Truncated Toeplitz operators

Mathematical aspects of the physics
with non-self-adjoint operators

Marseille, France 5-9 June 2017

TTO : D. Sarason - Algebraic properties of truncated Toeplitz operators, 2007

S. Garcia and W.T. Ross - Recent progress on TTO, 2012

S. Garcia, J. Mashreghi and W.T. Ross - Introduction to Model Spaces and their Operators, 2016

TTO appear in connection with many topics in mathematics, physics and engineering:
complex symmetric operators, finite time convolution eq. probability, approximation theory, elastodynamics, diffraction problems, signal processing, control theory, etc.

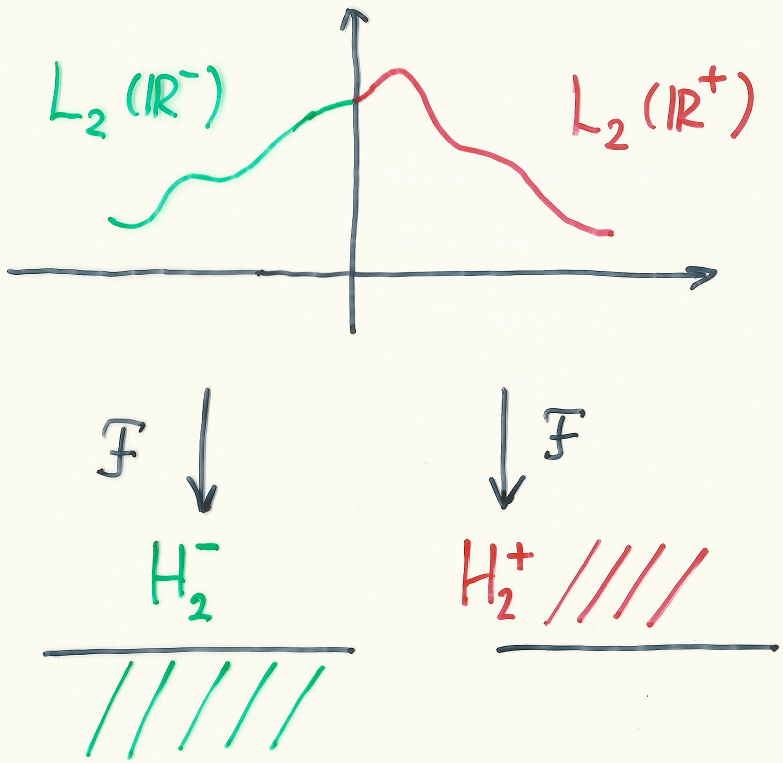
ATTO are a natural generalisation of rectangular Toeplitz matrices and truncated Toeplitz operators.

1. Spectral properties of TTO by equivalence after extension, JMAA 2016
2. ATTO and Toeplitz operators with matrix symbols, JOT 2017

