Multidimensional p-adic continued fraction algorithms

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Notation

Let p be a prime number, $\mathbb{Q}_p$ be the p-adic number field and $\mathbb{Z}_p$ be the p-adic integers.
Namely:
For $q \in \mathbb{Q}$ we define $|q|_p = p^{-m}$, $ord_p(q) := m$, where $q = p^m r$ and r is relatively prime to p.

We define the distance of $q_1, q_2 \in \mathbb{Q}$ by $|q_1 - q_2|_p$. $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ associated with the distance.
$\mathbb{Z}_p = \{x \in \mathbb{Q}_p | |x|_p \leq 1\}$.
For $\alpha = \sum_{n \in \mathbb{Z}} c_n p^n \in \mathbb{Q}_p \backslash \{0\}$ $(c_n \in \{0, 1, \ldots, p-1\})$ we define $\omega_p(\alpha) := c_0$ and $\lfloor \alpha \rfloor_p := \Sigma_{n \in \mathbb{Z}_{\leq 0}} c_n p^n$.

p-adic continued fraction algorithm

Schneider [1968] introduced the following p-adic continued fraction algorithm. Let $\xi_1 = \xi \in p\mathbb{Z}_p$. We define $\xi_n \in p\mathbb{Z}_p$ for $n \in \mathbb{Z}_{\geq 2}$ recursively by

$$\xi_n = \frac{p^{ord_p(\xi_{n-1})}}{\xi_{n-1}} - a_{n-1},$$

where $a_n \in \{1, \ldots, p-1\}$. Then, we have

$$\xi_1 = \cfrac{p^{ord_p(\xi_1)}}{a_1 + \cfrac{p^{ord_p(\xi_2)}}{a_2 + \cfrac{p^{ord_p(\xi_3)}}{\ddots}}}.$$

Ordinary continued fraction algorithm

Lagrange's Theorem[1770]

α is a real quadratic number if and only if α has an eventually periodic continued fraction expansion.

p-adic continued fraction algorithm

Weger [1988] showed that some quadratic elements are not eventually periodic by Schneider's continued fraction algorithm. Ruban [1970] also gave a different p-adic continued fraction algorithm. Ooto [2014] found a similar result related to the algorithm given by Ruban.

New p-adic continued fraction algorithms

We proposed a new p-adic continued fraction algorithm denoted by STY algorithm.

Saito, Tamura, Y[2016,preprint]

$\alpha \in \mathbb{Q}_p$ is a quadratic element over $\mathbb{Q}$ if and only if α has an eventually periodic STY continued fraction expansion.

Bekki [2017] has also shown p-adic version of Lagrange's theorem for imaginary irrationals in $\mathbb{Q}_p$ on his continued fraction algorithm.

Multidimensional continued fraction algorithms

There are many (ordinary) multidimensional continued fraction algorithms:for example

1. Jacobi-Perron algorithm and its modifications(Brun, Selmer,...)
2. Klein polyhedra ([Karpenkov],[Bruno],...)
3. Dynamical LLL algorithm([Pethő, Pohst, Bertḱ],[Bosma,Smeets]).

But we can not find a multidimensional p-adic continued fraction algorithm which has been studied besides Tamura's work [2012].

Tamura's work

By disclosing a link between the hermitian canonical forms of certain integral matrices and p-adic numbers, he has shown that a multidimensional p-adic continued fraction converges to $(x, x^2, \ldots, x^{n-1})$ in the p-adic sense without considering algorithms of continued fraction expansion, where x is the root of some polynomial of degree n.

Multidimensional p-adic continued fraction algorithms

The main topic of this talk is to propose some definite multidimensional p-adic continued fraction algorithms.

Overview of our talk

(1)We introduce a class of p-adic continued fraction algorithms via c-**maps**.

(2)We propose some continued fraction algorithms $(\Phi_0, \Phi_1, \Phi_2, \Phi_3)$ in the class.

(3)We give some theorems and numerical experiments related to these algorithms.

(4)We give affine c-map which is extension of c-map.

(5)We propose a new algorithm Φ_{3A} via affine c-map.

(6)We give numerical experiments and a conjecture related to these algorithms.

c-map

Let $K \subset \mathbb{Q}_p$ be a finite extension of $\mathbb{Q}$ of degree d. We define $s \in \mathbb{Z}$ by $s := d - 1$ for $d \geq 2$ and $s := 1$ for $d = 1$. Ind denotes the set $\{1, 2, \ldots, s\}$. D denotes K^s and E denotes $(p\mathbb{Z}_p)^s \cap K^s$.

