Limit-point / limit-circle classification of second-order differential operators and $\mathcal{PT}$-QM

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joint work with C. Trunk

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Real Spectra in Non-Hermitian Hamiltonians Having $\mathcal{PT}$ Symmetry

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The condition of self-adjointness ensures that the eigenvalues of a Hamiltonian are real and bounded below. Replacing this condition by the weaker condition of $\mathcal{PT}$ symmetry, one obtains new infinite classes of complex Hamiltonians whose spectra are also real and positive. These $\mathcal{PT}$ symmetric theories may be viewed as analytic continuations of conventional theories from real to complex phase space. This paper describes the unusual classical and quantum properties of these theories. [S0031-9007(98)06371-6]

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Several years ago, Bessis conjectured on the basis of numerical studies that the spectrum of the Hamiltonian $H = p^2 + x^2 + ix^3$ is *real and positive* [1]. To date there is no rigorous proof of this conjecture. We claim that the reality of the spectrum of H is due to $\mathcal{PT}$ symmetry. Note that H is invariant *neither* under parity $\mathcal{P}$, whose effect is to make spatial reflections, $p \to -p$ and $x \to -x$, *nor* under time reversal $\mathcal{T}$, which replaces $p \to -p$, $x \to -x$, and $i \to -i$. However,

depinning in type-II superconductors [2], or even to study population biology [3]. Here, initially real eigenvalues bifurcate into the complex plane due to the increasing external field, indicating the unbinding of vortices or the growth of populations. We believe that one can also induce dynamic delocalization by tuning a physical parameter (here N) in a self-interacting theory.

Furthermore, it was found that quantum field theories analogous to the quantum-mechanical theory in

Great interest: S. Albeverio, T. Azizov, F. Bagarello, C. Bender, M.V. Berry, S. Böttcher, S.F. Brandt, J. Brody, E. Caliceti, F. Cannata, J.-H. Chen, F. Correa, P. Dorey, C. Dunning, A. Fring, H.B. Geyer, E.M. Graefe, S. Graffi, U. Günther, D. Heiss, G.S. Japaridze, Y. Joglekar, H. Jones, O. Kirillov, D. Krejčiřík, S. Kuzhel, H. Langer, G. Lévai, V. Lotoreichik, P. Mannheim, P. Meisinger, K.A. Milton, A. Mostafazadeh, M.C. Ogilvie, K.C. Shin, P. Siegl, J. Sjöstrand, F. Stefani, F.-H. Szafraniec, T. Tanaka, R. Tateo, C. Tretter, M. Znojil, Q. Wang,...

$\mathcal{PT}$-symmetric Hamiltonian

Hamiltonian

$$H := p^2 + z^2(iz)^N, \quad z \in \Gamma, \quad N \in \mathbb{N}$$

Eigenvalue problem

$$H\phi = \lambda\phi$$

Schrödinger eigenvalue differential expression

$$-\phi''(z) + z^2(iz)^N\phi(z) = \lambda\phi(z), \quad z \in \Gamma$$

$\Gamma \subset$ *Stokes-wedges*, e. g. ϕ vanishes exponentially as $|z| \to \infty$, boundaries are *Stokes-lines*.

$\mathcal{PT}$-symmetry

Definition

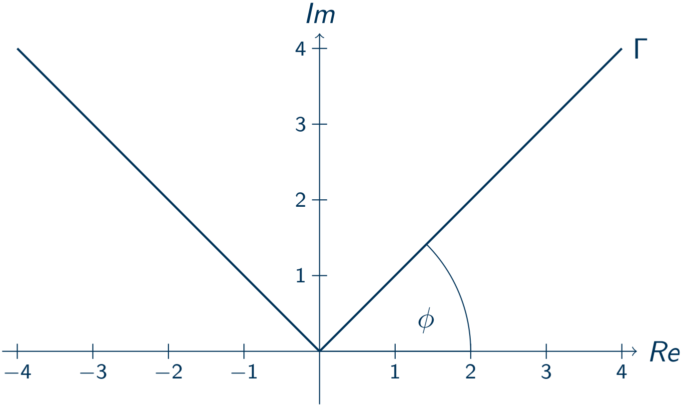
Closed, densely defined operator H on $L^2(\mathbb{R})$ is $\mathcal{PT}$-symmetric, if for all $f \in D(H)$

$$\mathcal{PT}f \in D(H) \quad \text{and} \quad \mathcal{PT}Hf = H\mathcal{PT}f.$$