Spaces and projections

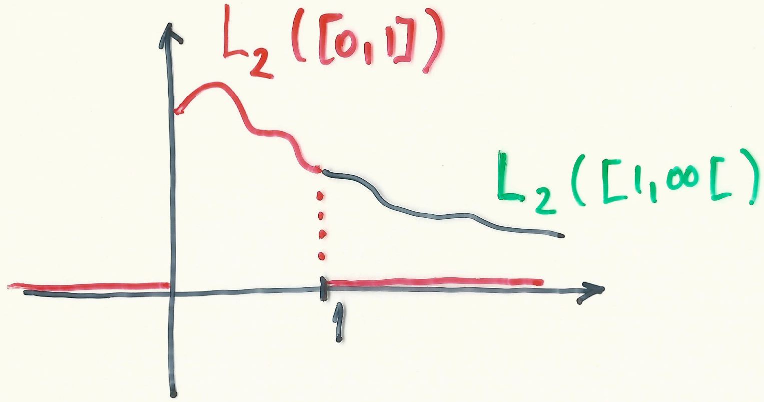
$$L_2(\mathbb{R}) =: L_2$$

$$\mathcal{F}f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{ixt}\, dt$$



$$\begin{tikzcd}
L_2 \arrow[d, "\mathcal{F}"] & = & L_2(\mathbb{R}^-) \arrow[d, "\mathcal{F}"] & \oplus & L_2(\mathbb{R}^+) \arrow[d, "\mathcal{F}"] \\
L_2 \arrow[rr, bend right, "P^-"'] \arrow[rrrr, bend right, "P^+"'] & = & H_2^- & \oplus & H_2^+
\end{tikzcd}$$

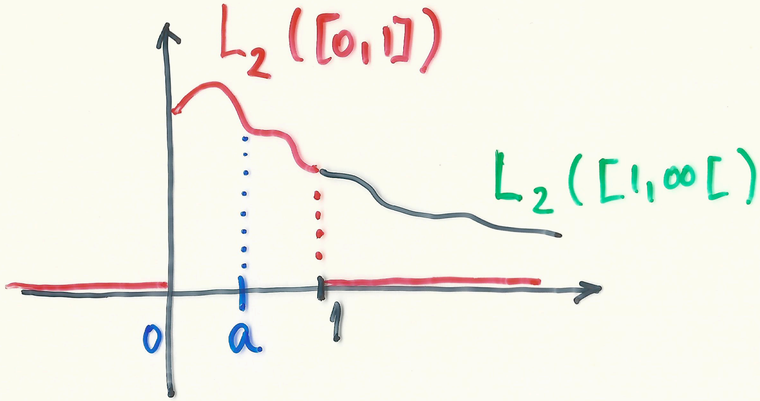
$$H_2^{\pm} = \mathcal{F} L_2(\mathbb{R}^{\pm})$$ Hardy spaces



$$\begin{array}{ccccc} L_2(\mathbb{R}^+) & = & \underline{L_2([0,1])} & \oplus & \underbrace{L_2([1,\infty[)}_{\mathcal{J}_1 \, L_2(\mathbb{R}^+)} \\ \downarrow \mathcal{F} & & \downarrow & & \downarrow \mathcal{F} \\ H_2^+ & = & K_{e^{ix}} & \oplus & \underline{\underline{e}}^{ix} \, H_2^+ \end{array}$$

model space

$$P_{e^{ix}} : H_2^+ \xrightarrow{\text{onto}} K_{e^{ix}}$$



$$\begin{array}{ccccc} L_2(\mathbb{R}^+) & = & \underline{L_2([0,1])} & \oplus & \underbrace{L_2([1,\infty[)}_{J_1 L_2(\mathbb{R}^+)} \\ \downarrow \mathcal{F} & & \downarrow & & \downarrow \mathcal{F} \\ H_2^+ & = & K_{e^{ix}} & \oplus & \underline{\underline{e}}^{ix} H_2^+ \end{array}$$

model space

$$P_{e^{ix}} : H_2^+ \xrightarrow{\text{onto}} K_{e^{ix}}$$

e^{ix} is analytic and bounded in $\mathbb{C}^+$, $|e^{ix}| = 1$ on $\mathbb{R}$: inner function

If $0 \leq a \leq 1$ then e^{iax} is an inner function that divides e^{ix}

$$K_{e^{iax}} = \mathcal{F} L_2([0,a]) \qquad K_{e^{iax}} \subset K_{e^{ix}}$$

Other inner functions: $1, \ \frac{x-i}{x+i}, \ \left(\frac{x-i}{x+i}\right)^2, \dots$