We consider a map $\Phi : D \to Ind \times L(D) \times GL(s, \mathbb{Z}_p \cap \mathbb{Q})$: for $\overline{\alpha} = (\alpha_1, \ldots, \alpha_s) \in D$, $\Phi(\overline{\alpha}) := (\phi(\overline{\alpha}), F_{\overline{\alpha}}, A_{\overline{\alpha}})$, where $L(D)$ is the set of linear fractional transformations and $F_{\overline{\alpha}} = (f_1, \ldots, f_s)$ satisfies that:

If $\alpha_{\phi(\overline{\alpha})} \neq 0$, for $1 \leq i \leq s$ and $(x_1, \ldots, x_s) \in D$

if $i = \phi(\overline{\alpha})$, for some $u_i, v_i \in V_p \cap \mathbb{Q}$

$$f_i(x_1, \ldots, x_s) = \frac{u_i p^{ord_p(\alpha_i)}}{x_i} - v_i,$$

if $i \neq \phi(\overline{\alpha})$, for some $u_i \in V_p \cap \mathbb{Q}$, $v_i \in \mathbb{Z}_p \cap \mathbb{Q}$

$$f_i(x_1, \ldots, x_s) = \frac{u_i p^k x_i}{x_{\phi(\overline{\alpha})}} - v_i,$$

where $k = \max\{ord_p(\alpha_{\phi(\overline{\alpha})}) - ord_p(\alpha_i), 0\}$,

and

$$f_i(\overline{\alpha}) \in p\mathbb{Z}_p.$$

If $\alpha_{\phi(\overline{\alpha})} = 0$, F is the identity map.

We call this map c-map.

Example

Let $s = 2$.

For $(\alpha_1, \alpha_2) \in D$ we put $\phi(\alpha_1, \alpha_2) := 1$ and in the case of $\alpha_1 \neq 0$ we set

$F_{(\alpha_1,\alpha_2)}(x_1, x_2) := \left(\frac{p^{ord_p(\alpha_1)}}{x_1} - \omega_p\left(\frac{p^{ord_p(\alpha_1)}}{\alpha_1}\right), \frac{p^k x_2}{x_1} - \omega_p\left(\frac{p^k \alpha_2}{\alpha_1}\right)\right)$,

where $k = \max\{ord_p(\alpha_1) - ord_p(\alpha_2), 0\}$,

and in the case of $\alpha_1 = 0$ we set $F_{(\alpha_1,\alpha_2)}(x_1, x_2) := (x_1, x_2)$.

We put $A_{(\alpha_1,\alpha_2)} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

We define

$\Phi(\alpha_1, \alpha_2) = (\phi(\alpha_1, \alpha_2), F_{(\alpha_1,\alpha_2)}, A_{(\alpha_1,\alpha_2)})$.

Then Φ is a c-map.

p-adic continued fraction via c-map

Let $\Phi(\overline{\alpha}) = (\phi(\overline{\alpha}), F_{\overline{\alpha}}, A_{\overline{\alpha}})$ for $\overline{\alpha} = (\alpha_1, \ldots, \alpha_s)^T \in D$ be a c-map.
Let $\overline{\alpha} = (\alpha_1, \ldots, \alpha_s)^T \in D$. We set $\overline{\alpha}^{(0)} := \overline{\alpha}$. $\overline{\alpha}^{(1)}, \overline{\alpha}^{(2)}, \ldots$ are defined inductively as follows: We suppose that $\overline{\alpha}^{(n)}$ for $n \in \mathbb{Z}_{\geq 0}$ is defined. We set $\overline{\alpha}^{(n+1)} := A_{\overline{\alpha}} F_{\overline{\alpha}}(\overline{\alpha}^{(n)})$.
We say that $\overline{\alpha}$ has Φ continued fraction expansion $\{\Phi(\overline{\alpha}^{(0)}), \Phi(\overline{\alpha}^{(1)}), \ldots\}$.
$\overline{\alpha}^{(n)} = (\alpha_1^{(n)}, \ldots, \alpha_s^{(n)})$ is referred to the n-th remainder of $\overline{\alpha}$.

Convergence of the algorithm

For an integer $n \geq 0$ we define the n-th convergent $\pi(\overline{\alpha}; n)$ as

$$\pi(\overline{\alpha}; n) := F_{\overline{\alpha}^{(0)}}^{-1} A_{\overline{\alpha}^{(0)}}^{-1} \cdots F_{\overline{\alpha}^{(n-1)}}^{-1} A_{\overline{\alpha}^{(n-1)}}^{-1}(0).$$

Theorem 1 Let $\overline{\alpha} \in D$. Let $\phi(\overline{\alpha}^{(n)})$ be not equal to 0 for every $n \in \mathbb{Z}_{\geq 0}$. Then, $\lim_{n\to\infty} \pi(\overline{\alpha}; n) = \overline{\alpha}$.

We remark that we do not have any result about quality of convergence at present.

Condition (H)

$\overline{\mathbb{Q}}$ denotes all algebraic elements over $\mathbb{Q}$ in $\mathbb{Q}_p$.

Let n be a positive integer. For $\beta \in (\overline{\mathbb{Q}} \setminus \mathbb{Q}) \cap p\mathbb{Z}_p$, if β has a minimal polynomial $x^n + a_1 x^{n-1} + \ldots + a_n$ over $\mathbb{Q}$ such that $a_i \in \mathbb{Z}_p$ for $1 \leq i \leq n$, $ord_p(a_{n-1}) = 0$ and $ord_p(a_n) > 0$, then we say that β satisfies the condition (H).

