Recent advances on the study of the discrete spectrum of a non-selfadjoint operator

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Conference on Non-Selfadjoint Operators

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CIRM

Plan of the talk

1. A general scheme.

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2. Zeros of holomorphic functions from different classes.

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4. A tentative list of recent references on the subject.
5. Several open problems.

A general scheme

Let $A_0 : H \to H$ be an operator on a Hilbert space, and $K : H \to H$ be an operator lying in $\mathcal{S}_p,\ 1 \le p < \infty$. Recall that

$$\mathcal{S}_p = \{K \in \mathcal{S}_\infty : ||K||_p^p := ||K||_{\mathcal{S}_p}^p = \sum_k s_k(A)^p < \infty\},$$

where $s_k(K) = \lambda_k(K^*K)^{1/2}$.

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Problem

What can one say about $\sigma_d(A)$ and its distributional characteristics ?

A general scheme

One needs to look at the so-called regularised determinant. For $K \in \mathcal{S}_p$ and $p \in \mathbb{N}^*$, define

$$\det{}_p(I+K) = \prod_k (1+\lambda_k) \exp\left(\sum_{j=1}^{p-1} \frac{(-1)^j}{j} \lambda_k^j\right),$$

where $\lambda_k = \lambda_k(K)$.

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Furthermore, consider the regularised perturbation determinant, *i.e.,*

$$F(\lambda) = \det{}_p(A-\lambda I)(A_0-\lambda I)^{-1} = \det{}_p(I+K(A_0-\lambda I)^{-1}).$$

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- $F \in Hol(\varrho(A_0))$, where $\varrho(A_0) = \bar{\mathbb{C}} \backslash \sigma(A_0)$,
- $Z(F)$, the zero set of F, coincides with $\sigma_d(A)$ up to mutiplicities,
- there is a special bound of F on $\varrho(A_0)$, *i.e.,*

$$\log|F(\lambda)| \le \Gamma_p ||K(A_0 - \lambda)^{-1}||_p^p \ (\le \Gamma_p ||K||_p^p ||(A_0 - \lambda)^{-1}||^p),$$

with $\lambda \in \varrho(A_0)$.

A general scheme

Let $\mathbb{D} = \{z : |z| < 1\}$ and $\mathbb{T} = \partial\mathbb{D} = \{z : |z| = 1\}$.

Now, let $\varphi : \mathbb{D} \to \varrho(A_0)$ and $\psi : \varrho(A_0) \to \mathbb{D}$ be the conformal maps of the corresponding domains, $\psi = \varphi^{-1}$. Make a "change of variables" $\lambda = \varphi(z), \ z \in \mathbb{D}$.

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One comes to $f(z) = F(\varphi(z)) \in Hol(\mathbb{D})$ such that

$$\log|f(z)| \le \frac{K}{d^p(z,\mathbb{T})} \frac{d^r(z,\mathcal{E})}{d^q(z,\mathcal{F})}, \quad z \in \mathbb{D},\ p,q,r \ge 0,$$

and $\mathcal{E},\mathcal{F} \subset \mathbb{T},\ \mathbb{T} = \{z : |z| = 1\},\ \#\mathcal{E}, \#\mathcal{F} < \infty$ and $\mathcal{E} \cap \mathcal{F} = \emptyset$. Of course,

$$d(z,\mathcal{E}) = \inf_{t\in\mathcal{E}} |z - t|,$$

so, for instance, $d(z,\mathbb{T}) = (1 - |z|),\ z \in \mathbb{D}$.

Zeros of holomorphic functions from different classes

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- Let $\mathcal{F} \subset \mathbb{T}, \ \#\mathcal{F} < \infty$.

Theorem (Borichev-Golinskii-K' 2009)

Let $f \in Hol(\mathbb{D})$, $|f(0)| = 1$, satisfy the growth condition

$$\log|f(z)| \leq \frac{K}{(1-|z|)^p \, d^q(z, \mathcal{F})}$$

for $z \in \mathbb{D}$ and $p, q \geq 0$. Then for each $\tau > 0$ there is a positive constant C_1 such that

$$\sum_{\zeta \in Z(f)} (1-|\zeta|)^{p+1+\tau} \, d^{(q-1+\tau)_+}(\zeta, \mathcal{F}) \leq C_1 \cdot K.$$

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where $z \in \mathbb{D}$ and $p, q, r \geq 0$. Then for each $\tau > 0$, there is a positive constant C_2 such that

$$\sum_{\zeta \in Z(f)} (1-|\zeta|)^{p+1+\tau} \frac{d^{(q-1+\tau)_+}(\zeta,\mathcal{F})}{d^{\min(p,r)}(\zeta,\mathcal{E})} \leq C_2 \cdot K.$$

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What happens when the set $\mathcal{F} \subset \mathbb{T}$ is countable ?

