# The Solvability of Differential Equations

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# Preliminaries

Consider $x \in \mathbf{R}^n$, the results are local and generalize to manifolds. Take the complex derivative $D = \frac{1}{i}\partial$ then

$$P(x,D)u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} P(x,\xi)\hat{u}(\xi)\, d\xi \qquad u \in C_0^\infty(\mathbf{R}^n)$$

where $P(x,\xi)$ is called the symbol of the operator.

This is a PDO if $\xi \mapsto P(x,\xi)$ is polynomial, and a classical ΨDO if $P(x,\xi)$ is a sum of terms

$$p_m(x,\xi) + p_{m-1}(x,\xi) + \dots$$

homogeneous of degree $m$, $m-1$, ... in $\xi$. Here $m$ is the *order*, $p_m = p$ is the *principal symbol* and $p_{m-1} = p_s$ the *subprincipal symbol* of $P$.

By using more general symbols $P$ one can localize in phase space $(x,\xi) \in T^*\mathbf{R}^n$, so called microlocal analysis.

# Local solvability

## Definition

$P$ is locally solvable near $x_0$ if

$$Pu = f$$

has a local weak (distribution) solution $u$ near $x_0$ for all $f \in C^\infty$ in a set of finite codimension.

A weak solution $u$ to the equation satisfies

$$\int f(x)\overline{\varphi(x)}\,dx = \int u(x)\overline{P^*\varphi(x)}\,dx \qquad \forall\,\varphi \in C_0^\infty$$

near $x_0$, where $P^*$ is the *adjoint*. When $P$ is not locally solvable the range has infinite codimension.

Observe that in the analytic category all non-degenerate PDO's are locally solvable by the Cauchy-Kovalevsky theorem.

# Estimates

Local solvability is equivalent to *a priori* estimates for the adjoint $P^*$:

$$\|u\|_{(0)} \le C(\|P^*u\|_{(N)} + \|u\|_{(-n)} + \|Au\|_{(0)}) \qquad \forall\, u \in C_0^\infty(\mathbf{R}^n)$$

where $x_0$ is **not** in the support of the function $A$ and

$$\|u\|_{(k)}^2 = \int_{\mathbf{R}^n} (1 + |\xi|^2)^k |\hat{u}(\xi)|^2 \, d\xi$$

is the square of the usual $L^2$ Sobolev norm.

To prove local non-solvability one constructs local approximate solutions to $P^*u = 0$, which are called *pseudomodes*.

# History

**Constant coefficients** PDO are locally solvable. (Ehrenpreis and Malgrange 1955)

**Variable coefficients:** Principal symbol $p(x,\xi)$ is invariant as a function of $(x,\xi) \in T^*\mathbf{R}^n$.

**Elliptic case:** $p(x,\xi) \neq 0$ for $\xi \neq 0$ are solvable. (Lax-Milgram 1954)

**The Hamilton field** of $p$ is $H_p = \sum_j \partial_{\xi_j} p \partial_{x_j} - \partial_{x_j} p \partial_{\xi_j}$.

**Definition**

The operator $P$ is of **principal type** if $H_p$ does not have the radial direction $\langle \xi, \partial_\xi \rangle$ when $p = 0$, thus $dp \neq 0$ when $p = 0$.

Then $P$ has simple characteristics, which is a generic condition for non-elliptic operators.

# The Lewy counterexample

**Example** The Cauchy-Riemann operator $P(D) = D_{x_1} + iD_{x_2}$ is locally solvable.

**Hans Lewy's counterexample (1957)** The operator

$$P = D_{x_1} + iD_{x_2} + i(x_1 + ix_2)D_{x_3} = D_{x_1} - x_2 D_{x_3} + i(D_{x_2} + x_1 D_{x_3})$$

is **not** locally solvable anywhere in $\mathbf{R}^3$.

In suitable coordinates this is the tangential Cauchy-Riemann operator on the boundary of the strictly pseudoconvex set

$$\{ (z_1, z_2) : \ |z_1|^2 + 2\,\mathrm{Im}\, z_2 < 0 \} \subset \mathbf{C}^2$$

By the Cauchy-Kovalevsky theorem $P$ is solvable in the analytic category, so the equation is solvable up to an arbitrarily small error near any given point.

# The bracket condition

**Theorem**

*Local solvability implies that* $\{\operatorname{Re} p, \operatorname{Im} p\} = H_{\operatorname{Re} p} \operatorname{Im} p = 0$ *on* $p^{-1}(0)$*, where* $p$ *is the principal symbol of* $P$*. (Hörmander 1960)*

Thus almost all non-elliptic PDO are not locally solvable!