We remark that for a polynomial $p(x) = x^n + a_1 x^{n-1} + \ldots + a_n \in \mathbb{Q}[x]$ with $a_i \in \mathbb{Z}_p$ for $1 \leq i \leq n$, $ord_p(a_{n-1}) = 0$ and $ord_p(a_n) > 0$, Hensel's lemma says that there exists $\alpha \in \mathbb{Q}_p$ such that $p(\alpha) = 0$ and $ord_p(\alpha) = ord_p(a_n)$.

Main Lemma

Lemma 2 Let $K \subset \mathbb{Q}_p$ be a finite extension of $\mathbb{Q}$. Then, there exists $\alpha \in K$ which satisfies the condition (H) and $K = \mathbb{Q}(\alpha)$.

Sketch of the proof. Let α satisfy that $K = \mathbb{Q}(\alpha)$. We define a transformation T_{Sch} on $\mathbb{Q}_p$ as follows: for $\alpha \in \mathbb{Q}_p$ with $\alpha \neq 0$,

$$T_{Sch}(\alpha) := \frac{p^{ord_p(\alpha)}}{\alpha} - \omega_p\left(\frac{p^{ord_p(\alpha)}}{\alpha}\right),$$

Then, we can show that for some $n \geq 0$ $T^n_{Sch}(\alpha)$ satisfies the condition (H).

c-**map** $\Phi_0^{[\epsilon]}$

For $\overline{\alpha} = (\alpha_1, \ldots, \alpha_s)^T \in D$ we define linear fractional transformations $G_j^{[\overline{\alpha},\epsilon]} = (g_1^{[\overline{\alpha},\epsilon];(j)}, \ldots, g_s^{[\overline{\alpha},\epsilon];(j)})$ on D for $\epsilon \in \{-1, 1\}$ and $j = 1, \ldots, s$ by
for $\overline{x} = (x_1, \ldots, x_s)^T \in D$ if $\alpha_j \neq 0$, then we set

if $i = j$,

$$g_i^{[\overline{\alpha},\epsilon];(j)}(\overline{x}) = \frac{\epsilon p^{ord_p(\alpha_i)}}{x_i} - \omega_p \left(\frac{\epsilon p^{ord_p(\alpha_i)}}{x_i} \right),$$

if $i \neq j$,

$$g_i^{[\overline{\alpha},\epsilon];(j)}(\overline{x}) = \frac{\epsilon p^k x_i}{x_j} - \omega_p \left(\frac{\epsilon p^k \alpha_i}{\alpha_j} \right),$$

where $k = \max\{ord_p(\alpha_j) - ord_p(\alpha_i), 0\}$.

$S \in GL(s, \mathbb{Z}_p \cap \mathbb{Q})$ denotes the matrix

$$S = (s_{ij}) := \begin{cases} (\delta_{(i+1)j}) & \text{for } 1 \le i \le s-1,\ 1 \le j \le s, \\ (\delta_{1j}) & \text{for } i = s,\ 1 \le j \le s, \end{cases}$$

where $\delta_{ii} := 1$ and $\delta_{ij} := 0$ for $i \ne j$ for $i, j \in Ind$. Let us define the c-maps $\Phi_0^{[\epsilon]}$. We define $\Phi_0^{[\epsilon]}(\cdot) := (1, G_1^{[\cdot,\epsilon]}, S)$. We remark that for $s = 1, 2$ $\Phi_0^{[1]}$ continued fraction coincides with Schneider's continued fraction.

c-map $\Phi_1^{[\epsilon,z]}$

We assume that $K \neq \mathbb{Q}$. Let $z \in K$ satisfy the condition (H) and $K = \mathbb{Q}(z)$. We define linear fractional transformations $H_j^{[\overline{\alpha},\epsilon,z]} = (h_1^{[\overline{\alpha},\epsilon,z];(j)}, \ldots, h_s^{[\overline{\alpha},\epsilon,z];(j)})$ on D for $\overline{\alpha} = (\alpha_1, \ldots, \alpha_s)^T \in D$ and $j = 1, \ldots, s$ as follows:
$g_i^{[\overline{\alpha},\epsilon];(j)}(\overline{\alpha})$ is written uniquely $g_i^{[\overline{\alpha},\epsilon];(j)}(\overline{\alpha}) = a_0 + a_1 z + \ldots + a_s z^s$ where $a_i \in \mathbb{Q}$ for $0 \leq i, j \leq s$. Let $a' > 0$ be the greatest common factor of the denominators of $a_i (1 \leq i \leq s)$ which is not divided by p. We define $h_i^{[\overline{\alpha},\epsilon,z];(j)}(\overline{x})$ for $j = 1, \ldots, s$ by

$$h_i^{[\overline{\alpha},\epsilon,z];(j)}(\overline{x}) := \frac{g_i^{[\overline{\alpha},\epsilon,z];(j)}(\overline{x})}{a'} - \frac{a_0}{a'} + \left\lfloor \frac{a_0}{a'} \right\rfloor_p .$$

