The Invariant Subspace Problem: A Concrete Operator Theory Approach

C.I.R.M.

Luminy, June 2017

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Joint work with Carl C. Cowen (Purdue University-Indiana University)

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$$\left.\begin{array}{l} T:\mathcal{H}\to\mathcal{H}, \text{linear and bounded} \\ \\ T(M)\subset(M), \text{closed subspace} \end{array}\right\} \implies M=\{0\} \text{ or } M=\mathcal{H}?$$

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2. Non-separable Hilbert spaces.

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$\ker(S - \lambda I) = \{0\}$ for any $\lambda \in \mathbb{C}$. That is, $\sigma_p(S) = \emptyset$.

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Characterization of the invariant subspaces of S?

Classical Beurling Theory:
Inner-Outer Factorization of the functions in the Hardy space.

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  ∗ 1954, Aronszajn and Smith (general case)

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    - ∗ 1967, Halmos
    - ∗ 1960's, Gillespie, Hsu, Kitano...

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There exists a separable Banach space $\mathcal{B}$ and a linear, bounded operator T acting on $\mathcal{B}$, **inyective and with dense range**, *without no non-trivial closed invariant subspaces.*

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• **1985, C. Read**, Construction of a linear bounded operator on ℓ^1 without non-trivial closed invariant subspaces.

The big open question

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Does every linear bounded operator T acting on a separable, reflexive complex Banach space $\mathcal{B}$ (or a Hilbert space $\mathcal{H}$) have a non-trivial closed invariant subspace?

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Theorem (Lomonosov) Let T be a linear bounded operator on $\mathcal{H}$, $T \neq \mathbb{C}Id$. If T commutes with a non-null compact operator, then T has a non-trivial closed invariant subspace.

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Theorem **(Lomonosov)** Let T be a linear bounded operator on $\mathcal{H}$, $T \neq \mathbb{C}Id$. If T commutes with a non-null compact operator, then T has a non-trivial closed invariant subspace. Moreover, T has a non-trivial closed hyperinvariant subspace.

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Theorem (Lomonosov) Any linear bounded operator T, not a multiple of the identity, has a nontrivial invariant closed subspace if it commutes with a non-scalar operator that commutes with a nonzero compact operator.

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- Does every operator satisfy “Lomonosov Hypotheses”?

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Construction of a "quasi-analytic" shift S on a weighted ℓ^2 space which has the following property: if K is a compact operator which commutes with a nonzero, non scalar operator in the commutant of S, then $K = 0$.

A "Concrete Operator Theory" approach

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- **Example.** Adjoint of a unilateral shift of infinite multiplicity. It may be regarded as S^* in $(\ell^2(\mathcal{H}))$ defined by

$$\mathrm{S}^*((\mathrm{h}_0, \mathrm{h}_1, \mathrm{h}_2, \cdots)) = (\mathrm{h}_1, \mathrm{h}_2, \cdots)$$

for $(\mathrm{h}_0, \mathrm{h}_1, \mathrm{h}_2, \cdots) \in \ell^2(\mathcal{H})$.

A "Concrete Operator Theory" approach

- **Universal Operators** (in the sense of G. C. Rota)

- **Example.** Let $a > 0$ and $T_a : L^2(0,\infty) \to L^2(0,\infty)$ defined by

$$T_a f(t) = f(t + a), \qquad \text{for } t > 0.$$

T_a is universal.

A "Concrete Operator Theory" approach: universal operators

• **Proposition.** Let $\mathcal{H}$ be a Hilbert space and U a linear bounded operator. Suppose that U is a universal operator. The following conditions are equivalent:

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Providing universal operators

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1. $\mathrm{Ker}(\mathrm{U})$ is infinite dimensional,
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Then U is universal.