**Note** that $[P, P^*] \sim \frac{1}{i}\{p, \overline{p}\} + \dots$ and for the Lewy counterexample:

$$\{\operatorname{Re} p, \operatorname{Im} p\} = 2\xi_3 \not\equiv 0.$$

where $p(x, \xi) = \xi_1 - x_2\xi_3 + i(\xi_2 + x_1\xi_3)$ could vanish when $\xi_3 \neq 0$.

For ΨDO we get $\{\operatorname{Re} p, \operatorname{Im} p\} \le 0$ on $p^{-1}(0)$, but for PDO's the bracket is *odd* in $\xi$ so it has to vanish on $p^{-1}(0)$ (switch $\xi \leftrightarrow -\xi$).

Non-zero bracket means that the principal symbol satisfies a topological winding number condition, for example $\tau \pm it$.

# Pseudospectrum

Let

$$P(h) = -h^2\Delta + V = p(x, hD)(x)$$

be the Schrödinger operator with potential $V \in C^\infty(\mathbf{R}^n)$. For

$$z \in \{|\xi|^2 + V(x) : \{\operatorname{Re} p, \operatorname{Im} p\} = 2\langle \xi, \nabla \operatorname{Im} V(x)\rangle \neq 0\}$$

we have

$$\|(P(h) - z)^{-1}\| \geq C_N h^{-N} \qquad \forall N$$

By Sard's theorem, this holds for almost all values when $\operatorname{Im} V \not\equiv 0$. (D., Sjöstrand and Zworski 2004). This has been generalized to systems (D. 2008).

The "almost eigenvalues" are called *pseudospectrum* and the "almost eigenfunctions" are called *pseudomodes*.

# Non-linear equations

The bracket condition also gives instability for the $C^\infty$ Cauchy problem for **quasilinear analytic vector fields**.

The Cauchy-Kovalevsky theorem gives local solvability for any analytic data on a non-characteristic analytic initial surface.

But for **almost all** analytic data there exists smooth data with the same Taylor expansion at a given point for which the Cauchy problem has **no** $C^2$ solution. (Lerner, Morimoto and Xu 2010)

For example, the non-homogeneous Burger's equation

$$\partial_t u + u\partial_{x_1} u = f(t,x,u) \qquad (t,x) \in \mathbf{R} \times \mathbf{R}^n$$

with analytic $f$ has **no** $C^2$ solution for **almost all** non-analytic Cauchy data $u(0,x)$. For example: when the bracket of the linearization $[\partial_t + \operatorname{Re} u\,\partial_x, \operatorname{Im} u\,\partial_x] = \big(\operatorname{Im}\partial_t u + \operatorname{Re} u\,\partial_x \operatorname{Im} u - \operatorname{Im} u\,\partial_x \operatorname{Re} u\big)\,\partial_x \neq 0$, thus $\operatorname{Im} f(t,x,u) - 2\operatorname{Im} u\,\partial_{x_1} \operatorname{Re} u \neq 0$.

# The Nirenberg-Treves conjecture

Nirenberg and Treves replaced conditions on the bracket with conditions on the sign changes of the symbol.

**Condition** $(\Psi)$**:** $\operatorname{Im} p$ does not change sign from $-$ to $+$ along the oriented bicharacteristics of $\operatorname{Re} p$ (the positive flow-out of $H_{\operatorname{Re} p}$).

This condition is *invariant* and implies that $H_{\operatorname{Re} p} \operatorname{Im} p \le 0$ at $p^{-1}(0)$.

**Example** For $D_t + if(t, x, D_x)$ with $f$ real and first order, $H_{\operatorname{Re} p} \operatorname{Im} p = \partial_t f(t, x, \xi)$ and condition $(\Psi)$ means that $t \mapsto f(t, x, \xi)$ cannot change sign from $-$ to $+$ for increasing $t$ and fixed $(x, \xi)$.

**The NT conjecture (1969)**

A principal type ΨDO is locally solvable if and only if the principal symbol satisfies condition $(\Psi)$.

# Resolution of the Nirenberg-Treves conjecture

Condition $(\Psi)$ is **necessary** for local solvability for ΨDO. (Hörmander 1980)

Condition $(\Psi)$ is **sufficient** for local solvability for ΨDO in two variables. (Lerner 1988)

**Theorem**

*If $P$ is a principal type ΨDO with principal symbol satisfying condition $(\Psi)$ then $P$ is locally solvable. (D. 2006)*

Thus condition $(\Psi)$ is **equivalent** to local solvability for ΨDO of principal type.

This has been generalized to systems with constant characteristics, but little is known in general for operators that are *not* of principal type.