We define $\Phi_1^{[\epsilon,z]}(\cdot) := (1, H_1^{[\cdot,\epsilon,z]}, S)$.

c-map $\Phi_2^{[\epsilon,z],(n)}$

For $\alpha \in K$ α is written uniquely $\alpha = a_0 + a_1 z + \ldots + a_s z^s$ where $a_i \in \mathbb{Q}$ for $0 \le i \le s$, and we define $denom_z(\alpha)$ for $\alpha \in K$ by

$$denom_z(\alpha) := \min\{|d| \ |d \in \mathbb{Z}, d(a_0 + a_1 x + \ldots + a_s x^s) \in \mathbb{Z}[x]\},$$

and we define $denom_z(\overline{\alpha})$ $\overline{\alpha} = (\alpha_1, \ldots, \alpha_s)^T \in K^s$ by

$$denom_z(\overline{\alpha}) := \max\{denom_z(\alpha_i) | 1 \le i \le s\}.$$

We define $v_{[\epsilon,z]}^{(1)} : D \to \mathbb{Z}$ by for $\overline{\alpha} = (\alpha_1, \ldots, \alpha_s)^T \in D$

$$v_{[\epsilon,z]}^{(1)}(\overline{\alpha}) := \min\{denom_z(H_i^{[\overline{\alpha},\epsilon,z]}(\overline{\alpha})) | 1 \le i \le s\}.$$

We define $v_{[\epsilon,z]}^{(n)} : D \to \mathbb{Z}$ for $n = 2, 3, \ldots$ recursively by

$$v_{[\epsilon,z]}^{(n)}(\overline{\alpha}) := \min\{denom_z(H_i^{[\overline{\alpha},\epsilon,z]}(\overline{\alpha}))v_{[\epsilon,z]}^{(n-1)}(H_i^{[\overline{\alpha},\epsilon,z]}(\overline{\alpha})) | 1 \le i \le s\}.$$

We define $\phi^{(n)}_{[\epsilon,z]} : D \to Ind$ for $n \in \mathbb{Z}_{\geq 1}$ by for $\overline{\alpha} = (\alpha_1, \ldots, \alpha_s)^T \in D$

$$\phi^{(n)}_{[\epsilon,z]}(\overline{\alpha}) := \min\{i \in Ind | v^{(n+1)}_{[\epsilon,z]}(\overline{\alpha}) = denom_z(H_i^{[\overline{\alpha},\epsilon,z]}(\overline{\alpha}))v^{(n)}_{[\epsilon,z]}(H_i^{[\overline{\alpha},\epsilon,z]}(\overline{\alpha}))\}.$$

We define $\Phi_2^{[\epsilon,z],(n)}$ for $n \in \mathbb{Z}_{\geq 1}$ by for $\overline{\alpha} \in D$

$$\Phi_2^{[\epsilon,z],(n)}(\overline{\alpha}) := (\phi^{(n)}_{[\epsilon,z]}(\overline{\alpha}), H^{[\overline{\alpha},\epsilon,z]}_{\phi^{(n)}_{[\epsilon,z]}(\overline{\alpha})}, id),$$

where id is the identity map.

p-reduced matrix

$M_n(\mathbb{Q}_p)$ denotes all $n \times n$ $\mathbb{Q}_p$ matrices for $n \in \mathbb{Z}_{\geq 1}$. Let n be a positive integer for a while.

For $\alpha = \Sigma_{n \in \mathbb{Z}} c_n p^n \in \mathbb{Q}_p$ and $m \in \mathbb{Z}$ We define $\lfloor \alpha : m \rfloor_p$ and $\langle \alpha : m \rangle_p$ by

$$\lfloor \alpha : m \rfloor_p := \Sigma_{n \leq m, n \in \mathbb{Z}} c_n p^n,$$
$$\langle \alpha : m \rangle_p := \Sigma_{n > m, n \in \mathbb{Z}} c_n p^n.$$

$M_n(\mathbb{Q})$ denotes all $n \times n$ $\mathbb{Q}$ matrices for $n \in \mathbb{Z}_{\geq 1}$. We define that $M = (m_{ij}) \in M_n(\mathbb{Q})$ is p-reduced, if M satisfies that for every integer i with $1 \leq i \leq n$ there exist uniquely an integer $j (:= u(i))$ with $0 \leq j \leq n$ such that

(1) for every integer k with $1 \leq k \leq u(i)$ $m_{ik} = 0$,
(2) if $u(i) \neq n$, then $m_{i\ u(i)+1} \in \{p^l | l \in \mathbb{Z}\}$,
and for every integer k with $i < k \leq n$ $m_{k\ u(i)+1} = 0$
(3) if $i > 1$, $u(i) \geq u(i-1)$,
and
(4) if $u(i) \neq n$, then for every integer j with $1 \leq j < i$

$$\langle m_{j\ u(i)+1} : ord_p(m_{i\ u(i)+1}) - 1 \rangle_p = 0.$$

Example.

$$\begin{pmatrix} p^2 & p + p^2 & \frac{1}{p^3} \\ 0 & p^3 & \frac{1}{p^2} \\ 0 & 0 & \frac{1}{p} \end{pmatrix}$$

A matrix of $M_n(\mathbb{Q})$ is uniquely converted to a p-reduced matrix by using the following row operations:

(a) Switch two rows,

(b) Multiply a row by an element of $V_p \cap \mathbb{Q}$,

(c) Add a row to another one by multiplied by an element of $\mathbb{Z}_p \cap \mathbb{Q}$.

