Essential spectrum of mixed-order systems of differential operators

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Based on:

[1] O.O.Ibrogimov and C.Tretter: *Essential spectrum of elliptic systems of pseudo-differential operators on* $L^2(\mathbb{R}^N) \oplus L^2(\mathbb{R}^N)$, J. Pseudo-Differ. Oper. Appl. 8(2), 147–166 (2017)

[2] O.O.Ibrogimov: *Essential spectrum of non-self-adjoint singular matrix differential operators*, J. Math. Anal. Appl. 451(1), 473–496 (2017)

[3] O.O.Ibrogimov, P.Siegl and C.Tretter: *Analysis of the essential spectrum of singular matrix differential operators*, J. Differ. Equ. 260(4), 3881–3926 (2016)

Setting

In a Hilbert space $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$, consider closable linear operator

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \mathrm{Dom}(\mathcal{A}) = W_1 \oplus W_2 \quad \text{dense in} \quad \mathcal{H}$$

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- Examples: Stokes system, Ekman problem, ...

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$$\mathcal{A} = \begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{d}t} p_1 \frac{\mathrm{d}}{\mathrm{d}t} + q_1 & \frac{\mathrm{d}}{\mathrm{d}t} p_2 + q_2 \\ -p_2 \frac{\mathrm{d}}{\mathrm{d}t} + q_2 & p_3 \end{pmatrix}$$

Motivation: Astrophysics [Beyer, *J. Math. Phys.* 36, 9 (1995)]

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$$\theta''(t) + \frac{2}{t}\theta'(t) = -\frac{1}{\alpha^2}\theta(t)^n, \quad t \in (0,\infty)$$

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- Conjecture: $\sigma_{\mathrm{ess}}(\mathcal{A}) = \{0\}$

$$\mathcal{A} = \begin{pmatrix} \phi_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2} + \phi_1 \frac{\mathrm{d}}{\mathrm{d}t} + \phi_0 & \psi_3 \frac{\mathrm{d}^3}{\mathrm{d}t^3} + \psi_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2} + \psi_1 \frac{\mathrm{d}}{\mathrm{d}t} + \psi_0 \\ -\frac{\mathrm{d}}{\mathrm{d}t} & c_0 \frac{\mathrm{d}}{\mathrm{d}t} \end{pmatrix}, \quad \mathcal{H} = L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$$

Motivations: Fluid dynamics [Pradas et al., *Phys. of Fluids* 23, (2011)]

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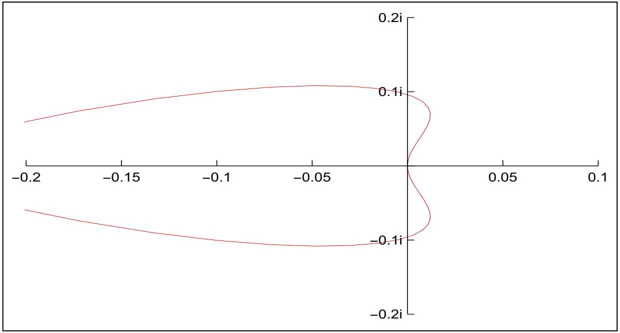

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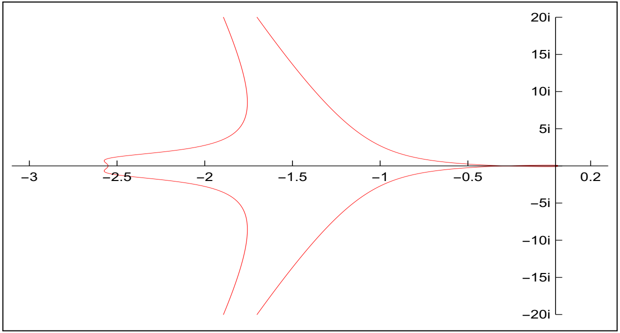

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$$\mathcal{A}_m := \begin{pmatrix} -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} & -\dfrac{\mathrm{d}}{\mathrm{d}x} \\ \dfrac{\mathrm{d}}{\mathrm{d}x} & \mathrm{e}^{-x^2} \end{pmatrix}, \qquad \mathcal{H}_m = L^2(-m,m) \oplus L^2(-m,m), \quad m \in \mathbb{N}$$