# Why condition $(\Psi)$?

$P$ is solvable $\Leftrightarrow$ $P^*$ has a finite-dimensional kernel

**Example** Let $(t, x) \in \mathbf{R} \times \mathbf{R}^n$, $|D_x| = \Delta_x^{1/2} \geq 0$ and

$$P = D_t + it|D_x| \qquad \text{with} \qquad \sigma(P) = \tau + it|\xi|$$

Thus, $\sigma(P)$ does **not** satisfy $(\Psi)$ and

$$P^* = D_t - it|D_x| = \frac{1}{i}(\partial_t + t|D_x|)$$

We have that $P^* u = 0$ if

$$u(t, x) = \int e^{i\langle x, \xi\rangle - t^2|\xi|/2} \phi(\xi)\, d\xi \qquad \forall\ \phi \in C_0^\infty$$

so $P^*$ has an infinite-dimensional kernel and $P$ is **not** solvable.

# Non-principal type operators

Where the principal symbol $p_m$ vanishes of at least second order, the *subprincipal symbol* $p_s = p_{m-1}$ becomes an important invariant.

**Example** Let

$$P = D_1 D_2 + B(x, D)$$

where $B$ has order one. The principal symbol $\xi_1 \xi_2$ is real and vanishes of second order at the double characteristics $\Sigma_2 = \{\, \xi_1 = \xi_2 = 0 \,\}$ and the subprincipal symbol is equal to the principal symbol $b$ of $B$.

If $x_j \mapsto \operatorname{Im} b$ changes sign then $P$ is not solvable and if $\pm \operatorname{Im} b > 0$ then $P$ is solvable (Mendoza–Uhlmann 1983-84).

Corresponds to condition $(\Psi)$ on subprincipal symbol — get both directions for second order principal symbol.

**Conjecture:** Is $P$ solvable if $\operatorname{Im} b$ does not change sign in $(x_1, x_2)$?

# Limit characteristics

Let $P$ be second order operator with principal symbol

$$\tau^2 - t^2|\xi|^2 - |\eta|^2$$

$P$ is *effectively hyperbolic* when $\xi \neq 0$ and then locally solvable for *any lower order terms*. Thus condition $(\Psi)$ cannot be necessary in general.

Observe that when $\xi \neq 0$ we find that $\Sigma_2 = \{\, \tau = t = |\eta| = 0 \,\}$ is *non-involutive* since $\{\, \tau, t \,\} = 1$.

**Definition**

We say that $\Gamma$ is a *limit bicharacteristic* (possibly a point) if there exist bicharacteristics $\Gamma_j$ that converge to it as smooth curves.

Then the normalized Hamilton fields $|\nabla p|^{-1} H_p \in C^\infty$ uniformly on $\Gamma_j$.

# Examples

Let $p = \prod_j p_j$ with real $p_j$ of principal type, $p_j = 0$ on $\Sigma_2$ and $p_j \neq p_k$ on $p^{-1}(0) \setminus \Sigma_2$. Then limit bicharacteristics of $p$ are the bicharacteristics of $p_j$ on $\Sigma_2$ for any $j$.

Let $p(x, \xi)$ be real and vanish of order $k \geq 2$ at $\Sigma_2 = \{\, \xi' = 0 \,\}$. If the localization at $\Sigma_2$

$$\eta \mapsto \sum_{|\alpha|=k} \partial_{\xi'}^{\alpha} p(x, 0, \xi'') \eta^{\alpha} / \alpha!$$

is of principal type when $\eta \neq 0$ (thus simple zeroes) then the limit bicharacteristics are the bicharacteristics of the localization at $\Sigma_2$.

*Assume* the bicharacteristic $\Gamma \in C^\infty$ uniformly, thus the normalization $H_{\widetilde{p}} = |\nabla p|^{-1} H_p \in C^\infty$ on $\Gamma$.

**A Lagrangean space** $L$ is *grazing* to $\Gamma$ if $L \subset Tp^{-1}(0)$ and the linearization of $H_{\widetilde{p}}$ is tangent to $L$, thus $T\Gamma \subseteq L$.

**Curvature condition** there exist grazing Lagrangean spaces $L$ such that

$$\left| dH_{\widetilde{p}}(w)\big|_{L(w)} \right| \le C \qquad w \in \Gamma$$

This gives uniform bounds on the curvature of $p^{-1}(0)$ and on the evolution of $L$ at $\Gamma$.

The curvature condition is satisfied by the earlier examples, but not by effectively hyperbolic operators.

Bicharacteristics satisfying these conditions are called *uniform* and then limit bicharacteristics exist.