When $M \in GL(n, \mathbb{Q})$ is converted by the p-reduced algorithm to $M' \in GL(n, \mathbb{Q})$, there exists $N \in GL(n, \mathbb{Z}_p \cap \mathbb{Q})$ such that $M' = NM$ and we denote by $pr(M)$ N.

Example

$p = 2$

$$\begin{pmatrix} 4 & 16 \\ 2 & 4 \end{pmatrix} \to \begin{pmatrix} 2 & 4 \\ 4 & 16 \end{pmatrix} \to \begin{pmatrix} 2 & 4 \\ 0 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 3/2 \\ -5 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$$

c-**map** $\Phi_3^{[\epsilon,z]}$

We recall that $z \in K$ satisfy the condition (H) and $K = \mathbb{Q}(z)$. For $\overline{\alpha} = (\alpha_1, \ldots, \alpha_s)^T \in D$ the $M_{\overline{\alpha}} = (m_{ij}) \in M_{s\times(s+1)}(\mathbb{Q})$ is defined by $\overline{\alpha} = M_{\overline{\alpha}}(z, z^2, \ldots, z^s, 1)^T$ and $M'_{\overline{\alpha}} \in M_s(\mathbb{Q})$ defines $M'_{\overline{\alpha}} := (m_{ij})_{1\le i\le s, 1\le j\le s}$.

We define the map $\tau_z : D \to M_s(\mathbb{Z}_p \cap \mathbb{Q})$ by for $\overline{\alpha} \in D$

$$\tau_z(\overline{\alpha}) := pr(M'_{\overline{\alpha}}).$$

We define $\Phi_3^{[\epsilon,z]}$ by for $\overline{\alpha} \in D$

$$\Phi_3^{[\epsilon,z]}(\overline{\alpha}) := (1, H_1^{[\overline{\alpha},\epsilon,z]}, \tau_z(H_1^{[\overline{\alpha},\epsilon,z]})).$$

Periodic points

For $\Phi_0^{[1]}$ continued fraction algorithm about $\mathbb{Q}$, which coincides with Schneider's continued fraction Bundschuh [1977] gave results, in which every rational number has an infinite periodic expansion or a finite expansion.

Proposition 1
Let K be $\mathbb{Q}$. Then, for every rational number α α has a finite $\Phi_0^{[-1]}$ continued fraction expansion.

$\Phi_0^{[-1]}$ continued fraction for quadratic cases

Next, we consider quadratic cases. Weger [1988] showed that some quadratic elements have a non periodic Schneider's continued fraction expansion. For $\Phi_0^{[-1]}$ continued fraction we can not expect that every quadratic element have a periodic continued fraction expansion from the numerical experiments (later we will show those), but we can not give the proof.

$\Phi_1^{[\epsilon,z]}$ continued fraction expansion for quadratic cases

Theorem 2

Let K be a quadratic field over $\mathbb{Q}$. Let $z \in K$ satisfy the condition (H) and $K = \mathbb{Q}(z)$. Then, for every rational number α α has a finite $\Phi_1^{[\epsilon,z]}$ continued fraction expansion. For every $\alpha \in K$ with $\alpha \notin \mathbb{Q}$, α has a periodic $\Phi_1^{[\epsilon,z]}$ continued fraction expansion.

$\Phi_3^{[\epsilon,z]}$ continued fraction for quadratic cases

Theorem 3
Let K be a quadratic field over $\mathbb{Q}$. Then, for every rational number α α has a finite $\Phi_3^{[\epsilon,z]}$ continued fraction expansion. For every $\alpha \in K$ with $\alpha \notin \mathbb{Q}$, α has a periodic $\Phi_3^{[\epsilon,z]}$ continued fraction expansion.

Cubic cases

Dubois, Paysant-Le Roux[1975] showed that for every real cubic number field there is a pair of numbers which has a periodic Jacobi-Perron expansion.

Let $K \subset \mathbb{Q}_p$ be a cubic field over $\mathbb{Q}$. Let $z \in K$ satisfy the condition (H) and $K = \mathbb{Q}(z)$. For an integer m which is relatively prime to p mz satisfies above condition. We suppose that z is integral over $\mathbb{Q}$. Let

$$x^3 + a_1x^2 + a_2x + a_3p^k \tag{1}$$

be the minimal polynomial of z, where $a_i \in \mathbb{Z}$ for $1 \leq i \leq 3$, $ord_p(a_2) = ord_p(a_3) = 0$.

