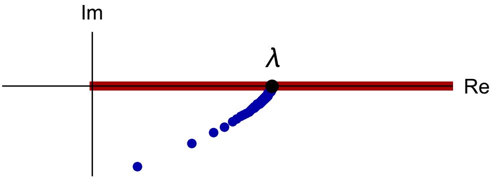
Schrödinger Operator with Non-Zero Accumulation Points of Complex Eigenvalues

Sabine Bögli (LMU Munich)



CIRM, 5 June 2017

Motivation 1: Pavlov (1960s)

$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V$ in $L^2(0,\infty)$ with BC $f'(0) = hf(0)$ for some $h \in \mathbb{C} \cup \{\infty\}$.
Assume $V \in L^\infty(0,\infty)$, decaying at $\infty \rightsquigarrow \sigma(H) = [0,\infty) \dot\cup \sigma_{\mathrm{dis}}(H)$.

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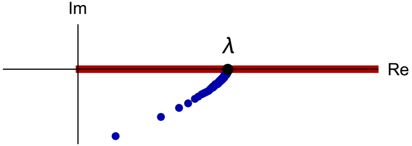
then $\#\sigma_{\mathrm{dis}}(H) < \infty$.
- Exponent $1/2$ sharp: $\forall\, \lambda \in (0,\infty)$, $\forall\, \beta \in (0, 1/2)$ $\exists\, \varepsilon > 0$ and $\exists\, h \in \mathbb{C} \backslash \mathbb{R}$, $\exists\, V \in L^\infty(0,\infty)$ real-valued such that
$$\sup_{x>0} |V(x)| \mathrm{e}^{\varepsilon x^{\beta}} < \infty$$
and $\exists\, (\lambda_n)_{n\in\mathbb{N}} \subset \sigma_{\mathrm{dis}}(H)$ with $\lim_{n\to\infty} \lambda_n = \lambda$.



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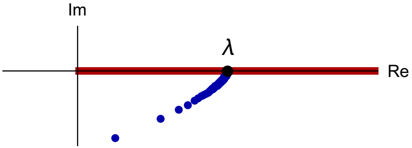

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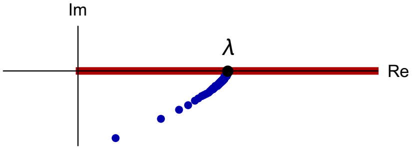

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Remark: For $h = 0$ (Neumann) and $h = \infty$ (Dirichlet), one could then extend V and eigenfunction to $\mathbb{R} \rightsquigarrow -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V$ in $L^2(\mathbb{R})$ has $(\lambda_n)_{n\in\mathbb{N}} \subset \sigma_{\mathrm{dis}}(H)$ with $\lim_{n\to\infty} \lambda_n = \lambda \in (0,\infty)$.

Motivation 2: Lieb-Thirring inequalities (1970s+)

Now arbitrary dimension $d \in \mathbb{N}$. Let $V \in L^p(\mathbb{R}^d)$ be real-valued with

$$\begin{cases} p \geq \frac{d}{2}, & \text{if } d \geq 3, \\ p > 1, & \text{if } d = 2, \\ p \geq 1, & \text{if } d = 1. \end{cases}$$

$H = -\Delta + V$ in $L^2(\mathbb{R}^d)$ is selfadjoint $\rightsquigarrow$ $\sigma_{\text{dis}}(H) \subset (-\infty, 0)$, and only possible accumulation point is 0.

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Lieb-Thirring inequalities:

$$\exists\, C_{d,p} > 0 : \quad \sum_{\lambda \in \sigma_{\text{dis}}(H)} |\lambda|^{p-\frac{d}{2}} \leq C_{d,p} \|V\|_p^p. \qquad (*)$$

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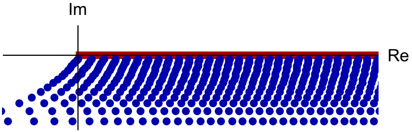
Remark: Then LHS of $(*)$ is ∞ while RHS is $< \infty$.

Main results

Theorem 1: Let $p > d$ and $\varepsilon > 0$. Then $\exists V \in L^\infty(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ with $\max\{\|V\|_\infty, \|V\|_p\} < \varepsilon$ and decaying at ∞ such that $H = -\Delta + V$ in $L^2(\mathbb{R}^d)$ has $(\lambda_n)_{n\in\mathbb{N}} \subset \sigma_{\mathrm{dis}}(H)$ with $\operatorname{Im} \lambda_n < 0$ and accumulating at every point of $[0, \infty)$.