For $r(x) = \frac{x-i}{x+i}$, r^n is an inner function $(n \in \mathbb{Z}_0^+)$

$$1 \lesssim r \lesssim r^2 \lesssim \cdots \lesssim r^n$$

$$H_2^+ = \quad \oplus \, r^n H_2^+$$

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$$1 \precsim r \precsim r^2 \precsim \cdots \precsim r^n$$

$$\begin{tikzcd}
H_2^+ = \arrow[r, bend right, "P_{r^n}"'] & K_{r^n} \oplus r^n H_2^+ \\
& \operatorname{span}\{e^{-t}\chi_+(t), \ te^{-t}\chi_+(t), \ldots, t^{n-1}e^{-t}\chi_+(t)\} \arrow[u, "\mathcal{F}"]
\end{tikzcd}$$

$$\dim K_{r^n} = n$$

Model spaces

Θ inner function — analytic and bounded in $\mathbb{C}^+$, $|\Theta| = 1$ a.e. on $\mathbb{R}$

$$H_2^+ = K_\Theta \oplus \Theta H_2^+$$

$$H_2^+ \xrightarrow{P_\Theta} K_\Theta$$

K_Θ is <u>finite dimensional</u> iff Θ is a <u>finite Blaschke product</u>

$$\Theta = \prod_{j=1}^{m} \frac{x - z_j^+}{x - \overline{z_j^+}}, \quad z_j^+ \in \mathbb{C}^+$$

rational, analytic in $\mathbb{C}^+$, $|\Theta| = 1$ on $\mathbb{R}$

Ex: K_{r^n}, $r(x) = \frac{x-i}{x+i}$.

Toeplitz operators (TO)

$$W_\gamma : L_2(\mathbb{R}^+) \longrightarrow L_2(\mathbb{R}^+)$$

$$f \longmapsto \chi_+ \underbrace{\int_{\mathbb{R}^+} \gamma(t-u) f(u)\, du}_{\gamma * f} \qquad \gamma \in L^1(\mathbb{R})$$

$\Big\downarrow \mathcal{F}$

Recall : $\mathcal{F} L_2(\mathbb{R}^+) = H_2^+$, $P^+ : L_2 \xrightarrow{\text{onto}} H_2^+$; $P^+ = \mathcal{F} \chi_+ \mathcal{F}^{-1}$

$\mathcal{F}\gamma = g$, $\mathcal{F}(\gamma * f) = g\, \mathcal{F}f = g\, F_+$

$$\mathcal{F} W_\gamma \mathcal{F}^{-1} = T_g : H_2^+ \longrightarrow H_2^+$$

$$F_+ \longmapsto P^+(g F_+)$$

- For $g \in L_\infty$, $T_g = P^+ g \big|_{P^+ L_2 = H_2^+}$ Toeplitz operator with symbol g

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$$\mathcal{F} W_\gamma \mathcal{F}^{-1} = T_g : H_2^+ \longrightarrow H_2^+$$

$$F_+ \longmapsto P^+(g F_+)$$

- For $g \in L_\infty^{n\times n}$, $T_g = P^+ g\big|_{P^+(L_2)^n = (H_2^+)^n}$ **Toeplitz operator with symbol** g

$$T_g : (H_2^+)^n \longrightarrow (H_2^+)^n$$

Truncated Toeplitz operators (TTO)

$$\begin{aligned} \tilde{W}_\gamma : L_2([0,1]) &\longrightarrow L_2([0,1]) \\ f &\longmapsto \chi_{[0,1]} \int_0^1 \gamma(t-u) f(u)\, du \end{aligned}$$

$\mathcal{F} \downarrow$

Recall: $\mathcal{F} L_2([0,1]) = K_{e^{ix}}$, $P_{e^{ix}} : L_2 \longrightarrow K_{e^{ix}}$;

$P_{e^{ix}} = \mathcal{F} \chi_{[0,1]} \mathcal{F}^{-1}$

$$\begin{aligned} \mathcal{F}\tilde{W}_\gamma\mathcal{F}^{-1} = A_g^{e^{ix}} : K_{e^{ix}} &\longrightarrow K_{e^{ix}} \\ F &\longmapsto P_{e^{ix}}(gF) \end{aligned}$$

