Random differences of arithmetic progressions in the primes

Tsz Ho Chan, Máté Wierdl (Both from University of Memphis)
Sunny morning in Marseille on May 24, 2017.

Plan

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- Motivation, history

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- Baby result

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- Proof of Baby result

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- Grownup results

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- Farewell

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Squares can be replaced by kth powers, so $p, p + d^k, p + 2d^k, \ldots, p + \ell \cdot d^k$ are all primes.

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In general, by the strong law of large numbers, R^ω is an infinite set iff $\sum_d \sigma_d = \infty$, and then $R^\omega(x) \sim \sum_{d \le x} \sigma_d$.

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$$\frac{1}{x} \sum_{x/2 \le d \le x} \frac{1}{x/\log^3 x} \sum_{n \le x} f(n) \cdot f(n+d) \cdot f(n+2d) > c > 0, \text{ for large } x.$$

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$$\frac{1}{\sum_{x/2 \le d \le x} \sigma_d} \sum_{x/2 \le d \le x} X_d(\omega) \cdot \frac{1}{x/\log^3 x} \sum_{n \le x} f(n) \cdot f(n+d) \cdot f(n+2d) \gg c,$$

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$$\frac{1}{\sum_{x/2\le d\le x}\sigma_d}\sum_{x/2\le n\le x} X_d(\omega)\cdot A_{x,d} \gg c \text{ with probability } 1.$$

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(Proof of Baby result cont'd)

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$$P\left(\left|\frac{1}{\sum_{x/2\le d\le x} \sigma_d} \sum_{x/2\le n\le x} (X_d(\omega) - \sigma_d)\cdot A_{x,d}\right| > \epsilon_x\right)$$

$$\le \frac{1}{\epsilon_x^2}\cdot \frac{(\log x)^4}{\cdot \sum_{x/2\le d\le x} \sigma_d} \le \frac{1}{\epsilon_x^2}\cdot \frac{(\log x)^4}{x^{1/2}}.$$

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$\liminf_{x\to\infty} \frac{\sum_{d\le x} \sigma_d}{(\log x)^{3+\epsilon}} > 0$. For $\ell = 2$, the theorem's assumption is
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Theorem: Random ℓ-intersector set

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Then the random sequence R^ω is an ℓ-intersector set: for any positive density subset A of the primes, there are infinitely many $p \in A$ and $r \in R^\omega$ for which the $\ell+1$ numbers $p, p+r, \ldots, p+\ell r$ are all in A.

This is not the best known for small ℓ. For $\ell = 1$, the theorem's assumption is that for some positive ϵ $\liminf_{x\to\infty} \frac{\sum_{d\le x}\sigma_d}{x^{1/2+\epsilon}} > 0$, so doesn't reach random squares, while we have
$\liminf_{x\to\infty} \frac{\sum_{d\le x}\sigma_d}{(\log x)^{3+\epsilon}} > 0$. For $\ell = 2$, the theorem's assumption is
$\liminf_{x\to\infty} \frac{\sum_{d\le x}\sigma_d}{x^{2/3+\epsilon}} > 0$, but enough to assume
$\liminf_{x\to\infty} \frac{\sum_{d\le x}\sigma_d}{x^{1/2+\epsilon}} > 0$.

(Grownup results cont'd)

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Conjecture: Sharpness for $\ell = 1$

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Farewell

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