Parity $\mathcal{P}$: $\mathcal{P}f(x) = f(-x)$
Time reversal $\mathcal{T}$: $\mathcal{T}f(x) = \overline{f(x)}$

Contour Γ

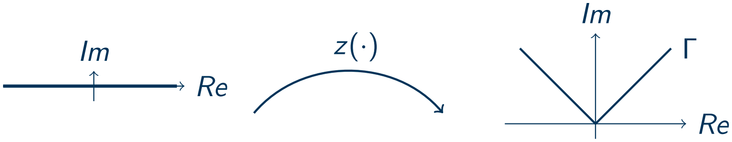
Today $\Gamma := \{xe^{i\phi \mathrm{sgn} x} : x \in \mathbb{R}\}$[1]:



[1] A. Mostafazadeh, Pseudo-Hermitian description of PT-symmetric systems defined on a complex contour, J. Phys. A: Math. Gen. **43** (2005), 3213-3234.

Back to the real line

Parameterize Γ: $z(x) := xe^{i\phi \mathrm{sgn} x}$, $w(x) := y(z(x))$:



$$A_+ w(x) := -e^{-2i\phi} w''(x) - e^{(N+2)i\phi}(ix)^{N+2} w(x), \; x \geq 0,$$
$$A_- w(x) := -e^{2i\phi} w''(x) - e^{-(N+2)i\phi}(ix)^{N+2} w(x), \; x \leq 0.$$

Sturm-Liouville Problem

$$A_\pm w(x) := -p_\pm(x) w''(x) + q_\pm(x) w(x) = \lambda w(x), \; x \in \mathbb{R}_\pm,$$

with $p_\pm(x) = e^{\mp 2i\phi}$ and $q_\pm(x) = -e^{\pm(N+2)i\phi}(ix)^{N+2}$.

Plan

1. Consider operator with boundary conditions on the semi axis
2. Study spectrum of operators on the semi axis
3. Study operator with matching conditions on $\mathbb{R}$
4. Spectrum

Semi axis: Limit-point/Limit-circle

Under certain assumptions:

$$-\left(p(x)y'(x)\right)' + q(x)y(x) = \lambda y, \quad \lambda \in \mathbb{C} \qquad (\star)$$

$$\int_0^\infty \mathrm{Re}\left(p|y'|^2 + q|y|^2\right)\, dx < \infty \qquad (\clubsuit)$$

$$y \in L^2(0,\infty) \qquad (\diamondsuit)$$

Theorem (Brown et. al '99)

Exactly one of the following holds

- **I** *There exists a unique solution of* $(\star)$ *satisfying* $(\clubsuit)$ *and this is the only solution satisfying* $(\diamondsuit)$ *(LPC).*
- **II** *There exists a unique solution of* $(\star)$ *satisfying* $(\clubsuit)$ *but all solutions satisfy* $(\diamondsuit)$ *(LPC).*
- **III** *All solutions of* $(\star)$ *satisfy* $(\clubsuit)$ *and* $(\diamondsuit)$ *(LCC).*

Limit-point/limit-circle classification of A_+

Linear independent solutions $y_\pm$ for $A_+ w(x) = \lambda w(x)$.

$$y_\pm(x) \sim [e^{(N+4)i\phi} s(x)]^{-1/4} exp\left(\pm \int_0^x \mathrm{Re}\, s(t)^{1/2}\, dt\right),$$

where $s(t) := -e^{(N+4)i\phi}(ix)^{(N+2)} - e^{2i\phi}\lambda$.

