Acoustic waves in the incompressible limits of viscous fluids

Eduard Feireisl
based on joint work with D. Donatelli (L'Aquila)

Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague

Mathematical aspects of non–selfadjoint operators, CIRM Marseille 5 - 9 June 2017

The research leading to these results has received funding from the European Research Council
under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC Grant Agreement 320078

A model arising in astrophysics

Primitive system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x (\varrho \Theta)^\gamma = \varepsilon^\alpha \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \frac{1}{\varepsilon^2} \varrho \nabla_x F$$

$$\partial_t(\varrho \Theta) + \mathrm{div}_x(\varrho \Theta \mathbf{u}) = 0$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \lambda \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \lambda \geq 0,$$

Gravitational potential

$$F \in C^\infty(R^3), \ F(x) > 0, \ |x|^2 |\nabla_x F(x)| + |x|^3 |\nabla_x^2 F(x)| \leq c \text{ for all } x \in R^3$$

$$\underline{F} \frac{1}{|x|} \leq F(x) \leq \overline{F} \frac{1}{|x|} \text{ for all } |x| > R$$

Far field conditions

$$\varrho \to \overline{\varrho} > 0, \ \mathbf{u} \to 0, \ \Theta \to 1 \text{ as } |x| \to \infty$$

Low Mach number approximation of (certain) supernovae

Target system

$$\mathrm{div}_x(\varrho_0 \mathbf{V}) = 0$$

$$\partial_t \mathbf{V} + \mathbf{V} \cdot \nabla_x \mathbf{V} + \nabla_x \Pi = -\frac{\varrho_0}{\mathcal{R}} \nabla_x F$$

$$\partial_t(\mathcal{R}) + \mathrm{div}_x(\mathcal{R}\mathbf{V}) = 0$$

Static density profile

$$\nabla_x \varrho_0^\gamma = \varrho_0 \nabla_x F \text{ in } R^3, \ \varrho_0 \to \overline{\varrho} > 0 \text{ as } |x| \to \infty$$

Initial state of the primitive system

Initial data

$$\varrho_{0,\varepsilon}(0,\cdot) \approx \varrho_0 + \varepsilon \varrho_0^1$$

$$\mathbf{u}_{0,\varepsilon}(0,\cdot) \approx \mathbf{V}_0 + \nabla_x \Phi_0$$

$$\Theta_{0,\varepsilon}(0,\cdot) = 1 + \varepsilon^2 \Theta_0^{(2)}$$

Generalized Helmholtz decomposition

$$\mathbf{v} = \mathbf{H}_{\varrho_0}[\mathbf{v}] + \nabla_x \Phi$$

$$\mathrm{div}_x\, (\varrho_0 \nabla_x \Phi) = \mathrm{div}_x(\varrho_0 \mathbf{v}),\ \ \Phi \in D^{1,2}(R^3) \text{ for } \mathbf{v} \in L^2(R^3, R^3).$$

Well vs ill prepared data

Well: $\varrho_0^1 = 0,\ \ \nabla_x \Phi = 0,\ \ \Theta_0^{(2)} \in L^1 \cap L^\infty$

Ill: otherwise

Expected conclusion

Convergence

$$\varrho_\varepsilon \approx \varrho_0 + \boxed{\varepsilon s_\varepsilon} \to \varrho_0$$

$$\mathbf{u}_\varepsilon \approx \mathbf{V} + \boxed{\nabla_x \Phi}_\varepsilon \to \mathbf{V}$$

Potential temperature

$$\Theta_\varepsilon \approx 1 + \varepsilon^2 \Theta_\varepsilon^{(2)},\ \Theta_\varepsilon^{(2)} \to \frac{\varrho_0}{\mathcal{R}}$$

