On the invertibility of block matrix operators

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Mathematical aspects of the physics with non-self-adjoint operators

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What is this talk about?

Is a bounded self-adjoint operator

$$B = \begin{pmatrix} A_0 & V \\ V^* & A_1 \end{pmatrix}$$

acting on the orthogonal sum of two Hilbert spaces $\mathfrak{H}_0 \oplus \mathfrak{H}_1$ continuously invertible? If yes, what is the norm of its inverse or, equivalently, what is the width of the spectral gap containing zero?

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Two examples:

(i) $B = \begin{pmatrix} I & V \\ V^* & -I \end{pmatrix}$ V arbitrary and (ii) $B = \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix}$, V bijection

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Perturbation arguments for (i): $\|V\| < 1 \Rightarrow (-1 + \|V\|, 1 - \|V\|) \subset \rho(B)$. Actually $(-1, 1) \subset \rho(B)$ [Davis, Kahan (1970)]

Outline

We consider

$$B = \begin{pmatrix} A_0 & V \\ V^* & A_1 \end{pmatrix}$$

as a perturbation of

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix}$$

- Perturbing an existing spectral gap containing zero
- Opening a new spectral gap containing zero

Perturbing an existing spectral gap

Notations:

$$-d_l := \max\{\lambda < 0 \,|\, \lambda \in \sigma(A)\}$$
$$d_r := \min\{\lambda > 0 \,|\, \lambda \in \sigma(A)\}$$

with $\min\emptyset := +\infty$. The open interval $(-d_l, d_r)$ is a spectral gap of the operator A containing zero.

Perturbing an existing spectral gap

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c_l &:= |\max\{\lambda < 0 \mid \lambda \in \sigma(A_0)\} - \max\{\lambda < 0 \mid \lambda \in \sigma(A_1)\}| \\
c_r &:= |\min\{\lambda > 0 \mid \lambda \in \sigma(A_0)\} - \min\{\lambda > 0 \mid \lambda \in \sigma(A_1)\}|
\end{aligned}
$$

with $\min\emptyset = \max\emptyset = +\infty$ and $\infty - \infty = 0$.

Theorem 1. Assume that the operator A is invertible. If

$$\|V\| < \min\left\{\sqrt{d_l(d_l+c_l)}, \sqrt{d_r(d_r+c_r)}\right\},$$

then the operator B is continuously invertible. Moreover, the open interval

$$(-d_l+\delta_l, d_r+\delta_r)$$

belongs to the resolvent set of the operator B, where

$$\delta_l := \|V\| \tan\left(\frac{1}{2}\arctan\frac{2\|V\|}{c_l}\right), \qquad \delta_r := \|V\| \tan\left(\frac{1}{2}\arctan\frac{2\|V\|}{c_r}\right)$$

with the natural conventions $1/t = 0$ if $t = +\infty$, $1/t = +\infty$ if $t = 0$, and $\arctan(+\infty) = \pi/2$.

Examples. (1) $B = \begin{pmatrix} I & V \\ V^* & -I \end{pmatrix}$. We have $c_l = c_r = \infty \Rightarrow \delta_l = \delta_r = 0$ and for any V the interval $(-1, 1)$ belongs to the resolvent set of the operator B.

Examples. (1) $B = \begin{pmatrix} I & V \\ V^* & -I \end{pmatrix}$. We have $c_l = c_r = \infty \Rightarrow \delta_l = \delta_r = 0$ and for any V the interval $(-1, 1)$ belongs to the resolvent set of the operator B.

(2) Consider the case when $\mathfrak{H}_0 = \mathfrak{H}_1 = \mathbb{C}^2$,

$$A_0 = \begin{pmatrix} -3/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} -1/2 & 0 \\ 0 & 3/2 \end{pmatrix}, \qquad V = \begin{pmatrix} \sqrt{3}/2 & 0 \\ 0 & \sqrt{3}/2 \end{pmatrix}.$$

Then $c_l = c_r = 1$, $d_l = d_r = 1/2$, and, hence, $\sqrt{d_{l,r}(d_{l,r} + c_{l,r})} = \sqrt{3}/2$. The norm of V equals $\sqrt{3}/2$. Hence, $\delta_{l,r} = 1/2$. The spectrum of $B = A + V$ consists of three eigenvalues -2, 0, and 2. Thus, the result of Theorem 1 is sharp if $c_{l,r} < \infty$.

Perturbing an existing spectral gap

(3) Let $\mathfrak{H}_0 = \mathbb{C}^2$, $\mathfrak{H}_1 = \mathbb{C}$,

$$A_0 = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_1 = -1, \qquad V = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}.$$

Then $c_l = d_l = d_r = 1$, $c_r = \infty$ so that $\sqrt{d_l(d_l + c_l)} = \sqrt{2}$. The norm of the operator V equals $\sqrt{2}$. Hence $\delta_l = 1$ and $\delta_r = 0$. The spectrum of $B = A + V$ consists of three eigenvalues -3, 0, and 1. Thus, the result of Theorem 1 is optimal if $c_r = \infty$ as well.

