Lieb-Thirring type bounds for Dirac and fractional Schrödinger operators with complex potentials

Jean-Claude Cuenin

Ludwig-Maximilians Universität München

June 5, 2017, Luminy

Statement of the problem

$H = T(p) + V(x)$ in $L^2(\mathbb{R}^d)$
V decaying and complex-valued

Kinetic energies

- $T(p) = (m^2 + |p|^2)^{1/2} - m$
- $T(p) = \sum_{j=1}^{d} \alpha_j p_j + m\beta$
- $T(p) = \sum_{j=1}^{d}(1 - \cos(p_j))$
- any 'reasonable' translation-invariant operator
- some non translation-invariant (magnetic) operators

Questions

- Where are the discrete (or embedded) eigenvalues of H located?
- Can the eigenvalues be controlled by an L^p norm of V?
- What is the rate of accumulation to $\sigma_{\text{ess}}(H)$?

Lieb-Thirring inequalities (in the self-adjoint case)

Consider $H = -\Delta + V$ with V real-valued. Then

$$\sum_{\lambda \in \sigma_{\mathrm{d}}(H)} |\lambda|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V_-(x)^{d/2+\gamma} \, \mathrm{d}\, x$$

holds for

$$\begin{aligned} d = 1: &\quad \gamma \geq 1/2 \\ d = 2: &\quad \gamma > 0 \\ d \geq 3: &\quad \gamma \geq 0 \end{aligned}$$

- Lieb-Thirring (1976): $d = 3$, $\gamma = 1$ (stability of matter!)
- Cwikel-Lieb-Rozenblum: $d \geq 3$, $\gamma = 0$ (number of eigenvalues!)
- Semiclassical interpretation (phase-space integral)
- Sharp constants $L_{\gamma,d}$ known for $d \geq 1$, $\gamma \geq 3/2$ and $d = 1$, $\gamma = 1/2$

Non-selfadjoint Schrödinger operators (Single eigenvalues)

Consider $H = -\Delta + V$ with V complex-valued. Then

$$\forall z \in \sigma_{\mathrm{d}}(H): \quad |z|^\gamma \le C_{d,\gamma} \int_{\mathbb{R}^d} |V(x)|^{d/2+\gamma} \, \mathrm{d}\, x$$

where

$$\begin{aligned} d = 1: &\quad \gamma = 1/2 \\ d = 2: &\quad 0 < \gamma \le 1/2 \\ d \ge 3: &\quad 0 \le \gamma \le 1/2 \end{aligned}$$

- Abramov, Aslanyan, Davies (2001): $d = 1$ ($C_{1,1/2} = 1/2$)
- Frank (2011): $d \ge 2$ (used uniform resolvent estimates of Kenig, Ruiz, Sogge 1987)
- For radial potentials, Frank and Simon (2015) proved the case $1/2 \le \gamma \le d/2$
- In the non-radial case this is still open (Laptev-Safronov conjecture)

Non-selfadjoint Schrödinger operators (Eigenvalue sums)

$H = -\Delta + V$

$V \in L^q(\mathbb{R}^d;\mathbb{C})$ with $d/2 < q \leq (d+1)/2$

Theorem (Frank, Sabin 2014)

$$\sum_{z \in \sigma_{\mathrm{d}}(H)} \frac{\operatorname{dist}(z,[0,\infty))}{|z|^{(1-\epsilon)/2}} \leq C_{d,q,\epsilon} \|V\|_{L^q(\mathbb{R}^d)}^{(1+\epsilon)q/(2q-d)}$$

for certain $\epsilon \geq 0$ (depending on d and q).

- In particular, the rate of accumulation to any $\lambda \in (0,\infty)$ is ℓ^1.
- Proof is based on the 'complex analysis method' and uses uniform resolvent estimates in Schatten spaces as input.
- Previous results by Frank, Laptev, Lieb, Seiringer (2006), Demuth, Hansmann, Katriel (2009) and Laptev, Safronov (2009).
- It is not clear what the 'correct' weight should be.

Dirac operator ($d = 1$)

$D_0 = -\mathrm{i}\sigma_1\partial_x + m\sigma_3$, $m \geq 0$
$V \in L^1(\mathbb{R};\mathbb{C}^4)$
$U := \int_{\mathbb{R}} V(x)\,\mathrm{d}\,x$

Theorem (Cuenin, Laptev, Tretter 2013)

If $\|V\|_1 < 1$, then the nonreal eigenvalues lie in the union of two disks.