$\phi \neq \frac{4k-N-2}{2N+8}\pi \to \mathrm{Re}\, s(t)^{1/2} \neq 0 \rightsquigarrow$ Case I, LPC
$\phi = \frac{4k-N-2}{2N+8}\pi \to \mathrm{Re}\, s(t)^{1/2} = 0 \rightsquigarrow$ Case III, LCC

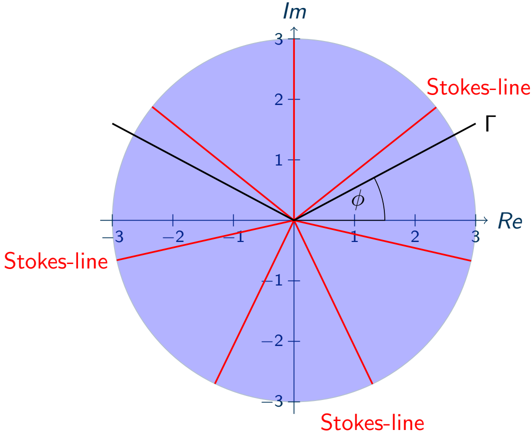
We get the same for A_-!

Theorem

Γ on Stokes-wedges $\cong$ LPC and Γ on Stokes-lines $\cong$ LCC

Stokes-Wedges and Stokes-lines for $N = 3$:

$$A_+ w(x) := -e^{-2i\phi} w''(x) - ie^{5i\phi} x^5 w(x), \ x \geq 0.$$



Spectrum

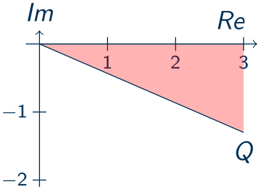
Restrict to case I: For example: $\phi = \pi/10$ and $N = 3$.

$$A_+ w(x) := -e^{-i\pi/5} w''(x) + x^5 w(x),\ x \geq 0$$

$$\mathrm{dom}(A_+) := \{u : u, pu' \in AC(0,\infty), u, A_+ u \in L^2(0,\infty), u(0) = 0\}$$

$$Q := \mathrm{clconv}\left\{ re^{-2i\phi} - e^{(N+2)i\phi}(ix)^{N+2} :\ x \in [0,\infty),\ 0 < r < \infty \right\}$$

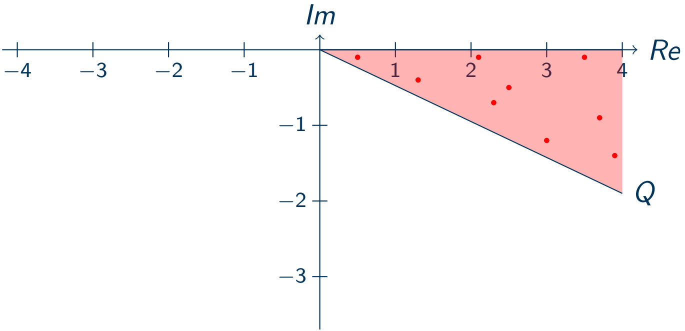
$$= \mathrm{clconv}\left\{ re^{-i\pi/5} + x^5 :\ x \in [0,\infty),\ 0 < r < \infty \right\}$$



Theorem (Brown et. al '99)

In I $\sigma(A_+) \subset Q$ (with only isolated eigenvalues of finite algebraic multiplicity in Q).

