Multiplier Methods for Spectral Theory

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Mathematical Aspects of the Physics with non Self-Adjoint Operators

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Index

Schrödinger Equation

$$\partial_t \psi = i\Delta\psi, \qquad \psi(0,x) = f(x) \in L^2(\mathbb{R}^d;\mathbb{C})$$

$$\psi(t,x) =: e^{it\Delta} f(x) = e^{i\frac{|\cdot|^2}{4t}} * f(x)$$

$$= \frac{1}{(4\pi i t)^{\frac{d}{2}}} e^{i\frac{|x|^2}{4t}} \int e^{-i\frac{x\cdot y}{2t}} e^{-i\frac{|y|^2}{4t}} f(y)\, dy.$$

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Dispersion

$$f_N(x) = e^{iNx} e^{-\frac{x^2}{2}} \qquad (d = 1)$$

$$e^{it\Delta} f_0(t,x) = \frac{1}{\sqrt{it+1}} e^{-\frac{x^2}{2}(it+1)}$$

$$e^{it\Delta} f_N(t,x) = e^{-itN^2 + iNx} f_0(t, x - Nt) \, .$$

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Observables

$$i\partial_t \psi = H\psi \qquad\qquad H \text{ - symmetric}$$

$$T(t) = \langle \psi, K\psi \rangle$$

$$\Rightarrow \dot{T}(t) = -i\langle \psi, [H, K]\psi \rangle$$

$$\Rightarrow \ddot{T}(t) = -\langle \psi, [H, [H, K]]\psi \rangle$$

In particular

$$[K, H] = 0 \qquad \Rightarrow \qquad \dot{T}(t) \equiv 0.$$

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Multipliers

Choose $K\psi = \phi\psi$, $H = -\Delta$ and compute $[H,[H,K]]$:

$$\begin{aligned}\frac{d^2}{dt^2}\int |\psi|^2|\phi|^2\,dx &= \frac{d}{dt} - 2\Im \int \psi\nabla\overline{\psi}\cdot\nabla\phi \\ &= 4\int \nabla\psi D^2\phi\nabla\overline{\psi}\,dx - \int(\Delta^2\phi)|\psi|^2\,dx.\end{aligned}$$

1. For $\phi = |x|^2$ (position)

$$\frac{d^2}{dt^2}\int |x|^2|\psi|^2\,dx = 8\int |\nabla\psi|^2\,dx = 16E$$

Position is a convex function of time (weak dispersion).

1. For $\phi = |x|^2$ (position)

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2. For $\phi = |x|$

$$\frac{d}{dt}\Im \int \overline{\psi}\partial^r \psi \, dx = 4 \int \frac{|\partial^\tau \psi|^2}{|x|} \, dx - \int |\psi|^2 \Delta^2 |x| \, dx.$$

which gives the Morawetz estimate

$$\int \int \frac{|\partial^\tau \psi|^2}{|x|} \, dx \, dt \leq C \|\psi(0)\|^2_{\dot{H}^{\frac{1}{2}}}, \qquad (n \geq 3)$$

3. For

$$\phi(|x|) \simeq \begin{cases} |x|^2, & |x| \leq 1 \\ |x|, & |x| > 1 \end{cases} , \qquad \phi_R(\cdot) = R\phi\left(\frac{\cdot}{R}\right)$$

one gets local smoothing (Kato, Constantin-Saut, Sjolin, Vega)

$$\sup_{R>0} \frac{1}{R} \int \int_{|x|<R} |\nabla \psi|^2 \, dx \, dt \leq C \|\psi(0)\|^2_{\dot{H}^{\frac{1}{2}}}.$$

For electromagnetic Hamiltonians

$$H = (i\nabla + A)^2 + V(x), \qquad A : \mathbb{R}^d \to \mathbb{R}^d, \qquad V : \mathbb{R}^d \to \mathbb{R}$$