Theorem (Cuenin, Siegl 2017)

If $m > 0$ and $\pm\mathrm{Re}\,U_{11} < 0$, then for $0 < \varepsilon \ll 1$, there exists an eigenvalue $z(\varepsilon)$ of $D_0 + \varepsilon V$ satisfying

$$z(\varepsilon) = \pm m \mp \frac{m}{2}U_{11}^2\varepsilon^2 + o(\varepsilon^2), \quad \varepsilon \to 0+\,.$$

- The estimate of [CLT13] becomes sharp in the weak coupling regime: $|z \mp m| \leq \frac{m}{2}\varepsilon^2\|V\|_1^2 + o(\varepsilon^2)$.
- $\exists$ inequality for eigenvalues sums, but we also need $V \in L^{1+\epsilon}$

Uniform resolvent estimates for general t.i. kinetic energies

$X := \mathbb{R}^d$ or $\mathbb{Z}^d$, $\Gamma^* \subset\subset X^*$,
$T \in \mathfrak{B}(X^*; \mathbb{R}) \cap C^\infty(\Gamma^*; \mathbb{R})$,
$\kappa(T) := \{\lambda \in \mathbb{R} : \exists \xi \in \Gamma^* \text{s.t. } T(\xi) = \lambda \text{ and } \nabla T(\xi) = 0\}$,
$M_\lambda := \{\xi \in \Gamma^* : T(\xi) = \lambda\}$, $\lambda \in \mathbb{R} \setminus \kappa(T)$.

Theorem

Assume that M_λ has at least k non-vanishing principal curvatures at every point and let $p \in [1, 2(k+2)/(k+4)]$. Then for every $f \in \mathcal{S}(X)$ s.t. $\mathrm{supp}(\widehat{f}) \subset \Gamma^*$, the following inequalities hold.

1. $\|\langle x\rangle^{-\frac{1}{2}-\epsilon}(T(D)-z)^{-1} f\|_{L^2} \leq C \|\langle x\rangle^{\frac{1}{2}+\epsilon} f\|_{L^2}$,
2. $\|\langle x\rangle^{-\frac{1}{2}-\epsilon}(T(D)-z)^{-1} f\|_{L^2} \leq C \|f\|_{L^p}$,
3. $\|(T(D)-z)^{-1} f\|_{L^{p'}} \leq C \|f\|_{L^p}$.

The constant depends on p, d, ϵ and $\mathrm{dist}(z, \kappa(T))$, but not on z.

(There are also Schatten space versions of these inequalities.)

Dirac and fractional Schrödinger operator ($d \geq 2$)

$D_0 = \sum_{j=1}^{d} \alpha_j p_j + m\beta$ and $\mathcal{J}_s = (p^2 + m^2)^{s/2}$, $0 < s < d$, $m \geq 0$

$$V \in \begin{cases} L^{(d+1)/2} \cap L^{d/s} & \text{if } s < 2d/(d+1) \\ L^{d/s \leq q \leq (d+1)/2} & \text{if } s \geq 2d/(d+1) \end{cases}$$

Theorem (Cuenin 2017)

- *For $s \geq 2d/(d+1)$, all eigenvalues of $\mathcal{J}_s + V$ are contained in a compact set. E.g. $|z|^{q-\frac{d}{s}} \leq C\|V\|_{L^q}^q$ for $m = 0$.*
- *For $s < 2d/(d+1)$ or for Dirac, $(|\mathrm{Im}\, z|/|\mathrm{Re}\, z|)^{d/s-1} \leq C\|V\|_{L^{d/s}}^{d/s}$*
- *In either case, any eigenvalue sequence accumulating to a noncritical value is in ℓ^1.*

- Open problem: For $s < 2d/(d+1)$ or for Dirac, do eigenvalues lie in a compact set (even for $V \in L^\infty \cap L^{(d+1)/2}$ and smooth)?

Magnetic Schrödinger operators

$H := (-\mathrm{i}\nabla + A)^2 + V$ in $L^2(\mathbb{R}^d)$, $d \geq 3$ (for simplicity)
$A \in L^\infty(\mathbb{R}^d, \mathbb{C}^d)$ s.t. $\langle x \rangle^{1+\delta}(|A(x)| + |\nabla A(x)|) \leq C_A < \infty$
$V \in L^q(\mathbb{R}^d; \mathbb{C})$ with $d/2 \leq q \leq (d+1)/2$

Theorem

If $C_A \ll 1$, then all eigenvalues of H (including embedded ones) lie in a compact set. More precisely,

$$\left(1 - \min(1, |z|)^{-1/2} \frac{C_A}{\epsilon_0}\right) |z|^{q-d/2} \leq C \|V\|_{L^q}^q$$

where ϵ_0, C only depend on d, q, δ.

This extends the result of Frank (2011).