Acoustic waves

Acoustic equation

$$\varepsilon \partial_t s_\varepsilon + \mathrm{div}_x \left[\varrho_0 \nabla_x \Phi_\varepsilon \right] = 0$$

$$\varepsilon \varrho_0 \partial_t \nabla_x \Phi_\varepsilon + \varrho_0 \nabla_x \left[\frac{p'(\varrho_0)}{\varrho_0} s_\varepsilon \right] = 0$$

or, in a concise form

$$\varepsilon^2 \partial^2_{t,t} \Phi_\varepsilon - \frac{p'(\varrho_0)}{\varrho_0} \mathrm{div}_x \left(\varrho_0 \nabla_x \Phi_\varepsilon \right) = 0$$

Initial data

$$s_\varepsilon(0,\cdot) = \varrho_0^{(1)},\ \nabla_x \Phi_\varepsilon(0,\cdot) = \nabla_x \Phi_0$$

Acoustic propagator

$$\mathcal{A}_{\varrho_0} : v \mapsto -\frac{p'(\varrho_0)}{\varrho_0} \mathrm{div}_x \left(\varrho_0 \nabla_x v \right)$$

Basic properties [DeBièvre and Pravica]

Self–adjoint operator

$$\mathcal{A}_{\varrho_0} : v \mapsto -\frac{p'(\varrho_0)}{\varrho_0} \mathrm{div}_x \left(\varrho_0 \nabla_x v \right)$$

$$v \in L^2(R^3), \ \langle u; v \rangle = \int_{R^3} uv \frac{\varrho_0}{p'(\varrho_0)} \ \mathrm{d}x$$

Spectrum

- the point spectrum of $\mathcal{A}_{\varrho_0}$ is empty
- the operator $\mathcal{A}_{\varrho_0}$ satisfies the *Limiting absorption principle*

Operators

$$\mathcal{V} \circ (\mathcal{A}_{\varrho_0} - \lambda \pm \mathrm{i}\eta)^{-1} \circ \mathcal{V} : L^2(\Omega) \to L^2(\Omega),$$

$$\mathcal{V}[v] = (1 + |x|^2)^{-s/2}, \ s > 1$$

are bounded uniformly for $\lambda \in [a, b], \ 0 < a < b, \ \eta > 0,$

Local decay properties

Theorem (Kato)

Let C be a closed densely defined linear operator and H a self-adjoint densely defined linear operator in a Hilbert space X. For $\lambda \notin R$, let $R_H[\lambda] = (H - \lambda \mathrm{Id})^{-1}$ denote the resolvent of H. Suppose that

$$\Gamma = \sup_{\lambda \notin R,\ v \in \mathcal{D}(C^*),\ \|v\|_X = 1} \| C \circ R_H[\lambda] \circ C^*[v] \|_X < \infty.$$

Then

$$\sup_{w \in X,\ \|w\|_X = 1} \frac{\pi}{2} \int_{-\infty}^{\infty} \| C \exp(-\mathrm{i} t H)[w] \|_X^2 \,\mathrm{d}t \le \Gamma^2.$$

Local decay estimates

$$\int_{-\infty}^{\infty} \left\| \varphi G(\mathcal{A}_{\varrho 0}) \exp\left(\pm \mathrm{i} \sqrt{\mathcal{A}_{\varrho 0}}\, t \right) [h] \right\|_{L^2(R^3)}^2 \,\mathrm{d}t \le c(\varphi, G) \|h\|_{L^2(R^3)}^2$$

$$\varphi \in C_c^\infty(R^3),\ G \in C_c^\infty(0, \infty)$$

Improved decay estimates

Strichartz estimates (frequency localized)

$$\int_{-\infty}^{\infty} \left\| G(\mathcal{A}_{\varrho 0}) \exp\left(\pm \mathrm{i}\sqrt{\mathcal{A}_{\varrho 0}}\, t\right) [h] \right\|^p_{L^q(R^3)} \, \mathrm{d}t \le c(G) \|h\|^p_{L^2(R^3)}$$

