On the distribution of eigenvalues of compactly perturbed operators

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Germany

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This talk is based on . . .

. . . **joint work** with M. Demuth, F. Hanauska (Clausthal) and G. Katriel (Karmiel).

Contents:

- Classical results on eigenvalues of **compact** operators
- New results on eigenvalues of **compactly perturbed** operators
- A sketch of proof
- Final remarks

Note: In the whole talk we consider operators on a (general) complex Banach space X.

Some more notation

- $\mathcal{B}(X)$ and $\mathcal{C}(X)$ denote bounded and compact operators on X, respectively.

For a closed operator Z in X:

- $\sigma(Z)$ denotes the spectrum of Z,
- $\sigma_d(Z) := \{\lambda \in \mathbb{C} : \lambda \text{ isolated eigenvalue of finite algebraic mult.}\}$,
- $\sigma_{ess}(Z) := \{\lambda \in \mathbb{C} : \lambda I - Z \text{ not Fredholm}\}$.

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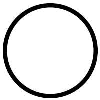

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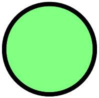

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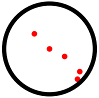
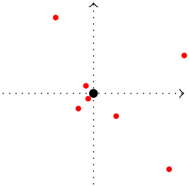


Eigenvalues of compact operators

(F. Riesz, 1916) Let X be infinite-dimensional and $K \in \mathcal{C}(X)$. Then



$$\begin{cases} \sigma_{ess}(K) & = & \{0\} \\ \sigma_d(K) & = & \sigma(Z) \setminus \{0\} \end{cases}$$

- If infinitely many discrete eigenvalues $\lambda_1(K), \lambda_2(K), \ldots$, then

$$|\lambda_n(K)| = \text{dist}(\lambda_n(K), \{0\}) \to 0 \quad (n \to \infty).$$

Can we say something about speed of convergence?

For instance, is $(\lambda_n(K))_n \in l_p(\mathbb{N})$ for some $p > 0$?

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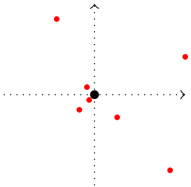

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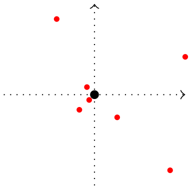

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For certain subclasses of $\mathcal{C}(X)$, the answer is YES.

- For example, if K is **nuclear**, i.e.

$$K = \sum_n x_n' \otimes x_n \quad \text{with} \quad \sum_n \|x_n'\|_{X'} \|x_n\|_X < \infty,$$

then $(\lambda_n(K))_n \in l_2(\mathbb{N})$. (A. Grothendieck'55)

For many other examples (Hilbert-Schmidt, absolutely-summing, ...) see, e.g., books by H. König or A. Pietsch.

Important for this talk:

Class $S_p(X)$ containing all $K \in \mathcal{B}(X)$ with $(a_n(K))_n \in l_p(\mathbb{N})$.

- Here the nth approximation number of $K \in \mathcal{B}(X)$ is defined as

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Special classes of compact operators

$$S_p(X) := \left\{K \in \mathcal{C}(X);\ \|K\|_p := \|(a_n(K))_n\|_{l_p} < \infty\right\}.$$

- **Degrees of compactness:**

$$p < q \ \Rightarrow\ l_p(\mathbb{N}) \subset l_q(\mathbb{N}) \ \Rightarrow\ S_p(X) \subset S_q(X)$$

Theorem (H. Weyl'49, H. König'78)

Let $p > 0$. If $K \in S_p(X)$, then $(\lambda_n(K))_n \in l_p(\mathbb{N})$. Moreover, there exists $c_p \geq 1$ such that

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Now we **generalize the problem**: Let

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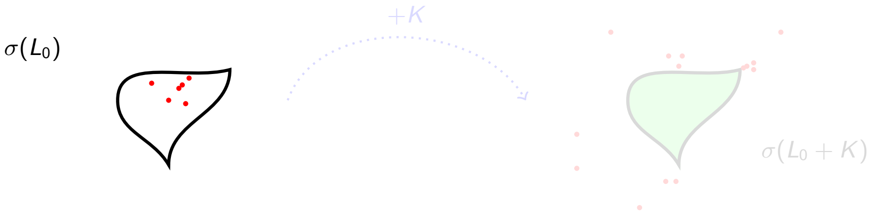