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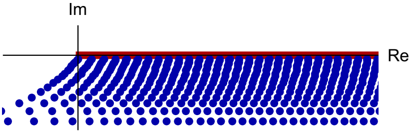

Theorem 2: In $\mathbb{R}^d_+ := \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : x_d > 0\}$.
Let $p > d$ and $\varepsilon > 0$ and $h \in \mathbb{R} \cup \{\infty\}$. Then $\exists V \in L^\infty(\mathbb{R}^d_+) \cap L^p(\mathbb{R}^d_+)$ with $\max\{\|V\|_\infty, \|V\|_p\} < \varepsilon$ and decaying at ∞ such that $H = -\Delta + V$ in $L^2(\mathbb{R}^d_+)$ with BC $\partial_{x_d} f = hf$ on $\partial\mathbb{R}^d_+$ has $(\lambda_n)_{n\in\mathbb{N}} \subset \sigma_{\text{dis}}(H)$ with $\operatorname{Im} \lambda_n < 0$ and accumulating at every point of $[0, \infty)$.

Proof idea part 1: Construction of a basic potential

For $c \in \mathbb{C}$, $t \in \mathbb{R}$ and $a > 0$ define

$$U_{c,t,a}(x) := \begin{cases} c, & |x - te_d| < a, \\ -\frac{(d-3)(d-1)}{4|x-te_d|^2}, & |x - te_d| \geq a. \end{cases}$$

Note: For $d = 1$ and $d = 3$: $U_{c,t,a}$ is c or 0.

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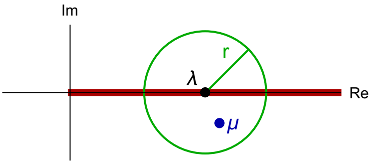
$$\forall\, t \in \mathbb{R}: \quad \|U_{c,t,a}\|_p < \varepsilon, \quad \|U_{c,t,a}\|_\infty < \delta, \quad \mu \in \sigma_{\mathrm{dis}}(-\Delta + U_{c,t,a}).$$



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Proof for $t = 0$: For $\tau \in \mathbb{C}$ and $a > 0$ define

$$f(x) := \begin{cases} \frac{\mathrm{e}^{\mathrm{i}ka}}{\sqrt{a} J_{\frac{d}{2}-1}(\tau a)} \frac{J_{\frac{d}{2}-1}(\tau |x|)}{|x|^{\frac{d}{2}-1}}, & 0 < |x| \leq a, \\ \frac{\mathrm{e}^{\mathrm{i}k|x|}}{|x|^{\frac{d-1}{2}}}, & |x| > a, \end{cases} \qquad k := -\mathrm{i}\frac{J_{\frac{d}{2}-2}(\tau a)}{J_{\frac{d}{2}-1}(\tau a)}\tau + \frac{\mathrm{i}(d-3)}{2a}.$$

Then $-\Delta f + U_{c,0,a} f = \mu f$ with $c := k^2 - \tau^2$, $\mu := k^2$.

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  - Proved for radial potentials (Frank-Simon, 2016) and for general $L^p(\mathbb{R}^d)$ potentials if $p \in \left(\frac{d}{2}, \frac{d+1}{2}\right]$ (Frank, 2011).
  - Open for $\left(\frac{d+1}{2}, d\right]$.

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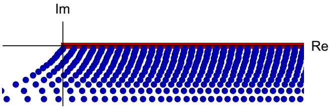
Remark (Kato): Let $\Lambda \subset \sigma_{\text{dis}}(-\Delta + V_1)$ be a finite set and $r > 0$. Then there exists $\gamma_{V_1,\Lambda,r} > 0$ such that if $\|V_2\|_\infty < \gamma_{V_1,\Lambda,r}$, then

$$\forall\, t > 0 : \quad \sup_{\lambda \in \Lambda} \operatorname{dist}(\lambda, \sigma(-\Delta + V_1 + V_2(\cdot - te_d)) < r.$$

Proof idea part 3: Construction of V satisfying the claims of Theorem 1

There exists a bijective map that send $n \in \mathbb{N}$ to $(q_n, m_n) \in (\mathbb{Q} \cap (0, \infty)) \times \mathbb{N}$ (is countable!).



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$$\operatorname{Im}\lambda_n < 0, \quad |\lambda_n - q_n| < \frac{1}{m_n}, \quad n \in \mathbb{N}.$$

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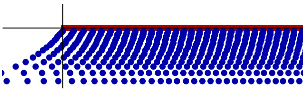

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The potential V is constructed inductively, as the limit $n \to \infty$ of

$$V_n := \sum_{j=1}^{n} U_{c_j, t_j, a_j},$$

where in each step we use Lemmas 1 and 2 to add another eigenvalue.

References

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Thank you for your attention!