Theorem 4

Let $\overline{\alpha} := (z, z^2)^T$. Then, $\overline{\alpha}$ has a periodic $\Phi_1^{[\epsilon,z]}$ continued fraction expansion.

Proof.

$$\overline{\alpha}_1 = \left(\epsilon z, \frac{-\epsilon z^2}{a_3} + \frac{-\epsilon a_1 z}{a_3}\right)^T,$$

$$\overline{\alpha}_2 = SH_1^{[\epsilon,z]}(\overline{\alpha}_1) = \left(\frac{-\epsilon z}{a_3}, \frac{-z^2}{a_3} + \frac{-a_1 z}{a_3}\right)^T,$$

$$\overline{\alpha}_3 = SH_1^{[\epsilon,z]}(\overline{\alpha}_2) = \left(z, z^2 + a_1 z\right)^T,$$

$$\overline{\alpha}_4 = SH_1^{[\epsilon,z]}(\overline{\alpha}_3) = \left(\epsilon z, \frac{-\epsilon z^2}{a_3} + \frac{-\epsilon a_1 z}{a_3}\right)^T.$$

Theorem 5
Let $K \subset \mathbb{Q}_p$ be a cubic field over $\mathbb{Q}$. There exists $z \in K$ which satisfies the condition (H) and $K = \mathbb{Q}(z)$ such that $(z, z^2)^T$ has a periodic $\Phi_2^{[-1,z],(1)}$ continued fraction expansion.

Multidimensional cases

Theorem 6
Let $K \subset \mathbb{Q}_p$ be a finite extension of $\mathbb{Q}$. Let $z \in K$ satisfy the condition (H) and $K = \mathbb{Q}(z)$. Let $u_1 := z$ and $u_i := \sum_{i \le j \le s} a_{ij} z^j$ for $2 \le i \le s$, where $a_{ij} \in \mathbb{Q} \cap \mathbb{Z}_p$ for $1 \le i \le s, i \le j \le s$ and $ord_p(a_{ii}) = 0$ for $1 \le i \le s$. Then, $\overline{\alpha} := (u_1, \ldots, u_s)^T$ has a periodic $\Phi_3^{[\epsilon,z]}$ continued fraction expansion.

Numerical Experiments

In Table 1 for the prime numbers p with $2 \leq p \leq 100$ and $\{z \in \mathbb{Q}_p | x^2 + ax + bp$ is the minimal polynomial of $z, a, b \in \mathbb{Z},\ 0 < a \leqq 10, -10 \leq b \leq 10, ord_p(a) = 0\}$ and 100 elements in D we observe periodicity by the $\Phi_0^{[-1,z]}$ continued fraction algorithm. The elements in D denoted by D' are generated by using pseudo-random number generation algorithm given by Saito, Yamaguchi [2016].

1^* is the number of the periodic points in D'. 2^* is the number of the points in D' such that the height of a remainder exceeds 10^{300} and we terminated the iteration.

Table 1 $\Phi_0^{[-1,z]}$, $z^2+az+bp=0$, $\alpha_1\in D'$

prime number	1*	2*	prime number	1*	2*
2	0	7800	43	0	20000
3	0	11700	47	0	20000
5	1	14399	53	0	20000
7	2	16598	59	0	20000
11	0	19000	61	0	20000
13	0	19200	67	0	20000
17	0	19600	71	0	20000
19	0	19800	73	0	20000
23	0	20000	79	0	20000
29	0	20000	83	0	20000
31	0	20000	89	0	20000
37	0	20000	97	0	20000
41	0	20000			

In Table 2 for the prime numbers p with $2 \leq p \leq 100$ and $\{z \in \mathbb{Q}_p | x^3 + ax + bp$ is the minimal polynomial of $z, a, b \in \mathbb{Z},\ 0 < a \leqq 10, -10 \leq b \leq 10, ord_p(a) = 0\}$ and 100 elements in D denoted by D' we observe periodicity by the $\Phi_1^{[-1,z]}$ continued fraction algorithm. 1^* is the number of the periodic points in D'. 2^* is the number of the points in D' such that the height of a remainder exceeds 10^{300} and we terminated the iteration.

Table 2 $\Phi_1^{[-1,z]}$, $z^3+az+bp=0$, $(\alpha_1,\alpha_2)\in D'$

prime number	1*	2*
2	7290	1110
3	10585	2015
5	12489	2311
7	14510	2690
11	16908	2292
13	17739	1461
17	18021	1379
19	17680	1920
23	18629	1171
29	18254	1546
31	18463	1337
37	19156	644
41	19314	486

prime number	1*	2*
43	18487	1313
47	19536	464
53	19298	502
59	19779	21
61	19319	681
67	19563	237
71	19407	393
73	19514	286
79	19979	21
83	19197	603
89	19171	629
97	19351	649