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The idea of the proof: The local version of geometric arguments from V. K, K.A. Makarov, A.K. Motovilov, [Trans. Amer. Math. Soc. **359** (2007), 77 – 89].

Opening a new spectral gap. The case when V is bijective

If $V : \mathfrak{H}_1 \to \mathfrak{H}_0$ is bijective, then $\mathfrak{H}_0$ and $\mathfrak{H}_1$ are isomorphic.

Theorem 2. Let the self-adjoint bounded operators A_0 and A_1 satisfy $A_0 \geq a_0$, $A_1 \leq a_1$ for some $a_0 \geq 0$ and $a_1 \leq 0$. If $V : \mathfrak{H}_1 \to \mathfrak{H}_0$ is a bijection, then the operator B is boundedly invertible and the interval (λ_-, λ_+) belongs to its resolvent set, where

$$\lambda_+ := -\frac{\|A_1\| - a_0}{2} + \sqrt{(\|A_1\| + a_0)^2/4 + \|V^{-1}\|^{-2}},$$
$$\lambda_- := \frac{\|A_0\| + a_1}{2} - \sqrt{(\|A_0\| - a_1)^2/4 + \|V^{-1}\|^{-2}}.$$

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In particular, if $a_0 > 0$ or $a_1 < 0$, that is the interval (a_1, a_0) belongs to the resolvent set of the diagonal operator

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix},$$

then $\lambda_+ > a_0$ and $\lambda_- < a_1$.

Opening a new spectral gap. The case when V is bijective

M. Winklmeier [Ph.D. Thesis (2005), Corollary 3.18] proved that $(-\lambda_W, \lambda_W) \in \rho(B)$ with

$$\lambda_W := -\frac{\|A_0\| + \|A_1\|}{2} + \sqrt{\frac{(\|A_0\| + \|A_1\|)^2}{4} + \|V^{-1}\|^{-2}}.$$

Theorem 2 gives a larger width of the spectral gap,

$$\lambda_W \leq \min\{\lambda_+, -\lambda_-\}$$

where

$$\lambda_+ := -\frac{\|A_1\| - a_0}{2} + \sqrt{(\|A_1\| + a_0)^2/4 + \|V^{-1}\|^{-2}},$$
$$\lambda_- := \frac{\|A_0\| + a_1}{2} - \sqrt{(\|A_0\| - a_1)^2/4 + \|V^{-1}\|^{-2}}.$$

The equality $\lambda_W = \min\{\lambda_+, -\lambda_-\}$ holds if and only if $A_0 = 0$ or $A_1 = 0$.

Opening a new spectral gap. The general case

Assumptions. Let the self-adjoint bounded operators A_0 and A_1 satisfy $A_0 \geq 0$, $A_1 \leq 0$. Let $V : \mathfrak{H}_1 \to \mathfrak{H}_0$ be a bounded operator with a closed range. If $\mathrm{Ker} V^* \neq \{0\}$ we assume that the operator

$$A_{0,\mathrm{Ker}} := P_{\mathrm{Ker}V^*} A_0 \big|_{\mathrm{Ker}V^*} : \mathrm{Ker}V^* \to \mathrm{Ker}V^*$$

is bijective. Similarly, if $\mathrm{Ker}V \neq \{0\}$ we assume that the operator

$$A_{1,\mathrm{Ker}} := P_{\mathrm{Ker}V} A_1 \big|_{\mathrm{Ker}V} : \mathrm{Ker}V \to \mathrm{Ker}V$$

is bijective.

Under these assumptions the operator

$$\mathcal{V} := P_{\mathrm{Ran}V} V \big|_{\mathrm{Ran}V^*} : \mathrm{Ran}V^* \to \mathrm{Ran}V$$

is continuously invertible.

Opening a new spectral gap. The general case

If $\mathrm{Ker} V^* \neq \{0\}$, the operator A_0 can be represented as the block operator matrix

$$A_0 = \begin{pmatrix} A_{0,\mathrm{Ker}} & W_0 \\ W_0^* & A_{0,\mathrm{Ran}} \end{pmatrix}$$

with respect to the orthogonal decomposition $\mathfrak{H}_0 = \mathrm{Ker} V^* \oplus \mathrm{Ran} V$.

Similarly, if $\mathrm{Ker} V \neq \{0\}$, the operator A_1 can be represented as the block operator matrix

$$A_1 = \begin{pmatrix} A_{1,\mathrm{Ker}} & W_1 \\ W_1^* & A_{1,\mathrm{Ran}} \end{pmatrix}$$

with respect to the orthogonal decomposition $\mathfrak{H}_1 = \mathrm{Ker} V \oplus \mathrm{Ran} V^*$.

