Complex Complexiton solutions for the KdV equation

Mathematical aspects of the physics with non-self-adjoint operators.

CIRM, Marseille 8th June 2017

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Historical origin of solitary waves

1834 *Scott Russel*'s first observation of a *Translation Wave* along the Union Canal - Gyle.

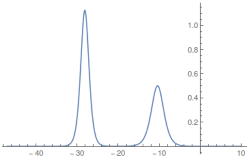
1895 *Korteg* and *de Vries* proposed the equation and explicit solutions.

$$u_t \quad + \quad 6\, u_x u \quad + \quad u_{xxx} \quad = \quad 0$$

$$\boxed{\text{Singularity formation}} \quad \Longleftrightarrow \quad \boxed{\text{Dispersive effect}}$$

1960 *Gardner* Integrability

$$D_t + F_x = 0$$



Let's define two linear problems: $\begin{cases} L\,\phi = \lambda(t)\,\phi \\ A\,\phi = \phi_t \end{cases}$

where: $L\,\phi = -\phi_{xx} - u(x,t)\phi, \quad A\,\phi = \phi u_x + 4\Big(\lambda(t) - 2u(x,t)\Big)\phi.$

$$\boxed{u(x,t) \text{ solves KdV eq.} \quad \Longleftrightarrow \quad L_t + [L, A] = 0, \quad \lambda_t = 0}$$

Classification for the solution:

- $\lambda > 0$ Positon
- $\lambda = 0$ Rational
- $\lambda < 0$ Negatons (solitons)
- $\lambda \in \mathbb{C}$ Complexitons (real or complex)

1968 The Lax formalism

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Darboux-Crum method

Fix $\lambda_1 \neq \lambda_2$ and consider the functions ϕ_1, ϕ_2 and $\phi_2[1]$ such that:

$$-\phi_i'' + V(x)\,\phi = \lambda_i \phi_i, \qquad \phi_2[1] = \frac{W(\phi_1, \phi_2)}{\phi_1}.$$

Then $\phi_2[1]$ is solution of the new problem:

$$-\phi_2[1]'' + V_{\text{new}}\,\phi = \lambda_2\,\phi_2[1],$$

where:

$$\boxed{V_{\text{new}} = V - 2\tfrac{d^2}{dx^2}\Big(\ln(\phi_1)\Big)}$$

Crum theorem: Consider $\phi_1, \ldots, \phi_N, \phi$, then:

$$V_{\text{new}} = V - 2\frac{d^2}{dx^2}\Big(\ln(W(\phi_1, \ldots, \phi_N))\Big)$$

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Construction of (complex) complexiton solution.

Seed solution for KdV: $u(x,t) \equiv 0$.

The equation for A suggests for $k \in \mathbb{C}$:

$$\lambda = -\frac{k^2}{4},$$

so that $\phi_i = \begin{cases} \cosh\left(\frac{k_i}{2}(x - k_i^2 t)\right) & \text{if } i \text{ odd}, \\ \sinh\left(\frac{k_i}{2}(x - k_i^2 t)\right) & \text{if } i \text{ even}, \end{cases}$ solves $-\phi_{xx} = \lambda\phi$.

Therefore solutions of the KdV are:

$$S_i(x,t) = -\frac{d^2}{dx^2}\Big(\ln(W(\phi_1, \ldots, \phi_i))\Big)$$
$$\Big(= F_i\left[\cosh(\cdot), \sinh(\cdot), \cos(\cdot), \sin(\cdot)\right](x,t)\Big)$$

Dynamics for $S_i(x,t)$:direction and velocity.

Since the wave numbers $k_i \in \mathbb{C} \Rightarrow$ non travelling solution.

In fact, it is readily seen from the dispersion relation coming from the equation for operator A that the velocity of the wave generated by k_i is

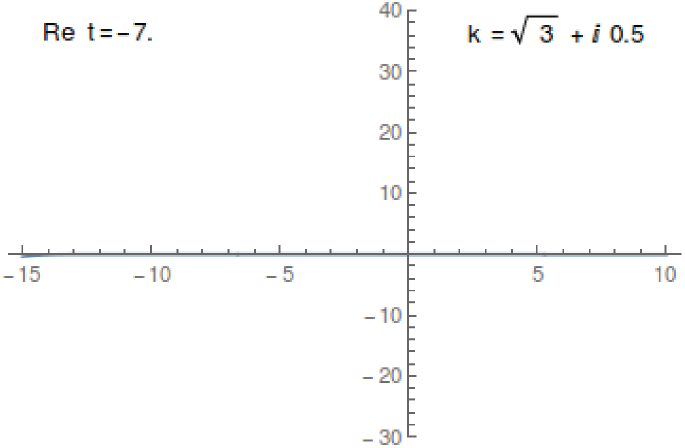
$$c(k_i) = \mathrm{Re(k_i)}^2 - 3\,\mathrm{Im(k_i)}^2$$

Therefore, for the values

$$\frac{\mathrm{Re(k_i)}}{\mathrm{Im(k_i)}} = \sqrt{3}$$

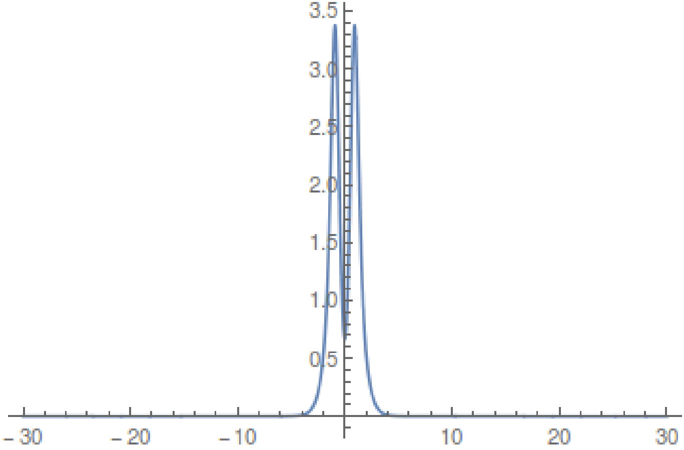
the solution $S_i(x,t)$ is standing (oscillating) wave.