Table 3 $\Phi_2^{[-1,z],(1)}$, $z^3 + az + bp = 0$, $(\alpha_1, \alpha_2) \in D'$

prime number	1*	2*
2	8104	296
3	12157	443
5	14353	447
7	16933	267
11	18915	285
13	18978	222
17	19278	122
19	19495	105
23	19741	59
29	19756	44
31	19788	12
37	19724	76
41	19737	63

prime number	1*	2*
43	19721	79
47	19995	5
53	19766	34
59	19749	51
61	19979	21
67	19795	5
71	19793	7
73	19785	15
79	19998	2
83	19793	7
89	19791	9
97	19962	38

Table 4 $\Phi_2^{[-1,z],(2)}$, $z^3 + az + bp = 0$, $(\alpha_1, \alpha_2) \in D'$

prime number	1*	2*
2	8391	9
3	12599	1
5	14792	8
7	17200	1
11	19200	1
13	19199	1
17	19400	0
19	19600	0
23	19800	0
29	19800	0
31	19800	0
37	19800	0
41	19800	0

prime number	1*	2*
43	19800	0
47	20000	0
53	19800	0
59	19800	0
61	20000	0
67	19800	0
71	19800	0
73	19800	0
79	20000	0
83	19800	0
89	19800	0
97	20000	0

Table 5 $\Phi_3^{[-1,z]}$, $z^3 + az + bp = 0$, $(\alpha_1, \alpha_2) \in D'$

prime number	1^*	2^*	prime number	1^*	2^*
2	8398	2	43	19800	0
3	12600	0	47	20000	0
5	14800	0	53	19800	0
7	17200	0	59	19800	0
11	19200	0	61	20000	0
13	19200	0	67	19800	0
17	19400	0	71	19800	0
19	19600	0	73	19800	0
23	19800	0	79	20000	0
29	19800	0	83	19800	0
31	19800	0	89	19800	0
37	19800	0	97	20000	0
41	19800	0			

Table 6 $\Phi_3^{[-1,z]}$, $z^4+az+bp=0$, $(\alpha_1,\alpha_2,\alpha_3)\in D'$

prime number	1*	2*
2	8099	1
3	12897	3
5	15200	0
7	17400	0
11	19600	0
13	19800	0
17	19700	0
19	19900	0
23	19900	0
29	19800	0
31	19800	0
37	19800	0
41	20000	0

prime number	1*	2*
43	20000	0
47	20000	0
53	20000	0
59	19800	0
61	19800	0
67	19800	0
71	19900	0
73	19900	0
79	20000	0
83	20000	0
89	20000	0
97	20000	0

Table 7 $\Phi_3^{[-1,z]}$, $z^5 + az + bp = 0$, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in D'$

prime number	1^*	2^*	prime number	1^*	2^*
2	8798	2	43	19800	0
3	13199	1	47	20000	0
5	15199	1	53	20000	0
7	17600	0	59	20000	0
11	19600	0	61	20000	0
13	19800	0	67	20000	0
17	19800	0	71	20000	0
19	19800	0	73	19999	1
23	19800	0	79	20000	0
29	19800	0	83	19800	0
31	20000	0	89	19800	0
37	20000	0	97	20000	0
41	19800	0			

Table 8 $\Phi_3^{[-1,z]}$, $z^6+az+bp=0$, $(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)\in D'$

prime number	1*	2*	prime number	1*	2*
2	9000	0	43	20000	0
3	13400	0	47	20000	0
5	15499	1	53	20000	0
7	17600	0	59	20000	0
11	19600	0	61	20000	0
13	19800	0	67	20000	0
17	19900	0	71	20000	0
19	19900	0	73	20000	0
23	19900	0	79	19900	0
29	19900	0	83	19900	0
31	19900	0	89	20000	0
37	19900	0	97	20000	0
41	19900	0			

Impression

We has not got an algorithm from c-map for which we can expect that Lagrange Theorem holds. In contrast we have some multidimensional continued fraction algorithm(in $\mathbb{C}^n$) for which Lagrange Theorem holds or we can expect that Lagrange Theorem holds.
For example, Pethő, Pohst, Bertḱ[2017], Tamura,Yasutomi[2009].

We will extend c-map.

Extension of c-map

Let $\Phi(\overline{\alpha}) = (\phi(\overline{\alpha}), F_{\overline{\alpha}}, A_{\overline{\alpha}})$ for $\overline{\alpha} = (\alpha_1, \ldots, \alpha_s)^T \in D$ be a c-map and $\gamma : D \to p\mathbb{Z}_p \cap \mathbb{Q}$ be a map. We propose a p-adic continued fraction via Φ and γ as follows: Let $\overline{\alpha} = (\alpha_1, \ldots, \alpha_s)^T \in D$. We set $\overline{\alpha}^{(0)} := \overline{\alpha}$. $\overline{\alpha}^{(1)}, \overline{\alpha}^{(2)}, \ldots$ are defined inductively as follows: We suppose that $\overline{\alpha}^{(n)}$ for $n \in \mathbb{Z}_{\geq 0}$ is defined. We set $\overline{\alpha}^{(n+1)} := A_{\overline{\alpha}} F_{\overline{\alpha}}(\overline{\alpha}^{(n)}) + \gamma(\overline{\alpha})$. We say that $\overline{\alpha}$ has the affine c-map continued fraction expansion $\{\Phi'(\overline{\alpha}^{(0)}), \Phi'(\overline{\alpha}^{(1)}), \ldots\}$, where $\Phi'(\overline{\beta}) = (\phi(\overline{\beta}), F_{\overline{\beta}}, A_{\overline{\beta}}, \gamma(\beta))$ for $\overline{\beta} \in D$. We can easily see that Theorem 1 holds for an affine c-map.

