{2}$



- spectrum of a graph in the previous figure, $a_1 = 2$, $a_2 = 1$, $b_1 = 0$, $b_2 = 0$

Example – star graph with nonequal lengths of the edges



- spectrum of a star graph with different lengths of the edges, $l_1 = 1,\ l_2 = 1,\ l_3 = 1.03$

Conclusion

- for an equilateral graph there is at most $2N$ high frequency abscissas
- for a bipartite equilateral graph there is at most N high frequency abscissas
- for a tree equilateral graph with odd degree of vertices there exists such a damping that there is at least N high frequency absissas

Reference

P. Freitas, J. Lipovsky: Eigenvalue asymptotics for the damped wave equation on metric graphs
J. Diff. Eq. **263** (2017), 2780–2811, in press, arXiv: 1307.6377 [math-ph]

Thank you for your attention!

Reference

P. Freitas, J. Lipovsky: Eigenvalue asymptotics for the damped wave equation on metric graphs
J. Diff. Eq. **263** (2017), 2780–2811, in press, arXiv: 1307.6377 [math-ph]

Thank you for your attention!