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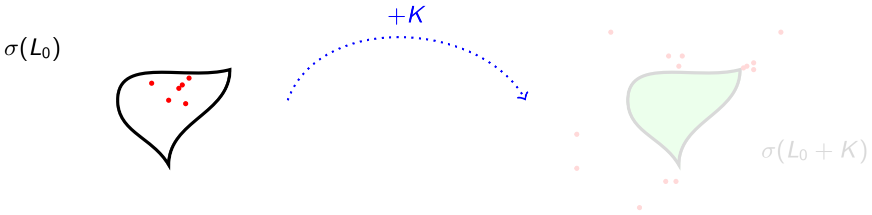

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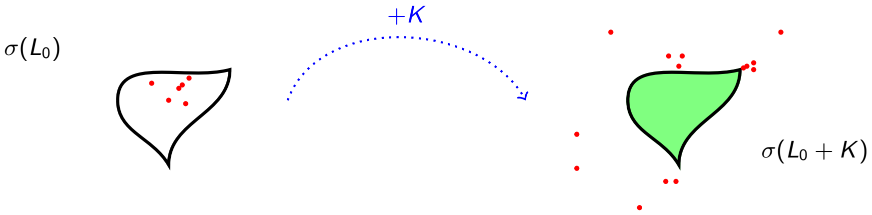

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Question: Let $K \in S_p(X)$ and $L_0 \in \mathcal{B}(X)$. Is it true that for some $q \geq p$

$$\Big(\operatorname{dist}(\lambda_n(L_0 + K), \sigma_{ess}(L_0)) \Big)_n \in l_q(\mathbb{N}) \text{ ?}$$

Recall that the answer is yes if $L_0 = 0$ (with $q = p$).

Unfortunately, in general, the answer is No! (Just take $K = 0$)

$\sigma(L_0)$:

Possible solutions: (1) Put more restrictions on L_0.
(2) Ignore some eigenvalues by replacing $\sigma_{ess}(L_0)$ with a larger set Ω (e.g. $\Omega = \sigma(L_0)$) and look at

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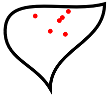

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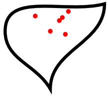

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Modified question (quantitative version): Does there exist $q \geq p$ and $C = C(X, p, q, L_0, \Omega)$ such that

$$\sum_{\lambda \in \sigma_d(L_0+K)} \operatorname{dist}(\lambda, \Omega)^q \leq C\|K\|_p^p \text{ ?} \tag{1}$$

- **Analogy** to (selfadjoint) Lieb-Thirring inequalities:

$$\sum_{\lambda \in \sigma_d(-\Delta+V), \lambda<0} \qquad \leq C_{p,d}\|V\|_{L_p}^p.$$

- Let $N(\Omega_s)$ denote the **number of eigenvalues** of $L_0 + K$ in $\Omega_s = \{\lambda : \operatorname{dist}(\lambda, \Omega) > s\}$. Then (1) implies

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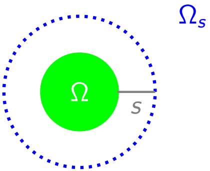

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- $\Omega = \sigma(L_0)$: In general, the answer is No.

  Note: If $X = \mathcal{H}$, however, the answer is Yes, if
  - L_0 and K are selfadjoint and $q = p \geq 1$ (T. Kato '87),
  - L_0 is selfadjoint and $q = p > 1$ (M.H.'13),
- $\Omega = \{w : |w| \leq \text{spec-rad}(L_0)\}$: In general, the answer is No.
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- $\Omega = \{w : |w| \leq \text{spec-rad}(L_0)\}$: In general, the answer is No.
- $\Omega = \{w : |w| \leq \|L_0\|\}$: The answer is YES!