Affine c-map

For the c-map $\Phi_3^{[\cdot,\epsilon,z]} := (1, H_1^{[\cdot,\epsilon,z]}, \tau_z(H_1^{[\cdot,\epsilon,z]}))$ we will define an affine c-map. Let $\overline{\alpha} = (\alpha_1, \ldots, \alpha_s)^T \in D$ and let $\overline{\beta} = (\beta_1, \ldots, \beta_s)^T := \tau_z(H_1^{[\epsilon,z]}(\overline{\alpha}))$. Then, there exists a unique $M = (m_{ij}) \in M_{s\times(s+1)}(\mathbb{Q})$ such that $\overline{\beta} = M(z, z^2, \ldots, z^s, 1)^T$. We define $\gamma'(\overline{\alpha}) \in E$ by

$$\gamma'(\overline{\alpha}) := (-\langle m_{1s+1}\rangle_p, \ldots, -\langle m_{ss+1}\rangle_p)^T.$$

We define the affine c-map $\Phi_{3A}^{[\epsilon,z]} := (1, H_1^{[\cdot,\epsilon,z]}, \tau_z(H_1^{[\cdot,\epsilon,z]}), \gamma'(\cdot))$.

Table 9 $\Phi_{3A}^{[-1,z]}$, $z^3 + az + bp = 0$, $(\alpha_1, \alpha_2) \in D'$

prime number	1*	2*	prime number	1*	2*
2	8400	0	43	19800	0
3	12600	0	47	20000	0
5	14800	0	53	19800	0
7	17200	0	59	19800	0
11	19200	0	61	20000	0
13	19200	0	67	19800	0
17	19400	0	71	19800	0
19	19600	0	73	19800	0
23	19800	0	79	20000	0
29	19800	0	83	19800	0
31	19800	0	89	19800	0
37	19800	0	97	20000	0
41	19800	0			

Table 10 $\Phi_3^{[-1,z]}$, $z^4+az+bp=0$, $(\alpha_1,\alpha_2,\alpha_3)\in D'$

prime number	1*	2*	prime number	1*	2*
2	8100	0	43	20000	0
3	12900	0	47	20000	0
5	15200	0	53	20000	0
7	17400	0	59	19800	0
11	19600	0	61	19800	0
13	19800	0	67	19800	0
17	19700	0	71	19900	0
19	19900	0	73	19900	0
23	19900	0	79	20000	0
29	19800	0	83	20000	0
31	19800	0	89	20000	0
37	19800	0	97	20000	0
41	20000	0			

Table 11 $\Phi_3^{[-1,z]}$, $z^5 + az + bp = 0$, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in D'$

prime number	1*	2*
2	8800	0
3	13200	0
5	15200	0
7	17600	0
11	19600	0
13	19800	0
17	19800	0
19	19800	0
23	19800	0
29	19800	0
31	20000	0
37	20000	0
41	19800	0

prime number	1*	2*
43	19800	0
47	20000	0
53	20000	0
59	20000	0
61	20000	0
67	20000	0
71	20000	0
73	20000	0
79	20000	0
83	19800	0
89	19800	0
97	20000	0

Table 12 $\Phi_{3A}^{[-1,z]}$, $z^6+az+bp=0$, $(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)\in D'$

prime number	1*	2*	prime number	1*	2*
2	9000	0	43	20000	0
3	13400	0	47	20000	0
5	15500	0	53	20000	0
7	17600	0	59	20000	0
11	19600	0	61	20000	0
13	19800	0	67	20000	0
17	19900	0	71	20000	0
19	19900	0	73	20000	0
23	19900	0	79	19900	0
29	19900	0	83	19900	0
31	19900	0	89	20000	0
37	19900	0	97	20000	0
41	19900	0			

Conjecture

Let p be any prime number and K be any finite extension of $\mathbb{Q}$ with $K \subset \mathbb{Q}_p$. Let $s+1$ be its degree over $\mathbb{Q}$ and $z \in K$ be any element with the condition (H) and $K = \mathbb{Q}(z)$. For every $\overline{\alpha} = (\alpha_1, \ldots, \alpha_s) \in D$ which satisfies that $1, \alpha_1, \ldots, \alpha_s$ is linearly independent over $\mathbb{Q}$ $\overline{\alpha}$ has a periodic $\Phi_{3A}^{[\epsilon,z]}$ continued fraction expansion, where $\epsilon \in \{-1, 1\}$.

Remark. The conjecture holds for $s = 1$.

Conjecture of Conjecture

The proof of Conjecture is not so difficult.

Thank you very much!