For this $\mathcal{F}$, let $\beta(\mathcal{F})$ be its Minkowski type, *i.e.,*

$$\beta(\mathcal{F}) = \sup\{\beta : m(\mathcal{F}_s) = O(s^\beta), \quad s \to 0+\},$$

and $\mathcal{F}_s = \{t \in \mathbb{T} : d(t, \mathcal{F}) < s\}$.

Zeros of holomorphic functions from different classes

Let $\mathcal{E}, \mathcal{F} \subset \mathbb{T}, \#\mathcal{E} < \infty$, $\mathcal{F}$ be countable with Minkowski dimension $\beta(\mathcal{F})$. Let $\bar{\mathcal{F}} \cap \mathcal{E} = \emptyset$.

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$$\sum_{\zeta \in Z(f)} (1-|\zeta|)^{p+1+\tau} \frac{d^{(q-\beta(\mathcal{F})+\tau)_+}(\zeta,\mathcal{F})}{d^{\min(p,r)}(\zeta,\mathcal{E})} \le C_3 \cdot K.$$

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A (fast & tentative) list of references

- To start with : **2001** - A. Abramov, A. Aslanyan, E. B. Davies ; **2006** - R. Frank, A. Laptev, E. Lieb, R. Seiringer ; **2009** - A. Borichev, L. Golinskii, SK ; M. Demuth, M. Hansmann, G. Katriel ; **2013** - M. Hansmann ; **2015** - A. Borichev, L. Golinskii, SK, ...

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- On Schrödinger operators : **2014 ...** - R. Frank, J. Sabin ; R. Frank, B. Simon ; R. Frank, A. Laptev, O. Safronov, ...

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- On Dirac (magnetic Schrödinger, Pauli) operators : **2014 ...** - J.C. Cuenin, A. Laptev, C. Tretter ; C. Dubuisson ; J.C. Cuenin ; S. Sambou ; J.C. Cuenin, P. Siegl, ...

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- On Banach spaces : **2015** - M. Demuth, F. Hanauska, M. Hansmann, G. Katriel ; M. Hansmann, ...

Example 1 : Jacobi matrix

Let

$$J = J(\{a_k\}, \{b_k\}, \{c_k\}) = \begin{bmatrix} b_0 & c_0 & 0 & \dots \\ a_0 & b_1 & c_1 & \dots \\ 0 & a_1 & b_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where $\{a_k\}, \{b_k\}, \{c_k\} \subset \mathbb{C}$.

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where $\{a_k\},\{b_k\},\{c_k\} \subset \mathbb{C}$.

Set also

$$J_0 = J(\{1\},\{0\}) = \begin{bmatrix} 0 & 1 & 0 & \dots \\ 1 & 0 & 1 & \dots \\ 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Note that $\sigma(J_0) = [-2,2]$.

Example 1 : Jacobi matrix

Recall that if $J - J_0 \in \mathcal{S}_\infty$, the ideal of compact operators (*i.e.*, $\lim_{j\to\infty}(|a_j - 1| + |c_j - 1| + |b_j|) = 0$), one has

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We require that $J - J_0 \in \mathcal{S}_p,\ p \geq 1$, *i.e.*,

$$\sum_j (|a_j - 1|^p + |b_j|^p + |c_j - 1|^p) < \infty.$$

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So, let $J - J_0 \in \mathcal{S}_p,\ p \geq 1$.

Theorem (Borichev-Golinskii-K' 2009)

For $p = 1$ and any $\tau > 0$, we have

$$\sum_{\lambda \in \sigma_d(J)} \frac{d(\lambda, \sigma(J_0))}{|\lambda^2 - 4|^{(1-\tau)/2}} \leq C_5 ||J - J_0||_{\mathcal{S}_1} (= C_5 \sum_j (|a_j - 1| + |c_j - 1| + |b_j|)).$$

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For $p > 1$ and $\tau > 0$ we have

$$\sum_{\lambda \in \sigma_d(J)} \frac{d(\lambda, \sigma(J_0))^{p+1+\tau}}{|\lambda^2 - 4|} \leq C_6 ||J - J_0||^p_{\mathcal{S}_p} (= C_6 \sum_j (|a_j - 1|^p + |c_j - 1|^p + |b_j|^p)$$

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Here is an improvement :

Theorem (Hansmann-Katriel' 2010)

For $p > 1$ and $\tau > 0$ we have

$$\sum_{\lambda \in \sigma_p(J)} \frac{d(\lambda, \sigma(J_0))^{p+\tau}}{|\lambda^2 - 4|^{1/2}} \le C_7 ||J - J_0||^p_{\mathcal{S}_p} (= C_7 \sum_j (|a_j - 1|^p + |c_j - 1|^p + |b_j|^p)).$$

Some open problems

- On the functional-theoretic side : a counterpart of BGK' 2015 when $\mathcal{E}, \mathcal{F} \subset \mathbb{T},\ \mathcal{E} \cap \mathcal{F} = \emptyset$, but $\mathcal{E} \cap \bar{\mathcal{F}} \neq \emptyset$.

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Thank you !