Compact perturbations

Theorem (M. Demuth, F. Hanauska, M.H., H. Katriel '15)

Let $p > 0$ and $K \in S_p(X)$.

(1) *Let $n_{L_0+K}(s)$ denote the number of eigenvalues $\lambda \in \sigma_d(L_0 + K)$ with $|\lambda| > \|L_0\| + s$. Then*

$$n_{L_0+K}(s) \leq C_p \frac{s + \|L_0\|}{s^{p+1}} \|K\|_p^p, \quad (s > 0). \tag{2}$$

(2) *If $q > p + 1$, then*

$$\sum_{\lambda \in \sigma_d(L_0+K), |\lambda| > \|L_0\|} (|\lambda| - \|L_0\|)^q \leq C(p, q, \|L_0\|) \|K\|_p^p. \tag{3}$$

- Is the additional '+1' really necessary ???
- In case $X = \mathcal{H}$ estimate (3) is true if $q = \max(p, 1)$.
- Actually, we prove a bound on $n_{L_0+K}(s)$ **only assuming that** $a_n(K) \to 0$.

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Idea of proof

Let $s > 0$. We are interested in eigenvalues $\lambda \in \sigma_d(L_0 + K)$ with $|\lambda| > \|L_0\| + s$.

1. Let $N \in \mathbb{N}_0$ such that $a_{N+1}(K) < s$.
2. Choose F with $\text{Rank}(F) \leq N$ such that

$$\lambda - (L_0 + K - F) = (\lambda - L_0)(I - (\lambda - L_0)^{-1}(K - F))$$

   is invertible for $|\lambda| > \|L_0\| + s$

   (note that $\|(\lambda - L_0)^{-1}\| \leq 1/(|\lambda| - \|L_0\|) < 1/s$).

3. The function

$$\lambda \mapsto d(\lambda) := \det_p(I - F[\lambda - (L_0 + K - F)]^{-1})$$

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Final remarks

- Optimal exponents for eigenvalues of compact operators (Weyl, König) are obtained with **determinant free methods**. Is this possible in the compactly perturbed case as well?
- Optimal exponents in Banach and Hilbert space will **differ**!
- We can obtain bounds on eigenvalues in other sets $\Omega \subset \mathbb{C} \setminus \sigma(L_0)$ as long as we have some control of $\|(\lambda - L_0)^{-1}\|$ (or $\|K(\lambda - L_0)^{-1}\|_p$) on Ω. (Pseudospectrum)
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Final remarks

- Let us review the properties of $S_p(X)$:
  1. $(S_p(X), \|.\|_p)$ is a (quasi-)normed space and $\|K\| \leq \|K\|_p$.
  2. The finite rank operators are dense in $(S_p(X), \|.\|_p)$.
  3. $S_p(X)$ is ideal in $\mathcal{B}(X)$ and $\|AKB\|_p \leq \|A\|\|K\|_p\|B\|$ if $A, B \in \mathcal{B}(X)$.
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Theorem: (M.H.'16) If $(\mathcal{I}_p, \|.\|_{\mathcal{I}_p})$ satisfies (1)-(4), and $K \in \mathcal{I}_p$, then

$$n_{L_0+K}(s) \leq C_{p,\mathcal{I}_p} \frac{s + \|L_0\|}{s^{p+1}} \|K\|_{\mathcal{I}_p}^p, \qquad (s > 0).$$

Examples: Absolutely p-summing ($p \geq 2$), nuclear operators ($p = 2$), ...

- **An application**: Let $X = L_p(\Omega, \mu), 2 < p < \infty$, and let H, H_0 be generators of C_0- and contraction semigroups on X, respectively. Assume $e^H - e^{H_0}$ is compact integral operator with kernel d. Then for $r > 0$

$$\mathcal{N}_H(\{\lambda : \mathrm{Re}(\lambda) > r\}) \leq C_p \frac{e^r}{(e^r - 1)^{p+1}} \int_\Omega \left(\int_\Omega |d(x,y)|^{p'} d\mu(y) \right)^{\frac{p}{p'}} d\mu(x).$$

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