**Example** Let

$$p = \tau - \left( \langle A(t)x, x\rangle + 2\langle B(t)x, \xi\rangle + \langle C(t)\xi, \xi\rangle \right)/2$$

where $A(t)$, $B(t)$ and $C(t) \in C^\infty$ are real $n \times n$, $A(t)$ and $C(t)$ are symmetric. Then

$$p^{-1}(0) = \{\, \tau = \langle A(t)x, x\rangle/2 + \langle B(t)x, \xi\rangle + \langle C(t)\xi, \xi\rangle/2 \,\}$$

and the linearization of the Hamilton field $H_p$ is

$$\partial_t + \langle A(t)y + B^*(t)\eta, \partial_\eta\rangle - \langle B(t)y + C(t)\eta, \partial_y\rangle$$

Take $L(t) = \{\, (s, y, 0, E(t)y) \,\}$ with symmetric $E(t) \in C^\infty$ then $L(t)$ is a Lagrangean space and is grazing if

$$E'(t) = A(t) + 2\,\mathrm{Re}\, B(t)E(t) + E(t)C(t)E(t)$$

which is uniformly bounded by the curvature condition.

Assume condition $(\Psi)$ on the subprincipal symbol $p_s$ *does not hold* at the limit, in the sense that:

$$\min_{\partial\Gamma_j} \int \operatorname{Im} p_s |H_p|^{-1}\, ds / |\log \kappa_j| \to \infty \qquad j \to \infty \qquad \operatorname{Lim}(\overline{\Psi})$$

where $\min_{\Gamma_j} |H_p| = \kappa_j \to 0$. One example is when $\operatorname{Im} p_s$ changes sign from $-$ to $+$ on the limits of the bicharacteristics, then the integrand is $\mathcal{O}(1/\kappa_j) \to \infty$.

**Theorem**

*If $P$ has real principal symbol and $\{\,\Gamma_j\,\}_{j=1}^{\infty}$ is a family of uniform bicharacteristics so that $Lim\,(\overline{\Psi})$ is satisfied on $\Gamma_j$, then $P$ is not locally solvable near any of the limit bicharacteristics. (D. 2016)*

This has been extended to complex principal symbols such that $H_{\widetilde{p}}$ converges to a real vector field at $\Sigma_2$. (D. 2016)

# Operators of subprincipal type

**Assume** $\Sigma_2$ is involutive with symplectic foliation given by the Hamilton fields, it is *non-radial* if *all* its Hamilton fields are.

**Example** $\Sigma_2 = \{ \eta = 0 \}$ with leaves $L = \{ (x_0, y, \xi_0, 0) : y \in \mathbf{R}^k \}$.

**Definition**

$P$ is of *subprincipal type* if $H_{p_s}\big|_{\Sigma_2} \subseteq T\Sigma_2$ is transversal to the leaves $L$ when $p_s = 0$.

**Examples** $\Delta_x + D_t$ is of subprincipal type, $\Delta_x + D_x$ is not. The linearized Navier-Stokes equations are of subprincipal type.

$H_{p_s}$ is well defined modulo $TL$. To define condition $(\Psi)$ we need:

$$\big|dp_s\big|_{TL}\big| \le C|p_s| \qquad \text{for leaves } L \text{ of } \Sigma_2$$

Then $p_s$ is constant on the leaves after multiplication with a factor $\neq 0$.

**Definition**

$P$ satisfies condition Sub$(\Psi)$ if $p_s$ satisfies condition $(\Psi)$ on $T^\sigma\Sigma_2 = T\Sigma_2/TL$, which is symplectic. Ex: $T^\sigma\Sigma_2 \cong \{\,(x, y_0, \xi, 0)\,\}$.

If Sub$(\Psi)$ does not hold with sign change of *infinite order* we assume

$$\|\operatorname{Hess} p\| + |dp_s \wedge d\overline{p}_s| \le C|p_s|^\varepsilon \qquad \varepsilon > 0$$

Then the pseudomodes do not get too dispersed and stay local.

**Theorem**

*If $P$ has principal symbol vanishing of second order at a non-radial involutive manifold, is of subprincipal type and does not satisfy condition Sub$(\Psi)$, then $P$ is not locally solvable. (D. 2017)*

Extended to $(\Psi)$ on the Taylor expansion of $p + p_s$ at $\Sigma_2$. (D. 2017)

# Counterexample to Uhlmann's conjecture

Let

$$P = D_{x_1} D_{x_2} + D_t + if(t, x, D_x)$$

with real first order $f(t, x, \xi)$ satisfying $\partial_{x_j} f = 0$ for $j = 1, 2$.

Then $P$ is of subprincipal type, so we obtain that $P$ is not solvable if $t \mapsto f(t, x, \xi)$ changes sign of *finite order* from $-$ to $+$ on the double characteristics $\Sigma_2 = \{ \xi_1 = \xi_2 = 0 \}$.

Observe that $f$ has constant sign on the leaves of $\Sigma_2$ and thus on the limit bicharacteristics.

Thus the solvability of $P$ also depends on the real part of the subprincipal symbol.

# Open problems

Case when limit characteristics do not converge in $C^\infty$, for example some weakly hyperbolic operators.

Complex limits of the normalized Hamilton fields.

Condition $(\Psi)$ on the *refined principal symbol* $p + p_s$ in general.

Systems of non-principal type.