Inverse problems for boundary triples with applications

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joint work with
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Motivation:

- Titchmarsh-Weyl m-function for Sturm-Liouville problems

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Method: make use of abstract theory of boundary triples to

- introduce M-function,
- relate resolvent to operators on the boundary,

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$$(A_{\max} u, v)_H - (u, \widetilde{A}_{\max} v)_H = (\Gamma_1 u, \widetilde{\Gamma}_0 v)_{\mathcal{H}} - (\Gamma_0 u, \widetilde{\Gamma}_1 v)_{\mathcal{K}}.$$

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$\{\mathcal{H} \oplus \mathcal{K}, (\Gamma_1, \Gamma_0), (\widetilde{\Gamma}_1, \widetilde{\Gamma}_0)\}$ is a boundary triple for the adjoint pair $A, \widetilde{A}$.

ODE Example

For $p \in C^1(0,1)$, $p > 0$, $q \in L^\infty(0,1)$, consider

$$Lu = \left(-\frac{d}{dx}p\frac{d}{dx} + q\right)u \quad \text{and} \quad \widetilde{L}u = \left(-\frac{d}{dx}p\frac{d}{dx} + \overline{q}\right)u \quad \text{on } (0,1).$$

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For $u, v \in H^2(0,1)$

$$\left\langle A_{\max}u, v\right\rangle_{L^2} - \left\langle u, \widetilde{A}_{\max}v\right\rangle_{L^2} = \left\langle \Gamma_1 u, \widetilde{\Gamma}_0 v\right\rangle_{\mathbb{C}^2} - \left\langle \Gamma_0 u, \widetilde{\Gamma}_1 v\right\rangle_{\mathbb{C}^2},$$

where

$$\Gamma_1 u = \widetilde{\Gamma}_1 u = \begin{pmatrix} -p(1)u'(1) \\ p(0)u'(0) \end{pmatrix}, \quad \Gamma_0 u = \widetilde{\Gamma}_0 u = \begin{pmatrix} u(1) \\ u(0) \end{pmatrix}$$

and $\mathcal{H} = \mathcal{K} = \mathbb{C}^2$.

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$$S_{\lambda,B} : \mathcal{H} \to \ker(A_{\max} - \lambda)$$

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- $\widetilde{S}_{\lambda,B}$ and $\widetilde{M}_B(\lambda)$ defined analogously.

Relation to resolvent

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Theorem (Kreĭn-type formula)

- $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, $\lambda \in \rho(A_B) \cap \rho(A_C)$. *Then*

$$(A_B - \lambda)^{-1} = (A_C - \lambda)^{-1} - S_{\lambda,C}(I + (B - C)M_B(\lambda))(C - B)\Gamma_0(A_C - \lambda)^{-1}$$

- $B = 0$, $\lambda \in \rho(A_0) \cap \rho(A_C)$, *then*

$$(A_0 - \lambda)^{-1} = (A_C - \lambda)^{-1} - S_{\lambda,C}(I - CM_0(\lambda))\Gamma_1(A_C - \lambda)^{-1}.$$

Results for poles

Theorem

Let $\mu \in \mathbb{C}$ be an isolated eigenvalue of finite algebraic multiplicity of the operator A_B. Assume unique continuation holds, i.e.

$$\ker(A_{\max} - \mu) \cap \ker(\Gamma_1) \cap \ker(\Gamma_0) = \ker(\widetilde{A}_{\max} - \bar{\mu}) \cap \ker(\widetilde{\Gamma}_1) \cap \ker(\widetilde{\Gamma}_0) = \{0\}.$$

Then μ is a pole of finite multiplicity of $M_B(\cdot)$ and the order of the pole of $R(\cdot, A_B)$ at μ is the same as the order of the pole of $M_B(\cdot)$ at μ.

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Theorem

- *Let $B \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, $\mu \in \mathbb{C}$.*
- *Assume there exists $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ such that $\mu \in \rho(A_C)$.*

Then μ is isolated eigenvalue of finite algebraic multiplicity of A_B iff μ is pole of finite multiplicity of $M_B(\cdot)$.
In this case, order of the pole of $R(\cdot, A_B)$ at μ is same as order of the pole of $M_B(\cdot)$ at μ.