Idea of the Proof

Write $\mathcal{K}$ =Ker U

1. $\mathrm{UV} = \mathrm{Id}$,
2. $\mathrm{UW} = 0$,
3. ker$\mathrm{W} = \{0\}$,
4. Im$\mathrm{W} = \mathcal{K}$ and Im$\mathrm{V} = \mathcal{K}^{\perp}$.

- $\mathcal{M}$ =Im J is a closed subspace of U.
- J is an isomorphism onto $\mathcal{M}$.

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Step 1: Construct $\mathrm{V}, \mathrm{W} \in \mathcal{L}(\mathcal{H})$ such that

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Let T be a linear bounded operator on $\mathcal{H}$.
Let $\lambda \neq 0$ such that $|\lambda| \, \|\mathrm{T}\| \, \|\mathrm{U}\| < 1$ and define

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Examples of universal operators

- **Theorem** (1987, Nordgren, Rosenthal y Wintrobe)

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Let φ be a hyperbolic automorphism of $\mathbb{D}$. For every λ in the interior of the spectrum of C_φ, $\mathrm{C}_\varphi - \lambda\mathrm{I}$ is universal in $\mathcal{H}^2$.

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- **Disc automorphisms**

$$\varphi(z) = e^{i\theta}\,\frac{p - z}{1 - \overline{p}z} \qquad (z \in \mathbb{D}).$$

where $p \in \mathbb{D}$ and $-\pi < \theta \leq \pi$.

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⋆ Hyperbolic. φ has two fixed points α and β, such that $\alpha, \beta \in \partial\mathbb{D}$ ($\Leftrightarrow |p| > \cos(\theta/2)$)

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where $p \in \mathbb{D}$ and $-\pi < \theta \leq \pi$.

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⋆ Hyperbolic. φ has two fixed points α and β, such that $\alpha, \beta \in \partial\mathbb{D}$ ($\Leftrightarrow |p| > \cos(\theta/2)$)

⋆ Elliptic. φ has two fixed points α and β, with $\alpha \in \mathbb{D}$ ($\Leftrightarrow |p| < \cos(\theta/2)$)

Examples of universal operators

- **Theorem** (1987, Nordgren, Rosenthal y Wintrobe)

Let φ be a hyperbolic automorphism of $\mathbb{D}$. For every λ in the interior of the spectrum of C_φ, $\mathrm{C}_\varphi - \lambda \mathrm{I}$ is universal in $\mathcal{H}^2$.

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We may assume that φ fixes 1 and -1. Then,

$$\varphi(\mathrm{z}) = \frac{\mathrm{z}+\mathrm{r}}{1+\mathrm{rz}}, \qquad 0 < \mathrm{r} < 1.$$

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Every linear bounded operator T has a closed non-trivial invariant subspace

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for any $\mathrm{f} \in \mathcal{H}^2$, not an eigenfunction of C_φ, there exists $\mathrm{g} \in \overline{\mathrm{span}}\{\mathrm{C}_\varphi^{\mathrm{n}}\mathrm{f} : \mathrm{n} \geq 0\}$ such that $\mathrm{g} \neq 0$ and

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the **minimal** non-trivial closed invariant

subspaces for C_φ are one-dimensional

Question

Which conditions on f ensure that $K_f := \overline{\text{span}}\{C_\varphi^n f : \ n \geq 0\}$ is (or not) minimal?

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Mortini (1995), Matache (1998), Chkliar (1997), Shapiro (2011), GG-Gorkin (2011), Mortini (2013)...

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Eigenfuntions of C_φ?

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- **2012, GG, Gorkin and Suárez**, Constructive characterization of eigenfunctions of C_φ in the Hardy spaces $\mathcal{H}^p$

Universal operators vs. Lomonosov Theorem

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• **Naive Question:** Does there exist a universal operator which conmutes with a non-null compact operator *in a non-trivial way*?

Universal operators

Suppose S is a multiplication operator on the Hardy space $\mathcal{H}^2$ whose symbol is a singular inner function or infinite Blaschke product.