for any any $G \in C_c^\infty(0,\infty)$

$$\frac{1}{p} + \frac{3}{q} = \frac{1}{2}.$$

Decay for the scaled equation

$$\int_0^T \left\| G(\mathcal{A}_{\varrho 0}) \exp\left(\pm \mathrm{i}\sqrt{\mathcal{A}_{\varrho 0}}\, \frac{t}{\varepsilon}\right) [h] \right\|^p_{L^q(R^3)} \, \mathrm{d}t$$

$$= \varepsilon \int_0^T \frac{1}{\varepsilon} \left\| G(\mathcal{A}_{\varrho 0}) \exp\left(\pm \mathrm{i}\sqrt{\mathcal{A}_{\varrho 0}}\, \frac{t}{\varepsilon}\right) [h] \right\|^p_{L^q(R^3)} \, \mathrm{d}t$$

$$= \varepsilon \int_0^{T/\varepsilon} \left\| G(\mathcal{A}_{\varrho 0}) \exp\left(\pm \mathrm{i}\sqrt{\mathcal{A}_{\varrho 0}}\, t\right) [h] \right\|^p_{L^q(R^3)} \, \mathrm{d}t \le \boxed{\varepsilon}\, c(G) \|h\|^p_{L^2(R^3)}$$

Strichartz estimates by Metcalfe and Tataru

A wave equation

$$\partial_{t,t}^2 V - \mathrm{div}_x \left(\tilde{A}(x)\nabla_x V\right) + \tilde{\mathbf{B}}(x)\cdot\nabla_x V = Z,\ \ V(0,\cdot) = V_0,\ \ \partial_t V(0,\cdot) = V_1$$

Strichartz estimates

$$\|V\|_{L^p(-\infty,\infty;L^q(R^3))} \le \left(\|\nabla_x V_0\|_{L^2(R^3;R^3)} + \|V_1\|_{L^2(R^3)} + \|Z\|_{L^r(-\infty,\infty;L^s(R^3))}\right)$$

$$\frac{1}{p} + \frac{3}{q} = \frac{1}{2} = \frac{1}{r} + \frac{3}{s} - 2$$

Flatness of coefficients

$$\sum_{j\in Z} \sup_{x\in A_j} \left(|x|^2|\nabla_x^2\tilde{A}(x)| + |x||\tilde{A}(x)| + |\tilde{A}(x) - \overline{A}|\right) \le \delta,\ \overline{A} > 0$$

$$\sum_{j\in Z} \sup_{x\in A_j} \left(|x|^2|\nabla_x\tilde{\mathbf{B}}(x)| + |x||\tilde{\mathbf{B}}(x)|\right) \le \delta$$

$$A_j = \{2^j \le |x| \le 2^{j+1}\}$$

Decomposition method

A new density profile

$\tilde{\varrho}$, $\tilde{\varrho}$ flat on a large ball $\{|x| < R\}$, $\tilde{\varrho} = \varrho_0$ for $\{|x| > 2R\}$

Modified operator

$$\mathcal{A}_{\tilde{\varrho}}[v] = \mathrm{div}_x \left(p'(\tilde{\varrho}) \nabla_x v \right) + \tilde{\varrho} Q''(\tilde{\varrho}) \nabla_x \tilde{\varrho} \cdot \nabla_x v,$$

$$\tilde{A}(x) = p'(\tilde{\varrho})(x), \ \tilde{\mathbf{B}}(x) = \tilde{\varrho} Q''(\tilde{\varrho}) \nabla_x \tilde{\varrho}.$$

Strichartz estimates

The operator $\mathcal{A}_{\varrho_0}$ satisfies the flatness condition of Metcalfe and Tataru

A method by Smith and Sogge

Desired estimates

$$\int_{-\infty}^{\infty} \left\| G(\mathcal{A}_{\varrho_0}) \exp\left(\pm \mathrm{i}\sqrt{\mathcal{A}_{\varrho_0}}\, t\right) [h] \right\|_{L^q(R^3)}^p \,\mathrm{d}t \leq c(G) \|h\|_{L^2(R^3)}^p$$