Opening a new spectral gap. The general case

Denote

$$\mathcal{A}_0 := \begin{cases} A_{0,\mathrm{Ran}} - W_0^* A_{0,\mathrm{Ker}}^{-1} W_0 & \text{if} \quad \mathrm{Ker} V^* \neq \{0\}, \\ A_0 & \text{if} \quad \mathrm{Ker} V^* = \{0\} \end{cases}$$

and

$$\mathcal{A}_1 := \begin{cases} A_{1,\mathrm{Ran}} - W_1^* A_{1,\mathrm{Ker}}^{-1} W_1 & \text{if} \quad \mathrm{Ker} V \neq \{0\}, \\ A_1 & \text{if} \quad \mathrm{Ker} V = \{0\}. \end{cases}$$

Opening a new spectral gap. The general case

Theorem 3. Under thee above assumptions the operator B is boundedly invertible and

$$\|B^{-1}\| \le \left(1+\max\left\{\|W_0^* A_{0,\mathrm{Ker}}^{-1}\|, \|W_1^* A_{1,\mathrm{Ker}}^{-1}\|\right\}\right)^2 \cdot$$
$$\cdot \max\left\{\|A_{0,\mathrm{Ker}}^{-1}\|, \|A_{1,\mathrm{Ker}}^{-1}\|, \lambda_+^{-1}, -\lambda_-^{-1}\right\},$$

where

$$\lambda_+ := -\frac{\|\mathcal{A}_1\| - a_0}{2} + \sqrt{(\|\mathcal{A}_1\| + a_0)^2/4 + \|\mathcal{V}^{-1}\|^{-2}},$$
$$\lambda_- := \frac{\|\mathcal{A}_0\| + a_1}{2} - \sqrt{(\|\mathcal{A}_0\| - a_1)^2/4 + \|\mathcal{V}^{-1}\|^{-2}}$$

with

$$a_0 = \min \operatorname{spec}(\mathcal{A}_0) \ge 0 \quad \text{and} \quad a_1 = \max \operatorname{spec}(\mathcal{A}_1) \le 0.$$

Opening a new spectral gap. The general case

I. and K. Veselić in [Operators and Matrices **9** (2015), 241 – 275] used a different decomposition of the Hilbert space:

Let the self-adjoint bounded operators A_0 and A_1 satisfy $A_0 \geq 0$, $A_1 \leq 0$. Let $V : \mathfrak{H}_1 \to \mathfrak{H}_0$ be bounded. In addition, assume that

(i) the subspaces $\mathrm{Ker} A_0 \subset \mathfrak{H}_0$ and $\mathrm{Ker} A_1 \subset \mathfrak{H}_1$ are isomorphic and

$$V_{\mathrm{Ker} A_0, \mathrm{Ker} A_1} := P_{\mathrm{Ker} A_0} V\big|_{\mathrm{Ker} A_1} : \mathrm{Ker} A_1 \to \mathrm{Ker} A_0$$

is a bijection;

(ii) the operators A_0 and A_1 have closed range. Therefore,

$$A_{0, \mathrm{Ran} A_0} := A_0\big|_{\mathrm{Ran} A_0} : \mathrm{Ran} A_0 \to \mathrm{Ran} A_0$$

and

$$A_{1, \mathrm{Ran} A_1} := A_1\big|_{\mathrm{Ran} A_1} : \mathrm{Ran} A_1 \to \mathrm{Ran} A_1$$

are bijections.

Opening a new spectral gap. The general case

Theorem. (I. and K. Veselić (2015)) Under the above assumptions the operator B is boundedly invertible with

$$\|B^{-1}\| \leq \left(1+\max\left\{\|V_{\mathrm{Ran}A_0,\mathrm{Ker}A_1}V^{-1}_{\mathrm{Ker}A_0,\mathrm{Ker}A_1}\|, \|V^{*}_{\mathrm{Ker}A_0,\mathrm{Ran}A_1}V^{-*}_{\mathrm{Ker}A_0,\mathrm{Ker}A_1}\|\right\}\right)^2 \max\left\{\|A^{-1}_{0,\mathrm{Ran}A_0}\|, \|A^{-1}_{1,\mathrm{Ran}A_1}\|, \|V^{-1}_{\mathrm{Ker}A_0,\mathrm{Ker}A_1}\|\right\}, \tag{1}$$

where

$$V_{\mathrm{Ran}A_0,\mathrm{Ker}A_1} := P_{\mathrm{Ran}A_0}V|_{\mathrm{Ker}A_1} : \mathrm{Ker}A_1 \to \mathrm{Ran}A_0,$$
$$V_{\mathrm{Ker}A_0,\mathrm{Ran}A_1} := P_{\mathrm{Ker}A_0}V|_{\mathrm{Ran}A_1} : \mathrm{Ran}A_1 \to \mathrm{Ker}A_0.$$

Proposition. If the operator $V : \mathfrak{H}_1 \to \mathfrak{H}_0$ has a closed range, then our assumptions are weaker than those of Veselić's.

Thank You for Your Attention!