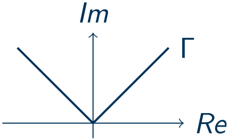
$$-e^{-i\pi/5} w''(x) + x^5 w(x) = \lambda w(x), \; x \geq 0:$$



$$Q = \operatorname{clconv}\left\{ re^{-i\pi/5} + x^5 : \; x \in [0, \infty), \; 0 < r < \infty \right\}$$

Back to $\mathcal{PT}$-symmetric Problem (Bender Boettcher) on Γ:

$$-\phi''(z) - (iz)^{N+2}\phi(z) = \lambda\phi(z), \quad z \in \Gamma$$



$$A_+ w(x) := -e^{-2i\phi} w''(x) - e^{(N+2)i\phi}(ix)^{N+2} w(x),\ x \geq 0$$
$$A_- w(x) := -e^{2i\phi} w''(x) - e^{-(N+2)i\phi}(ix)^{N+2} w(x),\ x \leq 0$$

with

$$\mathrm{dom}\, A_\pm = \left\{w, A_\pm w \in L^2(\mathbb{R}_\pm) : w, w' \in AC(\mathbb{R}_\pm), w(0\pm) = 0\right\}$$

Full line operator A:

$$Aw := \begin{cases} -e^{-2i\phi}w''(x) - (ix)^{N+2}e^{(N+2)i\phi}w(x), & x \geq 0 \\ -e^{2i\phi}w''(x) - (ix)^{N+2}e^{-(N+2)i\phi}w(x), & x \leq 0 \end{cases}$$

with domain

$$\operatorname{dom} A := \left\{ w, Aw \in L^2(\mathbb{R}) : \begin{array}{c} w|_{\mathbb{R}^\pm}, w'|_{\mathbb{R}^\pm} \in AC(\mathbb{R}^\pm), \\ w(0+) = w(0-), \\ w'(0+) = \alpha w'(0-) \end{array} \right\}$$

Theorem

- *y' is continous if and only if $\alpha = e^{2i\phi}$.*
- *A is $\mathcal{PT}$-symmetric if and only if $|\alpha| = 1$.*

Literature

- T.Ya. Azizov, C. Trunk,*On the limit point and limit circle classification for $\mathcal{PT}$ symmetric operators*
- T.Ya. Azizov and C. Trunk, *On domains of $\mathcal{PT}$ symmetric operators related to $-y''(x) + (-1)^n x^{2n} y(x)$*, J. Phys. A: Math. Theor. **43** (2010), 175303.
- T.Ya. Azizov and C. Trunk, *$\mathcal{PT}$ symmetric, Hermitian and $\mathcal{P}$-self-adjoint operators related to potentials in $\mathcal{PT}$ quantum mechanics operators related to $-y''(x) + (-1)^n x^{2n} y(x)$*, J. Math. Phys. **53** (2012), 012109.
- C. Bender, S. Boettcher, *Real Spectra in Non-Hermitian Hamiltonians Having $\mathcal{PT}$ Symmetry*, Phys. Rev. Lett. **80** (1998), 5243-5246.
- C.M. Bender, D.C. Brody, J.H. Chen, H.F. Jones, K.A. Milton and M.C. Ogilvie, *Equiv- alence of a complex $\mathcal{PT}$ -symmetric quartic Hamiltonian and a Hermitian quartic Hamil- tonian with an anomaly*, Phys. Rev. **74** (2006), 025016.
- B. Brown et.al.,*On the spectrum of second-order differential operators with complex coefficients*, Proc. R. Soc. Lond. 455 (1999), 1235–1257.
- A. Mostafazadeh, *Pseudo-Hermitian description of PT-symmetric systems defined on a complex contour*, J. Phys. A: Math. Gen. **38** (2005), 3213–3234.
- H.F. Jones and J. Mateo, *Equivalent Hermitian Hamiltonian for the non-Hermitian $-x^4$ potential*, Phys. Rev. **73** (2006), 085002.
- A.R. Sims,*Secondary Conditions for Linear Differential Operators of the Second Order*, J. of Math. and Mech. **6** (1957), 247–285.
- M.S.P. Estham,*The asymptotic solution of linear differential systems*, London Mathematical Society, 1989.
- F.W.J. Olver,*Asymptotics and special functions*, Academic Press New York, 1974.

Thank You!