Assume

$$q[\psi] := \int |(i\nabla + A)\psi|^2 + V|\psi|^2 \geq 0.$$

Define

$$B = DA - DA^t \in \mathcal{M}_{d\times d}(\mathbb{R})$$

$$i\partial_t\psi = H\psi \qquad K\psi = \phi\psi, \qquad \phi = \phi(|x|)$$

The analogous identity reads (F-Vega 2009)

$$\frac{d^2}{dt^2}\int |\psi|^2|\phi|^2\,dx = \frac{d}{dt} - 2\Im\int \psi\overline{\nabla_A\psi}\cdot\nabla\phi$$

$$= 4\int \nabla_A\psi D^2\phi\nabla_A\overline{\psi}\,dx - \int(\Delta^2\phi)|\psi|^2\,dx$$

$$+ 4\Im\int \phi'\psi B_\tau\cdot\overline{\nabla_A\psi}\,dx - 2\int \phi'(\partial^r V)|\psi|^2\,dx,$$

with

$$\nabla_A = \nabla - iA, \qquad B_\tau = \frac{x}{|x|}B \in \mathbb{S}^{d-1}.$$

We want to understand the previous identity as multiplication identities for the equation

$$i\partial_t \psi = H\psi.$$

Notice that

$$H, K \text{ - symmetric} \qquad \Rightarrow \qquad \begin{array}{l} [H, K] \text{ - skewsymmetric} \\ [H, [H, K]] \text{ - symmetric.} \end{array}$$

As a consequence ($i\partial_t \psi = H\psi$)

$$\Re\langle i\partial_t \psi, [H, K]\psi\rangle = -\Im\langle \partial_t \psi, [H, K]\psi\rangle = -\frac{1}{2}\frac{d}{dt}\langle \psi, [H, K]\psi\rangle$$

$$\begin{aligned} \Re\langle H\psi, [H, K]\psi\rangle &= \frac{1}{2}\left(\langle H\psi, [H, K]\psi\rangle + \langle [H, K]\psi, H\psi\rangle\right) \\ &= \frac{1}{2}\langle \psi, [H, [H, K]]\psi\rangle \end{aligned}$$

Resolvent estimates

Invertibility with continuous inverse $R(z) = (H - z)^{-1}$ related to the unique solvability of

$$H\psi - z\psi = f.$$

Let $H \geq 0$ symmetric and study

$$H\psi = k^2\psi, \qquad k \in \mathbb{R}.$$

Related to

$$Hu = i\partial_t u$$

by Fourier transform $\psi(k,x) := \mathcal{F}_{t\to k} u(\cdot, x)$. Multiply (in L^2) by $[H,\phi]\psi$ and take real parts

$$\frac{1}{2}\langle [H,[H,\phi]]\psi\rangle = k^2 \Re\langle \psi, [H,\phi]\psi\rangle = 0$$

since $[H,K]$ is skewsymmetric.

Example: $H = -\Delta,\ \phi = |x|^2$ so

$$\langle [H, [H, \phi]]\psi \rangle = 8 \int |\nabla \psi|^2$$

which proves that $\sigma_p(-\Delta) = \emptyset$.

Consider

$$-\Delta\psi - (k^2 \pm i\epsilon)\psi = f$$

Multiply by $[-\Delta, \phi]\psi = 2\nabla\phi \cdot \nabla\psi + \psi\Delta\phi$ and take real parts

$$2\int \nabla\psi D^2\phi\nabla\overline{\psi}\,dx - \frac{1}{2}\int(\Delta^2\phi)|\psi|^2\,dx \pm \epsilon\Im\int \psi\nabla\overline{\psi}\cdot\nabla\phi$$
$$= \Re\int \overline{f}(2\nabla\phi\cdot\nabla\psi + \psi\Delta\phi) \qquad (A)$$

$\Rightarrow$ need an L^2-control

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$\Rightarrow$ need an L^2-control

This comes by the multiplication of

$$-\Delta\psi - (k^2 \pm i\epsilon)\psi = f$$

by the symmetric operator $\eta\overline{\psi}$: taking the real parts one obtains

$$-\int \eta|\nabla\psi|^2 + \frac{1}{2}\int(\Delta\eta)|\psi|^2 + k^2\int \eta|\psi|^2 = -\Re\int f\eta\overline{\psi} \qquad (B)$$

Summing (A) and (B) we get

$$k^2 \int \eta|\psi|^2 + 2\int \nabla\psi D^2\phi\nabla\overline{\psi} - \int \eta|\nabla\psi|^2 + \frac{1}{2}\int \left(\Delta\eta - \Delta^2\phi\right)|\psi|^2$$
$$= \Re\int \overline{f}(2\nabla\phi\cdot\nabla\psi + \psi\Delta\phi - \eta\psi) \mp \epsilon\Im\int \psi\nabla\overline{\psi}\cdot\nabla\phi$$

which reduces with the choice $\eta = \Delta\phi$ to

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Agmon-Hörmander-L.A.P.