Schrödinger operators with unbounded background fields

$H := (-\mathrm{i}\nabla + A_0 + A_1)^2 + V_0 + W + V_1$ in $L^2(\mathbb{R}^d)$, $d \geq 3$
A_0, V_0 smooth, real-valued, $V_0 \geq 0$,
$|\partial^\alpha A_0(x)|\langle x\rangle^{|\alpha|-1} + |\partial^\alpha V_0(x)|\langle x\rangle^{|\alpha|-2} + |\partial^\alpha B_0(x)|\langle x\rangle^{\epsilon_\alpha - 1} \leq C_\alpha$
$V_1 \in L^q(\mathbb{R}^d; \mathbb{C})$ with $q \geq d/2$
$W \in L^\infty(\mathbb{R}^d; \mathbb{R})$
$A_1 \in L^\infty(\mathbb{R}^d, \mathbb{C}^d)$ s.t. $\langle x\rangle^{1+\delta}(|A_1(x)| + |\nabla A_1(x)|) \leq C_{A_1} < \infty$

Theorem (Cuenin, Kenig 2017)

If $C_{A_1} \leq \epsilon_0(d, q, \delta) \ll 1$, *then*

$$\sigma(H) \subset \left\{ |\mathrm{Im}\, z|^{1-\frac{d}{2q}} \lesssim_{d,q,\delta} 1 + \frac{1 + \|W\|_{L^\infty}}{1 - C_{A_1}/\epsilon_0} \|V_1\|_{L^q} \right\}$$

Landau Hamiltonian

$H := \left(-\mathrm{i}\partial_x + \frac{y}{2}\right)^2 + \left(-\mathrm{i}\partial_y - \frac{x}{2}\right)^2 + V(x,y)$ in $L^2(\mathbb{R}^2)$
$V \in L^q$ with $q \in (1,\infty]$
$\sigma_{\mathrm{ess}}(H) = \{\lambda_k := 2k+1 : k = 0,1,\ldots\}$

Theorem (Cuenin 2017)

$$\sigma(H) \cap \{z : |z-\lambda_k| \leq 1/2\} \subset \left\{z : |z-\lambda_k| \lesssim_{d,q} \|V\|_{L^q} \lambda_k^{\nu(q)}\right\},$$

$$\nu(q) := \begin{cases} \frac{1}{q} - 1, & 1 \leq q \leq \frac{3}{2}, \\ -\frac{1}{2q} & \frac{3}{2} \leq q \leq \infty. \end{cases}$$

- This uses sharp results of Koch and Ricci (2007) on L^p bounds of eigenfunctions.
- Open problem: Sharp L^q bounds for the harmonic oscillator.

Discrete Schrödinger operators on $\mathbb{Z}^d$

Standard Laplacian: $\Delta_{\mathrm{st}} f(x) := \frac{1}{2d} \sum_{|x-y|=1} f(y)$

Molchanov-Vainberg Laplacian: $\Delta_{\mathrm{MV}} f(x) := \frac{1}{2^d} \sum_{|x-y|=\sqrt{d}} f(y)$

Theorem

Any eigenvalue z *of* $-\Delta_{\mathrm{st}} + V$, *with* $V \in \ell^q(\mathbb{Z}^d)$ *for some* $q \in [1, 3/2]$, *satisfies*

$$1 \leq C_{d,q}(|V\|_{\ell^{3/2}} + \mathrm{dist}(z, \kappa(\Delta_{\mathrm{st}}))^{\frac{d}{2}-q}\|V\|_{\ell^q}).$$

Any eigenvalue z *of* $-\Delta_{\mathrm{MV}} + V$, *with* $V \in \ell^q(\mathbb{Z}^d)$ *for some* $q \in [1, (d+1)/2]$, *satisfies*

$$1 \leq C_{d,q}(1 + \mathrm{dist}(z, \kappa(\Delta_{\mathrm{MV}}))^{\frac{d}{2}-q})\|V\|_{\ell^q}$$

Examples of embedded eigenvalues

Theorem

Let $T \in C^\infty_{\rm pol}(\mathbb{R}^d;\mathbb{R})$ and let $\lambda \in \mathbb{R}$ be such that there exist $\eta \in \mathbb{R}^d$ with $T(\eta) = \lambda$ and $P_\nu \nabla T(\eta) \neq 0$ for some $\nu \in S^{d-1}$. Then there exists a sequence of smooth potentials $V_n : \mathbb{R}^d \to \mathbb{C}$, $n \in \mathbb{N}$, such that λ is an eigenvalue of $T(D) + V_n(x)$ in $L^2(\mathbb{R}^d)$ and such that

$$|V_n(x)| \leq C(n + |P_\nu x| + |P_\nu^\perp x|^2)^{-1}, \quad x \in \mathbb{R}^d.$$

In particular, for any $q > (d+1)/2$, we have that $\|V_n\|_{L^q} \to 0$ as $n \to \infty$.

- This is a generalization of an example due to Ionescu-Jerison (2003); see also Frank-Simon (2015)
- The resulting potentials will generally be complex-valued
- The eigenfunctions can be seen as superpositions of 'Knapp examples'

Thank you for your attention!