- For $g \in L_\infty$, θ inner, $A_g^\theta = P_\theta g \big|_{P_\theta L_2 = K_\theta}$ — Truncated Toeplitz oper. with symbol g

Truncated Toeplitz operators (TTO)

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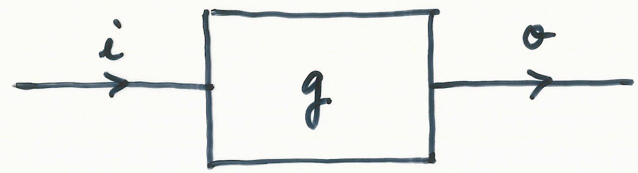
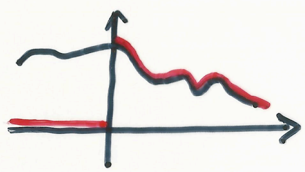
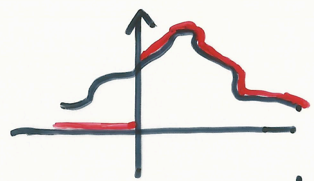
$P_\theta P^+ = P_\theta$

$$P_\theta (P^+ g) \big|_{P_\theta L_2 = K_\theta}$$

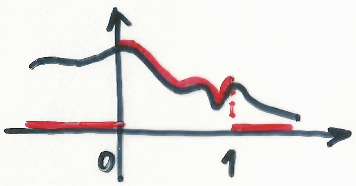
1) 



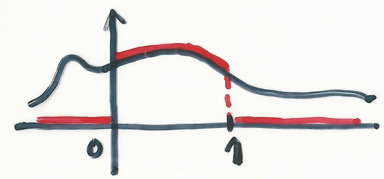
f_+



$\Phi_+ = P^+(g f_+) = T_g f_+$ TO



f_1

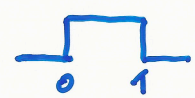


$\Phi_1 = P_{e^{ix}}(g f_1) = A_g^{e^{ix}} f_1$ TTO

2) <u>Volterra operator</u>

$V : L_2([0,1]) \longrightarrow L_2([0,1])$

$Vf(t) = \int_0^t f(u)\, du \quad , \quad 0 < t < 1$

$= \chi_{[0,1]}\Big($  $* f \Big)$

$\mathcal{F} V \mathcal{F}^{-1} = A_{\underline{g}}^{e^{ix}} : K_{e^{ix}} \to K_{e^{ix}} \quad , \quad \underline{g}(x) = \frac{e^{ix}-1}{ix}$

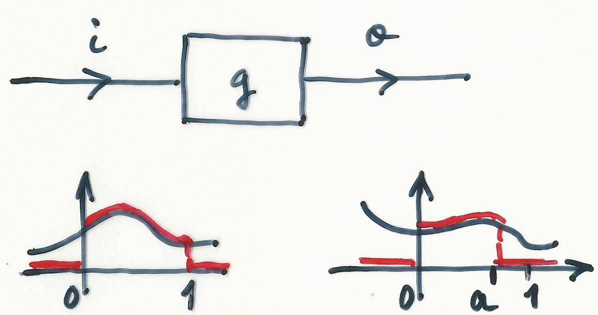
3) Toeplitz matrices

$$K_{r^n} = \text{span}\left\{ \frac{1}{x+i},\ r\frac{1}{x+i},\ \dots,\ r^{n-1}\frac{1}{x+i} \right\} \qquad r(x) = \frac{x-i}{x+i}$$

In the finite dimensional case, TTO are represented by $n\times n$ Toeplitz matrices

$$\begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} \\ a_1 & a_0 & a_{-1} & a_{-2} \\ a_2 & a_1 & a_0 & a_{-1} \\ a_3 & a_2 & a_1 & a_0 \end{bmatrix}$$