Ansatz

$$U = G(\mathcal{A}_{\varrho_0}) \exp\left(\pm \mathrm{i}\sqrt{\mathcal{A}_{\varrho_0}}\, t\right) [h]$$

$$\chi \in C_c^\infty(R), \ \chi(z) = \begin{cases} 1 \text{ for } |z| < R \\ 0 \text{ for } |z| > 2R, \end{cases}$$

$$U = \chi(|x|)U + (1 - \chi(|x|))U,$$

Local + Strichartz

$\chi(|x|)U$ satisies the localized frequency cut–off estimates as U does

Decay estimates for the far field component

Far field vector field

$$V = (1 - \chi(|x|))U$$

Wave equation

$$\partial_{t,t}^2 V - \mathrm{div}_x \left(\tilde{A}(x)\nabla_x V\right) + \tilde{\mathbf{B}}(x) \cdot \nabla_x V = Z,$$

$$Z = \tilde{A}\nabla_x\chi \cdot \nabla_x U + \mathrm{div}_x(\tilde{A}\nabla_x\chi U) - \tilde{\mathbf{B}} \cdot \nabla_x\chi U$$

$$V(0,\cdot) = (1-\chi)U_0,\ \partial_t V(0,\cdot) = (1-\chi)U_1.$$

Strichartz estimates

The result of Metcalfe and Tataru applies to V

Convergence proof

Relative energy

$$\mathcal{E}\left(\varrho, \Theta, \mathbf{u} \;\middle|\; r, \mathbf{U}\right)$$

$$= \int_{R^3} \left[\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + \frac{H(\varrho\Theta) - H'(r)(\varrho\Theta - r) - H(r)}{\varepsilon^2} \right] \mathrm{d}x$$

$$H(Z) = \frac{1}{\gamma - 1} Z^\gamma$$

Test functions

$$(r - \overline{\varrho}) \in C_c^\infty([0, T] \times R^3),\ r > 0,\ \mathbf{U} \in C_c^\infty([0, T] \times R^3; R^3)$$

Relative energy inequality

Inequality

$$\left[\mathcal{E}\left(\varrho,\Theta,\mathbf{u}\middle|r,\mathbf{U}\right)\right]_{t=0}^{t=\tau}+\varepsilon^{\alpha}\int_0^{\tau}\int_{R^3}\mathbb{S}(\nabla_x(\mathbf{u}-\mathbf{U})):\nabla_x(\mathbf{u}-\mathbf{U})\ \mathrm{d}x\mathrm{d}t$$

$$\leq\int_0^{\tau}\int_{R^3}\varrho\left(\partial_t\mathbf{U}+\mathbf{u}\cdot\nabla_x\mathbf{U}\right)\cdot(\mathbf{U}-\mathbf{u})+\varepsilon^{\alpha}\mathbb{S}(\nabla_x\mathbf{U}):\nabla_x(\mathbf{U}-\mathbf{u})\ \mathrm{d}x\mathrm{d}t$$

$$+\frac{1}{\varepsilon^2}\int_0^{\tau}\int_{R^3}\left[(r-\varrho\Theta)\partial_t H'(r)+\nabla_x H'(r)\cdot(r\mathbf{U}-\varrho\Theta\mathbf{u})\right]\ \mathrm{d}x\mathrm{d}t$$

$$-\frac{1}{\varepsilon^2}\int_0^{\tau}\int_{R^3}\left[\mathrm{div}_x\mathbf{U}\Big((\varrho\Theta)^{\gamma}-r^{\gamma}\Big)+\varrho\nabla_x F\cdot(\mathbf{U}-\mathbf{u})\right]\ \mathrm{d}x\mathrm{d}t$$

Ansatz

$$r=r_{\varepsilon}=\varrho_0+\underbrace{\varepsilon s_{\varepsilon}}_{acoustic\ \ component},\ \mathbf{U}=\mathbf{U}_{\varepsilon}=\mathbf{V}+\underbrace{\nabla_x\Phi_{\varepsilon}}_{acoustic\ \ component}$$