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  - general self-adjoint ODE case
  - in higher dimensions, non-self-adjoint case
  - in $\mathbb{R}^N$, pseudo-differential operator entries
- Problems of interest include:
  - when $\sigma^{s}_{\mathrm{ess}}(\mathcal{A}) \neq \emptyset$?
  - explicit description of $\sigma^{s}_{\mathrm{ess}}(\mathcal{A})$
  - “topological structure" of $\sigma_{\mathrm{ess}}(\mathcal{A})$
  - estimates one the essential spectral radius

Part I: The case of Ordinary Differential Operator entries

$$\mathcal{A} = \begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{d}t} p \frac{\mathrm{d}}{\mathrm{d}t} + q & -\frac{\mathrm{d}}{\mathrm{d}t}\overline{b} + \overline{c} \\ b\frac{\mathrm{d}}{\mathrm{d}t} + c & d \end{pmatrix}, \quad \mathrm{Dom}(\mathcal{A}) = C_0^2(\alpha,\beta) \oplus C_0^1(\alpha,\beta)$$

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- The Schur complement is given by, for $\lambda \in \mathbb{C} \setminus \sigma(\overline{d})$,

$$\tau_S(\lambda) = -\frac{\partial}{\partial t}\pi(\cdot, \lambda)\frac{\partial}{\partial t} + \mathrm{i}\Big(r(\cdot, \lambda)\frac{\partial}{\partial t} + \frac{\partial}{\partial t}r(\cdot, \lambda)\Big) + \varkappa(\cdot, \lambda)$$

$$\pi(\cdot, \lambda) := p - \frac{|b|^2}{d - \lambda}, \; r(\cdot, \lambda) := \mathrm{Im}\Big(\frac{\overline{b}c}{d - \lambda}\Big), \; \varkappa(\cdot, \lambda) := q - \lambda - \frac{|c|^2}{d - \lambda} + \frac{\partial}{\partial t}\mathrm{Re}\Big(\frac{\overline{b}c}{d - \lambda}\Big)$$

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3. If for every $\lambda \in \mathbb{R} \setminus \text{cl}\left\{\text{ran}\left(d - \frac{|b|^2}{p}\right)\right\}$, the limits exists and are finite

$$r_\beta(\lambda) := \lim_{t\nearrow\beta}(\beta - t)\frac{r(t,\lambda)}{\pi(t,\lambda)}, \qquad \varkappa_\beta(\lambda) := \lim_{t\nearrow\beta}(\beta - t)^2\frac{\varkappa(t,\lambda)}{\pi(t,\lambda)},$$

$$\Phi_1(t,\lambda) := (\beta - t)\frac{\frac{\partial}{\partial t}\pi(t,\lambda)}{\pi(t,\lambda)}, \qquad \Phi_2(t,\lambda) := (\beta - t)^2\frac{\frac{\partial}{\partial t}r(t,\lambda)}{\pi(t,\lambda)},$$

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$$r_\beta(\lambda) := \lim_{t \nearrow \beta}(\beta - t)\frac{r(t,\lambda)}{\pi(t,\lambda)}, \qquad \varkappa_\beta(\lambda) := \lim_{t \nearrow \beta}(\beta - t)^2\frac{\varkappa(t,\lambda)}{\pi(t,\lambda)},$$

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$$\sigma_{\mathrm{ess}}^{\mathrm{s}}(\mathcal{A}) = \Big\{\lambda \in \mathbb{R} \setminus \sigma_{\mathrm{ess}}^{\mathrm{r}}(\mathcal{A}) : \quad r_\beta(\lambda)^2 - \varkappa_\beta(\lambda) \geq \frac{1}{4}\Big\}.$$

Main results

1. $\sigma_{\text{ess}}^{\text{r}}(\mathcal{A}) = \text{cl}\Big\{\text{ran}\big(d - \frac{|b|^2}{p}\big)\Big\}$

2. If $\pi(t,\lambda) = \pi_0(\lambda) + \pi_1(\lambda)(t-\beta) + \mathcal{R}(t,\lambda)$ as $t \nearrow \beta$, then one has

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Remark: In the Astrophysics Model: $\pi_0(\lambda) \equiv 0$ and $\pi_1(\lambda) \equiv \frac{p_c}{\varrho_c}\Gamma_1(R)\theta'(R) \neq 0$.