Asymmetric truncated Toeplitz operators



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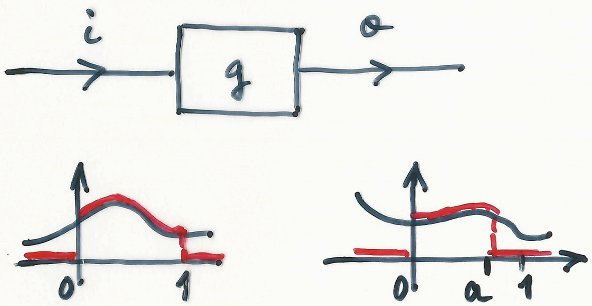

$g \in L_\infty$

θ, α inner

$$A_g^{\theta,\alpha} : K_\theta \longrightarrow K_\alpha$$

$$A_g^{\theta,\alpha} = P_\alpha g|_{P_\theta L_2 = K_\theta}$$

When $\alpha = \theta$, $A_g^{\theta,\alpha} = A_g^\theta$

- When is the operator equal to zero?

Toeplitz operators: $T_g = 0 \Leftrightarrow g = 0$ $\qquad T_{g_1} = T_{g_2} \Leftrightarrow g_1 = g_2$

ATTO: (C., Partington) $A_g^{\theta,\alpha} = 0 \Leftrightarrow g = \bar{\theta} g_- + \alpha g_+ \ , \ g_\pm \in (x \pm i) H_2^\pm$

TTO: Sarason, 2007

Remark: $g \in L_\infty \Rightarrow g = \bar{\theta} g_- + \underbrace{\overline{\psi_\theta} + \varphi_\alpha}_{} + \alpha g_+ \ , \ \psi_\theta \in (x+i) K_\theta$, $\varphi_\alpha \in (x+i) K_\alpha$

The ATTO "sees" the $\overline{K_\theta} + K_\alpha$ part of the symbol

$\Rightarrow$ If θ is a finite Blaschke product (i.e., K_θ is finite dimensional) any ATTO (or TTO) has a rational symbol

Equivalence after extension

Operator = bd. linear operator acting between complex Banach spaces

Def: $T: X \to \tilde{X}$, $S: Y \to \tilde{Y}$

$T \sim S$ $\quad$ T, S are equivalent $\Leftrightarrow$ $T = ESF$ with E, F invertible oper.

$T \overset{*}{\sim} S$ $\quad$ T, S are equivalent after extension $\Leftrightarrow$ there exist Banach spaces X_0, Y_0 and invertible operators $E: \tilde{Y} \oplus Y_0 \to \tilde{X} \oplus X_0$ and $F: X \oplus X_0 \to Y \oplus Y_0$ such that

$$\begin{pmatrix} T & 0 \\ 0 & I_{X_0} \end{pmatrix} = E \begin{pmatrix} S & 0 \\ 0 & I_{Y_0} \end{pmatrix} F$$

[Bart, Tsekanovskii, 1992]

Operators that are equivalent after extension share many properties.

If $T \overset{*}{\sim} S$:

$\ker T \simeq \ker S$

$\operatorname{Im} T$ is closed $\Leftrightarrow$ $\operatorname{Im} S$ is closed

$$\tilde{X}/_{\operatorname{Im} T} \simeq \tilde{Y}/_{\operatorname{Im} S}$$

T is invertible $\Leftrightarrow$ S is invertible

T is Fredholm $\Leftrightarrow$ S is Fredholm

$\dim \ker T = \dim \ker S$

$\operatorname{codim} \operatorname{Im} T = \operatorname{codim} \operatorname{Im} S$

<u>Theor</u>: (C., Partington) $A_g^{\theta,\alpha} \overset{*}{\sim} T_G$ with $G = \begin{bmatrix} \bar{\theta} & 0 \\ g & \alpha \end{bmatrix}$

$$T_G = P^+ G|_{(H_2^+)^2}$$

So we can reduce the study of kernels, Fredholmness, Fredholm properties, invertibility of ATTO / TTO to the corresponding study for <u>Toeplitz operators</u> with <u>matrix symbols</u> of a particular form