Examples of dynamics for $S_1(x,t)$: blow-up



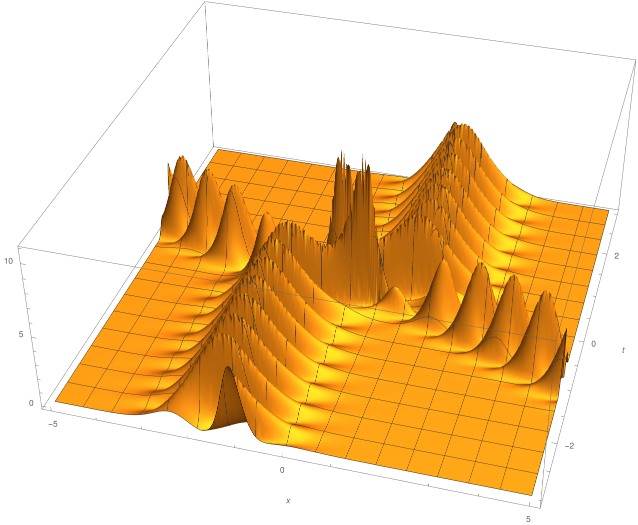
Plot: real part of the wave generated by single $k = \sqrt{3} + i/2$.

Examples of dynamics for $S_2(x,t)$: boundedness and localisation



Plot: Real part of the wave generated by $(\alpha + i\beta, 2 + i)$.

Interaction of two localised complexitons



Plot: Wave generated by $(\sqrt{3}+i, \sqrt{3}-i, \sqrt{3}+i3/2, \sqrt{3}-i3/2)$

Differences with the real solutions:

The solutions presented share with the real solutions some feature:

- Localization property for certain values of wavenumbers;
- Linear interaction up to a phase shift;
- Can catch the exotic *"breathers solutions"*.

There are nonetheless differences:

- Real solitons travel only leftwards;
- No blow-up for real solution.

Potential at time $t = 0$

The Darboux method provides a precise description of the spectrum and eigenfunctions. Set $H = -\frac{d^2}{dx^2} - S_i(x, 0)$, then

$$-\frac{k_j^2}{4} \in \sigma_d(H), \qquad e^{k_j}[i] \text{ associated eigenfunction}$$

For example, for the case $i = 2$ ($k \in \mathbb{C}$):

$$\begin{aligned} W(\phi_1, \dots, \phi_i, e^{kx}) =& (k - k_1)(k - k_2)(k_2 - k_1)e^{(k_1+k_2)x+kx} \\ &+ (k + k_1)(k + k_2)(k_2 - k_1)e^{-(k_1+k_2)x+kx} \\ &+ (k - k_1)(k + k_2)(k_2 + k_1)e^{(k_1-k_2)x+kx} \\ &+ (k + k_1)(k - k_2)(k_2 + k_1)e^{-(k_1-k_2)x+kx} \end{aligned}$$

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The scattering problem at time $t = 0$

$$-\frac{d^2}{dx^2}g - V(x)g = \lambda g$$

Consider now $k \in \mathbb{R}$:

$$\psi(x,k) \approx e^{ikx}, \qquad \tilde{\psi}(x,k) \approx e^{-ikx} \qquad \text{as } x \to +\infty;$$

$$\chi(x,k) = a(k)\tilde{\psi}(x,k) + b(k)\psi(x,k) \approx e^{ikx} \qquad \text{as } x \to -\infty.$$

The natural candidates for the functions above are of course:

$$\Psi_+(x,k) = \frac{\mathcal{W}(\phi_1, \phi_2, e^{ikt})}{\mathcal{W}(\phi_1, \phi_2)}$$

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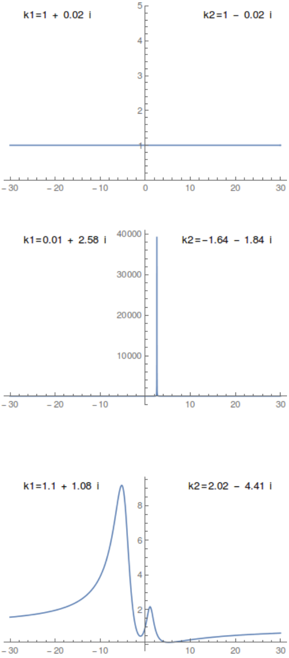
It turns out that:

$$\begin{cases} a(k) = \prod_{j=1}^{2} \frac{(-ik - k_j)}{(-ik + k_j)}, \\ b(k) = 0. \end{cases}$$

- Reflectionless potential
- In general $|a(k)|^{-1} \neq 1$ unless $\overline{k}_{2j} = k_{2j-1}$
- For high frequencies $|a(k)| \to 1$ as $|k| \to \infty$.
- $a(-k) = a(k)^{-1}$



Question and open problem

There are many open questions regarding complex solutions of the KdV equation:

- Physical meaning of a complex wavenumber.
- Integrability and conserved quantities.
- Dynamics and interaction.
- Stability and asymptotic stability for these solutions, studied for real solution by several authors *(Martel, Merle, Vega, Munoz, Alejo...)*. Very recently breathers stability have been proved.
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Thanks for you kind patience and attention :D

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