Part II: Pseudo-differential operators on $\mathbb{R}^N$

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- The Hörmander symbol class $\mathcal{S}^k = \mathcal{S}^k_{1,0}(\mathbb{R}^N \times \mathbb{R}^N)$, $k \in \mathbb{R}$, is defined to be the set of $\sigma \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ s.t. for all $\alpha, \beta \in \mathbb{N}_0^N$ there exists $C_{\alpha,\beta} > 0$ with

$$|(\partial_x^\beta \partial_\xi^\alpha)\sigma(x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{k-|\alpha|}, \quad (x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N.$$

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- ΨDO T_σ with symbol $\sigma \in \mathcal{S}^k$ on $\mathscr{S}(\mathbb{R}^N)$ is defined by

$$(T_\sigma\phi)(x) := \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} \mathrm{e}^{\mathrm{i}x\cdot\xi}\sigma(x,\xi)\,\widehat{\phi}(\xi)\,\mathrm{d}\xi, \quad \phi \in \mathrm{Dom}(T_\sigma) = \mathscr{S}(\mathbb{R}^N),$$

where $\widehat{\phi}$ is the Fourier transform of $\phi \in \mathscr{S}(\mathbb{R}^N)$,

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Matrix pseudo-differential operator

$$T_0 := \begin{pmatrix} T_a & T_b \\ T_c & T_d \end{pmatrix}, \quad \mathrm{Dom}(T_0) := \mathscr{S}(\mathbb{R}^N) \oplus \mathscr{S}(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \oplus L^2(\mathbb{R}^N) = \mathcal{H}$$

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- **Previous studies:** Grubb and Geymonat (1977), Rabier (2012)

The case of $q = 0$

Principal symbol of $T_0 - \lambda$ is

$$M_\lambda(x,\xi) = \begin{pmatrix} a_m(x,\xi) & b_n(x,\xi) \\ c_p(x,\xi) & d_q(x,\xi) - \lambda \end{pmatrix}, \qquad (x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N.$$

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Theorem

Let $q = 0$ and let T_a be uniformly elliptic on $\mathbb{R}^N$. Then

$$\{\lambda \in \mathbb{C} : T_0 - \lambda \quad \text{is not Douglis-Nirenberg elliptic}\} \subset \sigma_{\text{ess}}(T)$$

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2. show that the normalization of the sequence

$$\Big(-T_a^p(\lambda) T_b \phi_k,\, \phi_k\Big)^t \in \mathscr{S}(\mathbb{R}^n) \oplus \mathscr{S}(\mathbb{R}^n)$$

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Let $T_0 - \lambda$ be uniformly D.N. elliptic. Then $\lambda \in \sigma_{\text{ess}}(T) \iff 0 \in \sigma_{\text{ess}}(S(\lambda))$

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Remark

$T_0 - \lambda$ be uniformly D.N. elliptic $\implies$ $S(\lambda)$ is uniformly elliptic

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$$F_2(\lambda) := T_b (T_d-\lambda)^{-1}, \qquad \mathrm{Dom}(F_2(\lambda)) := H^{n-q}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N).$$

Essential spectrum due to singularity [Wong, *Comm. PDE*, 10 (1988)]

Grushin symbol class

A symbol $\sigma \in \mathcal{S}^k$ is said to be in the class $\mathcal{S}_0^k$ if, for all $\alpha, \beta \in \mathbb{N}_0^N$, there is a positive function $x \mapsto C_{\alpha,\beta}(x)$, $x \in \mathbb{R}^N$, such that

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$$\lambda \in \sigma_{\mathrm{ess}}(T) \quad \Longleftrightarrow \quad \sigma_{\lambda,\infty}(\xi) = 0 \quad \text{for some} \quad \xi \in \mathbb{R}^N.$$