A matrix differential operator, (Hain-Lüst operator)

$$A_{\max} = \begin{pmatrix} -\frac{d^2}{dx^2} + q(x) & \tilde{w}(x) \\ w(x) & u(x) \end{pmatrix}, \quad \widetilde{A}_{\max} = \begin{pmatrix} -\frac{d^2}{dx^2} + \overline{q(x)} & \overline{w(x)} \\ \overline{\tilde{w}(x)} & \overline{u(x)} \end{pmatrix},$$

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$$D(A_{\max}) = D(\widetilde{A}_{\max}) = H^2(0,1) \times L^2(0,1),$$

$$\Gamma_1 \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -y'(1) \\ y'(0) \end{pmatrix}, \quad \Gamma_0 \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} y(1) \\ y(0) \end{pmatrix},$$

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For $\mu_0 \notin \sigma(A_B)$, define the space (the bordered resolvent)

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Our aim is to study the relation between $M_B(\lambda)$ and the bordered resolvent $P_{\widetilde{\mathcal{S}}}(A_B - \lambda)^{-1}|_{\overline{\mathcal{S}}}$ where for any subspace M, P_M denotes the orthogonal projection onto M.

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Our results are of two types, concerning uniqueness and reconstruction. (e.g Borg Levinson compared with Gel'fand-Levitan type results)

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Our first theorem concerns uniqueness only.

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Let $\lambda \in \rho(A_B)$. Then $P_{\overline{\widetilde{\mathcal{S}}}}(A_B - \lambda)^{-1}|_{\overline{\mathcal{S}}}$ uniquely determines $M_B(\lambda)$. In particular, if also $\lambda \in \rho(A_C)$, then $P_{\overline{\widetilde{\mathcal{S}}}}(A_B - \lambda)^{-1}|_{\overline{\mathcal{S}}} = P_{\overline{\widetilde{\mathcal{S}}}}(A_C - \lambda)^{-1}|_{\overline{\mathcal{S}}}$ implies that $M_B(\lambda) = M_C(\lambda)$, and, if additionally $\lambda \in \rho(A_\infty)$, then $B = C$. Here, $A_\infty = \widetilde{A}^*|_{\ker \Gamma_0}$.

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In the completely abstract case we need further information to recover M_B.

Reconstruction from a single boundary-condition operators

Theorem

Assume we know $\overline{\mathcal{S}}$, $P_{\overline{\widetilde{\mathcal{S}}}}(A_B - \lambda)^{-1}|_{\overline{\mathcal{S}}}$ *and the two sets* $\overline{\mathrm{Ran}\,(S_{\mu,B})}$, $\overline{\mathrm{Ran}\,(\widetilde{S}_{\widetilde{\mu},B^*})}$ *for some* $(\mu, \widetilde{\mu})$ *with* $\mu, \overline{\widetilde{\mu}} \in \rho(A_B)$*. Then we can reconstruct* $M_B(\lambda)$ *uniquely if* B *is known.*

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However in the case of the Friedrichs model (a first order integrable operator) where the closed range of the solution operator is known, it is determined.

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Assume $P_{\overline{\tilde{\mathcal{S}}}}(A_B - \lambda)^{-1}|_{\overline{\mathcal{S}}}$ *and* $P_{\overline{\tilde{\mathcal{S}}}}(A_C - \lambda)^{-1}|_{\overline{\mathcal{S}}}$ *are known. In addition, assume that*
(i) $\Gamma_0(A_C - \lambda)^{-1}\overline{\mathcal{S}}$ *and* $\widetilde{\Gamma}_0(A_C - \lambda)^{-*}\overline{\widetilde{\mathcal{S}}}$ *are known,*
(ii) $\Gamma_0(A_C - \lambda)^{-1}\overline{\mathcal{S}}$ *is dense in* $\mathcal{H}$ *and* $\widetilde{\Gamma}_0(A_C - \lambda)^{-*}\overline{\widetilde{\mathcal{S}}}$ *is dense in* $\mathcal{K}$,
(iii) $\mathrm{Ran}\,(B - C)$ *is dense in* $\mathcal{H}$ *and* $\ker(B - C) = \{0\}$.
Then $M_B(\lambda)$ *can be recovered.*