Remark: $A_g^\theta - \lambda I_{K_\theta} = A_{g-\lambda}^\theta \overset{*}{\sim} T_{G_\lambda}$

$$G_\lambda = \begin{bmatrix} \bar\theta & 0 \\ g-\lambda & \theta \end{bmatrix}$$

Theor: T_G is invertible iff G admits a factorisation

[Mikhlin, Prössdorf] $\quad G = G_- G_+$ where $G_+^{\pm 1} \in (x+i) H_2^+$

$$G_-^{\pm 1} \in (x-i) H_2^-$$

$$T_G^{-1} \longleftarrow G_+^{-1} P^+ G_-^{-1} \text{ is bounded}$$

Theor: (C., Partington) A_g^θ is invertible iff T_G is invertible

$$(A_g^\theta)^{-1} = P_\theta \left[(P_\theta g P_\theta + Q_\theta)^{-1} \right]\Big|_{K_\theta}$$

$$(P_\theta g P_\theta + Q_\theta)^{-1} \psi_+ = P_1 \Big[\underbrace{G_+^{-1} P^+ G_-^{-1}}_{T_G^{-1}} \begin{pmatrix} T_{\bar\theta} \psi_+ \\ \psi_+ + P_\theta g Q_\theta \psi_+ \end{pmatrix}$$

- How are $\ker A_g^{\theta}$ and $\ker A_g^{\theta,\alpha}$ related? $(\alpha \lesssim \theta)$

$$\ker A_g^{\theta} \subset \ker A_g^{\theta,\alpha}$$

$$\varphi_+ \in \ker A_g^{\theta} \Leftrightarrow \varphi_+ \in \ker A_g^{\theta,\alpha}$$
$$(P_\theta - P_\alpha)\, g\, \varphi_+ = 0$$

$$g = g_1 + \alpha h_+ \qquad h_+ \in H_\infty^+ \subset (x+i) H_2^+ \quad , \ \alpha \lesssim \theta$$

$$\ker A^{\theta}_{g_1 + \alpha h_+} \subset \ker A^{\theta,\alpha}_{g_1 + \alpha h_+} = \ker A^{\theta,\alpha}_{g_1}$$ simpler!

$$\begin{bmatrix} \bar\theta & 0 \\ g_1 + \alpha h_+ & \theta \end{bmatrix} \begin{bmatrix} \varphi_{1+} \\ \varphi_{2+} \end{bmatrix} = \begin{bmatrix} \varphi_{1-} \\ \varphi_{2-} \end{bmatrix} \qquad \begin{bmatrix} \bar\theta & 0 \\ g_1 & \alpha \end{bmatrix} \begin{bmatrix} \varphi_{1+} \\ \varphi_{2+} \end{bmatrix} = \begin{bmatrix} \varphi_{1-} \\ \varphi_{2-} \end{bmatrix}$$

TTO with θ-separated symbols

A_g^θ with $g = \bar{\beta} h_- + \alpha h_+$ $\qquad (\alpha \beta = \theta, \ h_\pm \in H_\infty^\pm)$

1. $\ker A^\theta_{\bar{\beta} h_- + \alpha h_+} \subset \ker A^{\theta, \alpha}_{\bar{\beta} h_- + \alpha h_+} = \ker A^{\theta, \alpha}_{\bar{\beta} h_-}$

2. $\ker A^{\theta, \alpha}_{\bar{\beta} h_-} = K_{\gamma \beta}$, $\gamma = GCD(\bar{h}_-^i, \alpha)$