1. S is an isometric operator.
2. S^* has infinite dimensional kernel and maps $\mathcal{H}^2$ onto $\mathcal{H}^2$.

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  - S^* is universal.
  - Using the Wold Decomposition Theorem, such an operator can be represented as a block matrix on $\mathcal{H} = \oplus_{k=1}^{\infty} \mathrm{S}^k \mathrm{W}$, where $\mathrm{W} = \mathrm{H}^2 \ominus \mathrm{SH}^2$. Such a matrix is an upper triangular and has the identity on the super-diagonal:

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$$\mathrm{S}^* \sim \begin{pmatrix} 0 & \mathrm{I} & 0 & 0 & \cdots \\ 0 & 0 & \mathrm{I} & 0 & \cdots \\ 0 & 0 & 0 & \mathrm{I} & \cdots \\ & & & \ddots & \end{pmatrix}$$

Universal operators

An easy computation shows that every operator that commutes with S^* has the form

- This is an upper triangular block matrix whose entries on each diagonal are the same operator on the infinite dimensional Hilbert space W.
- Every block in such a matrix occurs infinitely often.
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$$A \sim \begin{pmatrix} A_0 & A_{-1} & A_{-2} & A_{-3} & \cdots \\ 0 & A_0 & A_{-1} & A_{-2} & \cdots \\ 0 & 0 & A_0 & A_{-1} & \cdots \\ & & & \ddots & \end{pmatrix}$$

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- So, the only compact operator that commutes with the universal operator S^* is 0, **not an interesting compact operator!**

Universal operators

- **Theorem** (2011, Cowen-GG)

Let φ be a hyperbolic automorphism of $\mathbb{D}$. Then C_φ^* is similar to the Toeplitz operator T_ψ, where ψ is the covering map of the unit disc onto the interior of the spectrum $\sigma(\mathrm{C}_\varphi)$.

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- **Straightforward consequence**

Known universal operators are not commuting with non-null compact operators.

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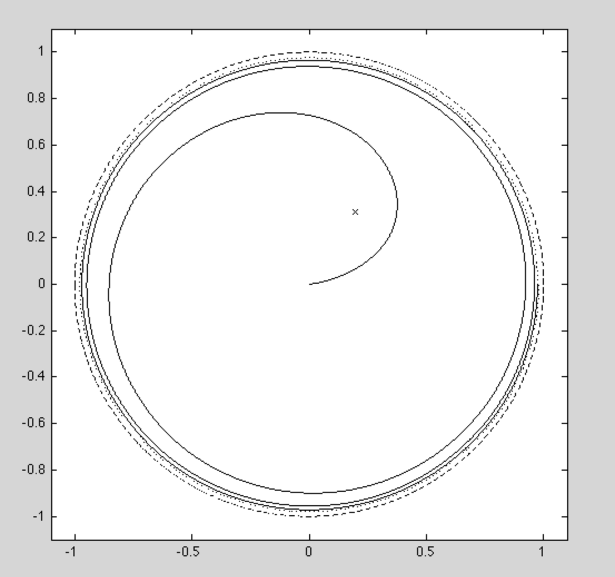
A universal operator which commutes with a compact operator



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Definition. Let $\mathcal{K}_\varphi$ be the set of compact operators that commute with T_φ^*, that is,

$$\mathcal{K}_\varphi = \{\mathrm{G} \in \mathcal{B}(\mathrm{H}^2) : \mathrm{G} \text{ is compact, and } \quad \mathrm{T}_\varphi^*\mathrm{G} = \mathrm{G}\mathrm{T}_\varphi^*\}$$

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Remark. $\mathcal{K}_\varphi \neq (0)$.

Compact operators commuting with universal operators

If F is a bounded operator on H^2, we will write $\{\mathrm{F}\}'$ for the commutant of F, the set of operators that commute with F, that is,

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For any operator F, the commutant $\{\mathrm{F}\}'$ is a norm-closed subalgebra of $\mathcal{B}(\mathrm{H}^2)$.