Choose $\Delta\phi \sim (1+|x|^2)^{-\frac{s}{2}}$, with $s = 1+\epsilon$ to get

$$k^2 \int \frac{|\psi|^2}{(1+|x|^2)^{\frac{1}{2}+}} + \int \frac{|\nabla\psi|^2}{(1+|x|^2)^{\frac{1}{2}+}} \leq C \int |f|^2 (1+|x|^2)^{\frac{1}{2}+}$$

$$k^2 \int (\Delta\phi)|\psi|^2 + 2\int \nabla\psi D^2\phi\nabla\overline{\psi} - \int (\Delta\phi)|\nabla\psi|^2$$
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We would like to get a uniform (in k) estimate for $\nabla\psi$. Choice $\phi = |x|^2$ fails at a first attempt:

$$2dk^2 \int |\psi|^2 + 2(1-d) \int |\nabla\psi|^2 = \Re \int \overline{f} 2\nabla\phi \cdot \nabla\psi \mp \epsilon \Im \int \psi\nabla\overline{\psi} \cdot \nabla\phi.$$

Consider

$$-\Delta\psi - k^2\psi = f \qquad u := e^{\pm ik|x|}\psi.$$

The function u solves

$$-\left(\nabla \mp ik\frac{x}{|x|}\right)^2 u - k^2 u = e^{\pm ik|x|} f$$

and notice that

$$|\nabla u|^2 = \left|\nabla\psi \pm ik\frac{x}{|x|}\psi\right|^2$$

Using this one gets (uniformly in k)

$$\int \frac{|\psi|^2}{|x|^2} \le C_H \int \left|\nabla(e^{\pm ik|x|}\psi)\right|^2 \le C \int |f|^2|x|^2.$$

Ikebe-Saito; Burq-Planchon-Stalker-Tahvildar Zadeh

Complex Potentials

Consider

$$-\Delta\psi - z\psi = V\psi \qquad V : \mathbb{R}^d \to \mathbb{C}, \qquad z = k_1 + ik_2.$$

Assume

$$\int |V||\psi|^2 \leq a \int |\nabla\psi|^2 \qquad (a < 1) \qquad (\star)$$

to be able to define $H = -\Delta + V$ as a m-sectorial operator in L^2.

CLAIM: $(\star) \;\Rightarrow\; \sigma(H) = \sigma(-\Delta)$.

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CLAIM: $(\star) \ \Rightarrow \ \sigma(H) = \sigma(-\Delta)$.

The claim is true in 3D (F-Krejcirik-Vega 2016). Suitable application of the Birman-Schwinger Principle and strongly using the fact that the Green function

$$G_z(x, y) = \frac{1}{4\pi} \frac{e^{-\sqrt{z}|x-y|}}{|x-y|}$$

satisfies $|G_z(x, y)| \leq G_0(x, y)$.

Not clear the claim in other dimensions. Using multipliers, we can prove the following.

Theorem (F-Krejcirik-Vega 2016)

Let $H = -\Delta + V$ *and*

$$\int |x|^2 |V|^2 |\psi|^2 \leq b^2 \int |\nabla \psi|^2, \qquad b < \frac{d-2}{5d-8}.$$

Then $\sigma_p(H) = \emptyset$*.*

We have an analogous result for

$$H = -\Delta_A + V$$

with $A \in L^2_{\text{loc}}$:

Theorem (F-Krejcirik-Vega 2016)

Assume

$$\int_{\mathbb{R}^d} (\Re V)_- \, |\psi|^2 \leq b_1^2 \int_{\mathbb{R}^d} |\nabla_A \psi|^2 \,,$$

$$\int_{\mathbb{R}^d} [\partial_r (r \, \Re V)]_+ \, |\psi|^2 \leq b_2^2 \int_{\mathbb{R}^d} |\nabla_A \psi|^2 \,,$$

$$\int_{\mathbb{R}^d} r^2 \left(|\Im V|^2 + \frac{1}{2} |B_\tau|^2 \right) \, |\psi|^2 \leq b_3^2 \int_{\mathbb{R}^d} |\nabla_A \psi|^2 \,.$$

with

$$b_1^2 < 1 - \frac{2 b_3}{d-2} \,, \qquad b_2^2 + b_3 \left[2 + \left(\frac{2}{d-2} \right)^{\frac{3}{2}} \right] < 1.$$

Then $\sigma_{\mathrm{p}}(H_{A,V}) = \varnothing$.