3. imposing the extra condition $h_+ \varphi_+ \in \beta H_2^+$, we conclude that

$$\boxed{\ker A_g^\theta = \beta \bar{\gamma}_1 K_{\gamma \gamma_1}} \quad , \quad \gamma_1 = GCD(h_+^i, \beta)$$

Analogously:

$$\boxed{\ker (A_g^\theta)^* = \alpha \bar{\gamma} K_{\gamma \gamma_1}}$$

$$\ker A_g^\theta = \beta\bar{\gamma}_1 \, k_{\gamma\gamma_1} \qquad\qquad \ker (A_g^\theta)^* = \alpha\bar{\gamma}\, k_{\gamma\gamma_1}$$

(i) $\ker A_g^\theta$ is finite dimensional $\Leftrightarrow$ $\gamma\gamma_1$ are finite Blaschke products

(ii) A_g^θ is injective $\Leftrightarrow$ γ, γ_1 are constants

(iii) If θ is a singular inner function (ex: $\theta(x) = e^{ix}$) then

$$A_g^\theta \text{ is Fredholm} \Leftrightarrow A_g^\theta \text{ is invertible}$$

(iv) $\ker A_g^\theta$ and $\ker (A_g^\theta)^*$ are simultaneously finite dimensional or not

By using the <u>equivalence after extension</u> to a Toeplitz operator with matrix symbol and results on the invertibility of these operators obtained in [C., dos Santos, 2000]:

(v) A_g^θ is <u>invertible</u> if $\inf_{z\in\mathbb{C}^+} \left(|\beta(z)| + |h_+(z)| \right) > 0$, $\inf_{z\in\mathbb{C}^+} \left(|\alpha(z)| + |\overline{h_-}(z)| \right) > 0$

In particular: $g = h_+ \in H^+_\infty$ (g cont. in $\mathbb{R}_\infty$)

(i) $A^\theta_{h_+}$ is invertible $\Leftrightarrow \inf\limits_{z \in \mathbb{C}^+} (|\theta(z)| + |h_+(z)|) > 0$

__Def__: $\Sigma(\theta) = \{ \xi \in \mathbb{C}^+ \cup \mathbb{R}_\infty : \lim\limits_{z \to \xi} \inf\limits_{z \in \mathbb{C}^+} |\theta(z)| = 0 \}$

(ii) $\sigma(A^\theta_{h_+}) = h_+(\Sigma(\theta))$

(iii) $\sigma_{ess}(A^\theta_{h_+}) = h_+(\Sigma(\theta) \cap \mathbb{R}_\infty)$

(iv) $\sigma_p(A^\theta_{h_+}) = \{ \lambda \in \mathbb{C} : \underbrace{GCD(\theta, (h_+ - \lambda)^i)}_{\beta_\lambda} \notin \mathbb{C} \}$

$$E_\lambda = \bar{\beta}_\lambda \theta K_{\beta_\lambda}$$

TTO with rational symbols

Remark : This is always the case for TTO on finite dimensional model spaces

$$R(x) = \frac{Ax+B}{x-z_0} \qquad (Az_0 + B \neq 0, \; z_0 \notin \mathbb{R})$$

$$A_R^\theta - \lambda I_{K_\theta} = A_{R-\lambda}^\theta \overset{*}{\sim} T_{G_\lambda} \qquad G_\lambda = \begin{bmatrix} \bar{\theta} & 0 \\ R-\lambda & \theta \end{bmatrix}$$

Th: Let Γ_R denote the closed contour defined by $w = R(x), x \in \mathbb{R}_\infty$

$$\sigma(A_R^\theta) = R(\Sigma(\theta)) = \sigma_{ess}(A_R^\theta) \cup \sigma_p(A_R^\theta)$$

$$\sigma_{ess}(A_R^\theta) = R(\Sigma(\theta)) \cap \Gamma_R$$

$$\sigma_p(A_R^\theta) = R(\Sigma(\theta)) \cap \operatorname{Int} \Gamma_R$$