Compact operators commuting with universal operators

Theorem [2015, Cowen, GG] The set $\mathcal{K}_\varphi$ is a closed subalgebra of $\{\mathrm{T}_\varphi^*\}'$ that is a two-sided ideal in $\{\mathrm{T}_\varphi^*\}'$. In particular, if G is a compact operator in $\mathcal{K}_\varphi$ and g and h are bounded analytic functions on the disk, then $\mathrm{T}_\mathrm{g}^*\mathrm{G}$, $\mathrm{G}\mathrm{T}_\mathrm{h}^*$, and $\mathrm{T}_\mathrm{g}^*\mathrm{G}\mathrm{T}_\mathrm{h}^*$ are all in $\mathcal{K}_\varphi$. Moreover, every operator in $\mathcal{K}_\varphi$ is quasi-nilpotent.

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- **Question.** Characterization of the bacwardshift invariant subspaces in the Bergman spaces A^2?

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• **Question.** Characterization of the bacwardshift invariant subspaces in the Bergman spaces A^2? **Well-Known:** Structure is extremely complicated (Borichev, Hedenmalm, Shimorim, Aleman-Richter-Sundberg...)

Consequences and Further Observations

Let A be a linear bounded operator on a Hilbert space and T a universal operator which commutes with a compact operator W.

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- $\mathcal{H} = M \oplus M^{\perp}$ and with respect to this decomposition

$$T \sim \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad \text{and} \quad W \sim \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

where A, C are invertible and P, Q, R, S are compact.

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- NOT $P = 0$ and $R = 0$ because kernel$(W) = (0)$.

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Observation: Since A is the operator of primary interest, Equation

$$\mathrm{AP} + \mathrm{BR} = \mathrm{PA}$$

is not so interesting if $\mathrm{P} = 0$.

Consequences, Further Observations, and a Question

- **Lemma.** If the universal operator $\mathrm{T} = \mathrm{T}_\varphi^*$ and the compact operator $\mathrm{W} = \mathrm{W}_{\psi,\mathrm{J}}^*$ have the representations

$$\mathrm{T} \sim \begin{pmatrix} \mathrm{A} & \mathrm{B} \\ 0 & \mathrm{C} \end{pmatrix} \quad \text{and} \quad \mathrm{W} \sim \begin{pmatrix} \mathrm{P} & \mathrm{Q} \\ \mathrm{R} & \mathrm{S} \end{pmatrix}$$

respect $\mathcal{H} = \mathrm{M} \oplus \mathrm{M}^\perp$, then there are a universal operator $\widetilde{\mathrm{T}}$ and an injective compact operator $\widetilde{\mathrm{W}}$ with dense range that commute for which $\widetilde{\mathrm{P}}$ in a replacement of P is not zero, that is, without loss of generality, we may assume $\mathrm{P} \neq 0$.

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- **Theorem [Cowen, GG]** Let the universal operator T and the commuting injective compact operator W with dense range having the representations with $\mathrm{P} \neq 0$. Then the following are true:

  - Either $\mathrm{R} \neq 0$ or A has a nontrivial hyperinvariant subspace.
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Theorem [Cowen, GG] Suppose L is an invariant subspace for the universal operator T_{φ}^{*} and the block matrix

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represents T_{φ}^{*} based on the splitting $\mathrm{H}^2 = \mathrm{M} \oplus \mathrm{M}^{\perp}$. Then, the projection of $\mathrm{L}^{\perp}$ onto M is an invariant linear manifold for A^{*}, the adjoint of the restriction of T_{φ}^{*} to M.

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Remark.

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Remark. Any of the linear manifolds provided by this Theorem are proper and invariant but, in principle, they are not necessarily non-dense.

Consequences, Further Observations, and a Question

Question: Is any of those proper A^*-invariant linear manifolds non-dense?

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Thank you for your attention