A Riemann-Hilbert approach to black hole solutions

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Introduction

In this talk we will study

gravity in terms of Riemann-Hilbert (RH) problem.

In gravity:

- spacetime is Lorentzian (pseudo-Riemannian) manifold (M, g)
- metric g: solution to Einstein's field equations $Ric = T$

  non-linear second-order PDE's for g, sourced by matter T
- Here: $T = 0$, vacuum solutions of Einstein's field equations
- Examples of exact solutions: Schwarzschild solution (outside region of a neutral static black hole)

  In general, finding exact solutions: hard.
- Different approach to constructing exact solutions: Riemann-Hilbert (RH) problem.

Introduction

Ingredients:

- Einstein's field equations in vacuum,   in dimensions $D = 4, 5$
- Restrict to subspace of solutions with $D - 2$ commuting isometries;

  in adapted coordinates, g only depends on two coordinates (v, ρ)

  $\Rightarrow$ non-linear PDE's in two space-like dimensions
- Metric g: symmetric 2-tensor in D dimensions.

  Assemble into matrix $M(v, \rho)$:

  values in non-compact Riemannian symmetric space G/H

  (G non-compact Lie group)
- $D = 5$: $G/H = SL(3, \mathbb{R})/SO(2, 1)$

- Matrix one-form $A = M^{-1} dM$
- Reduced field equations: $d\,(\rho \star A) = 0$

  Hodge dual $\star$, $(\star)^2 = -\mathrm{id}$

This sector of gravity is integrable: Lax pair, called

Breitenlohner-Maison linear system

`Annales de IHP, 1987`

Study by means of Riemann-Hilbert factorization problem.

Integrability: the linear system

Reformulate $d\,(\rho \star A) = 0$ in terms of an auxiliary linear system:

$$\tau\,(d + A)X = \star dX\ , \quad A = M^{-1}dM$$

- τ spectral parameter, $\tau \in \mathbb{C}$
- G-valued matrix $X(\tau; v, \rho)$. Demand:
  - X and X^{-1} to be analytic in interior of unit disc in plane τ, slightly beyond
  - $X(\tau = 0; v, \rho) = \mathbb{I}$
- $d\left(dX\,X^{-1}\right) = \left(dX\,X^{-1}\right) \wedge \left(dX\,X^{-1}\right)$
- Demanding $d\,(\rho \star A) = 0$: $\tau = \tau(w, v, \rho)\ , \quad w \in \mathbb{C}$

  Algebraic curve: $(w, \tau) \in \mathbb{C}^2$

$$w = v + \frac{\rho}{2\tau}(1 - \tau^2)$$

Monodromy matrix

But: since Lax pair has $A = M^{-1} dM$ as input, seem to have a fish-bites-each-tail problem ...

Not quite, as follows:

- $M \in G/H: \quad M = V^\natural \, V = M^\natural$, $\natural$ involutive anti-homomorphism
- $d\,(\rho \star A) = 0$, and X, $\Rightarrow \mathcal{M}(w) = \mathcal{M}^\natural(w)$, monodromy matrix
- Conversely, pick $\quad \mathcal{M}(w) = \mathcal{M}^\natural(w)$
- Recall $w = v + \frac{\rho}{2\tau}(1 - \tau^2)$; hence $\mathcal{M}(w(\tau; v, \rho))$

Riemann-Hilbert factorization problem

Perform canonical factorization in τ-plane (assuming its existence):

$$\mathcal{M}(w(\tau; v, \rho)) = M_-(\tau; v, \rho)\, M_+(\tau; v, \rho) \quad , \quad |\tau| = 1$$

- M_-, M_-^{-1} analytic and bounded in D_- $\quad (|\tau| > 1)$
- M_+, M_+^{-1} analytic and bounded in D_+ $\quad (|\tau| < 1)$
- $M_+(\tau = 0; v, \rho) = \mathbb{I}$. Unique.

Can show: `arXiv:1703.10366`

- $M_-(\tau = \infty; v, \rho) = M(v, \rho)$

  $A = M^{-1} dM, \quad d\,(\rho \star A) = 0$

- $M_+(\tau; v, \rho) = X(\tau; v, \rho)$

  solves Lax pair $\quad \tau\,(d + A)X = \star dX$

Riemann-Hilbert factorization problem

Questions:

1. When does $\mathcal{M}(w)$ have a canonical factorization?
2. How to pick $\mathcal{M}(w)$?
3. How to obtain explicit factorizations?

Question 1:

- $\mathcal{M}(w(\tau; v, \rho)$ invertible matrix function, Hölder continuous on unit circle $|\tau| = 1$
- Necessary and sufficient conditions for canonical RH factorization (Theorem (Cristina Câmara))

Question 3:

- Solve a vectorial factorization problem

$$\mathcal{M}\, M_+^{-1} = M_- \quad , \quad \mathcal{M}\, \phi_+ = \phi_- \qquad ,$$ Liouville's theorem

Question 2: Strategy:

- Pick known spacetime solution $M_{\rm seed}(v,\rho)$.

  Candidate $\mathcal{M}_{\rm seed}(w)$: $M_{\rm seed}(v=w,\rho=0)=\mathcal{M}_{\rm seed}(w)$.

  Delicate, but can be motivated. Works in all known cases.

- Deform $\mathcal{M}_{\rm seed}(w)$: $\mathcal{M}(w)=g^{\natural}(w)\,\mathcal{M}_{\rm seed}(w)\,g(w)\Rightarrow M_{new}(v,\rho)$.

- Ex:

$$\mathcal{M}(w)=\frac{1}{w^2}\begin{pmatrix} A & B\,w+\alpha & C\,w^2 \\ -B\,w-\alpha & D\,w^2 & 0 \\ C\,w^2 & 0 & 0 \end{pmatrix}$$

  $\alpha=0$: Seed describes near-horizon region of extremal rotating black hole solution in $D=4$

  $\alpha\neq 0$: $M_{new}(v,\rho)$, difficult from field equations.

Outlook: $\mathcal{M}(w)\leftrightarrow T^{-1}_{\mathcal{M}(w)}$. Study $d(\rho\star A)=0\leftrightarrow T_{\mathcal{M}(w)}\,f=g$.

Thanks!