On the largest prime factors of consecutive integers

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Definitions and notations

p, p_1, p_2, p' : prime number

ε : any sufficiently small positive constant

x : tend to infinity

$\varphi(n)$: the Euler function

$P^+(n)$: the largest prime factor of n
with the convention $P^+(1) = 1$

$P^-(n)$: the smallest prime factor of n
with the convention $P^-(1) = +\infty$

$P_y^+(n)$: the largest prime factor $p(\leqslant y)$ of n
with the convention $P_y^+(n) = 1$ if $P^-(n) > y$

Definition and notation

$\rho(u)$: the Dickman function defined as the unique continuous solution to the differential-difference equation

$$\begin{cases} \rho(u) = 1 & \text{if } \ 0 \leqslant u \leqslant 1, \\ u\rho'(u) = -\rho(u-1) & \text{if } \ u > 1. \end{cases}$$

$\omega(u)$: the Buchstab function defined as the unique continuous solution to the differential-difference equation

$$\begin{cases} u\omega(u) = 1 & \text{if } \ 1 \leqslant u \leqslant 2, \\ \{u\omega(u)\}' = \omega(u-1) & \text{if } \ u > 2. \end{cases}$$

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Introduction

Two fundamental structures : additive, multiplicative

Two typical examples of shifted number :

Fermat number : $2^{2^k}+1$, Twin prime numbers : $p, p+2$

Conjecture (De Koninck & Doyon hypothesis, 2011)

For any fixed integer $k \geqslant 2$ and n, let $a_1, a_2, \ldots, a_k$ be any permutation of the numbers $0, 1, \ldots, k-1$. Then

$$\mathrm{Prob}[P^+(n+a_1) < P^+(n+a_2) < \cdots < P^+(n+a_k)] = \frac{1}{k!},$$

i.e.,

$$\frac{1}{x}\big|\{n \leqslant x :\ P^+(n+a_1) < \cdots < P^+(n+a_k)\}\big| \to \frac{1}{k!}.$$

The simplest case $(k=2, a_1=0, a_2=1)$:

$$\big|\{n \leqslant x :\ P^+(n) < P^+(n+1)\}\big| \sim \frac{1}{2}x.$$

Three consecutive integers

Erdős & Pomerance(1978) :

(i). $P^+(n-1) > P^+(n) < P^+(n+1)$ for infinitely many n.

(ii). $P^+(n-1) < P^+(n) > P^+(n+1)$ for infinitely many n.

(iii). $P^+(n-1) < P^+(n) < P^+(n+1)$ for infinitely many n.

$$\left(n = p^{2^{k_0}}, \quad k_0 = \inf\{k : \ P^+(p^{2^k}+1) > p\}\right)$$

Balog(2001) :

(iv). $\left|\{n \leqslant x : \ P^+(n-1) > P^+(n) > P^+(n+1)\}\right| \gg x^{1/2}.$

$$(\text{Conjecture} : \sim \tfrac{1}{6}x)$$

Three consecutive integers

Theorem 1.

(i). $\left|\left\{n \leqslant x: P^+(n-1) > P^+(n) < P^+(n+1)\right\}\right| > 1.06 \times 10^{-7}x,$

(ii). $\left|\left\{n \leqslant x: P^+(n-1) < P^+(n) > P^+(n+1)\right\}\right| > 8.84 \times 10^{-4}x.$

Corollary 1. (Upper bounds of four patterns)

$$\left|\left\{n \leqslant x: P^+(n-1) > P^+(n) < P^+(n+1)\right\}\right| < 2x/3,$$

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Two consecutive integers

Conjecture 1. (Erdős-Pomerance, 1978) return

$$\left|\left\{n \leqslant x : \ P^{+}(n) < P^{+}(n+1)\right\}\right| \sim \frac{1}{2}x.$$

Erdős & Pomerance (1978) :

$$\left|\left\{n \leqslant x : \ P^{+}(n) < P^{+}(n+1)\right\}\right| > 0.0099x.$$

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Two consecutive integers

La Bretèche, Pomerance & Tenenbaum (2005) :

$$\left|\left\{n \leqslant x : \ P^+(n) < P^+(n+1)\right\}\right| > 0.05544x.$$

(Fouvry, $0.05544 \to 0.05866$)

Wang (2016) :

$$\left|\left\{n \leqslant x : \ P^+(n) < P^+(n+1)\right\}\right| > 0.1063x.$$

$P_y^+(n)$: $\max\{p | n : p \leqslant y\}$

Rivat (Theorem 2, 2001) : For $3 \leqslant y \leqslant \exp\left(\frac{\log x}{100 \log\log x}\right)$,

$$\left| \sum_{1 \leqslant n \leqslant x} f_y(n) \right| \ll x \exp\left(\frac{-\log x}{10 \log y}\right) \ll x(\log x)^{-10}$$

where

$$f_y(n) := \begin{cases} 1 & \text{if} \quad P_y^+(n+1) > P_y^+(n), \\ -1 & \text{if} \quad P_y^+(n+1) < P_y^+(n). \end{cases}$$

$$\Rightarrow \left| \left\{ n \leqslant x : P_y^+(n+1) > P_y^+(n) \right\} \right| \sim \frac{1}{2} x.$$

($y = x \Rightarrow$ Conjecture of Erdős & Pomerance)

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$P_y^+(n)$: $\max\{p|n : p \leqslant y\}$

Theorem 2.

For $y = x^\alpha$, $0 < \alpha \leqslant 1$, we have

$$\left|\left\{n \leqslant x : \ P_y^+(n+1) > P_y^+(n)\right\}\right| \geqslant C(\alpha)x$$

where $C(\alpha)$ is a positive constant.

($C(\alpha)$ has an explicit but complicated definition).

Three examples of values of $C(\alpha)$:

$$C\left(\frac{1}{3}\right) > 0.0506, \quad C\left(\frac{1}{2}\right) > 0.0914, \quad C\left(\frac{2}{3}\right) > 0.0948.$$

$P_y^+(n)$: $\max\{p|n : p \leqslant y\}$

In particular, $C(1) > 0.1356$ by taking $\alpha = 1$. So we have

Corollary 2.

$$\left|\left\{n \leqslant x : \ P^+(n+1) > P^+(n)\right\}\right| > 0.1356x.$$

So we have improved the previous constant 0.1063.

Remark : Under the Elliott-Halberstam conjecture and the Elliott-Halberstam conjecture for friable integers :

$$0.1356 \to 0.411$$

Several consecutive integers

Similar to the proofs of Theorem 1, we have

Theorem 3.

For any fixed integer $J \geqslant 3$ and $j_0 \in \{0, \dots, J-1\}$, we have

$$\left|\left\{n \leqslant x : P^+(n+j_0) = \min_{0 \leqslant j \leqslant J-1} P^+(n+j)\right\}\right| \geqslant C_3(J)x + o(x)$$

where

$$C_3(J) := \max_{0<\alpha<\frac{1}{2(J-1)}} \rho\left(\frac{1}{\alpha}\right)\left(\alpha \log \frac{1}{2\alpha(J-1)}\right)^{J-1} > 0.$$

Several consecutive integers

Theorem 4.

For any fixed $J \geqslant 3$ and $j_0 \in \{0, \ldots, J-1\}$, with "min" replaced by "max" we have

$$\left|\left\{n \leqslant x : P^+(n+j_0) = \max_{0 \leqslant j \leqslant J-1} P^+(n+j)\right\}\right| \geqslant C_4(J)x + o(x)$$

where

$$C_4(J) := \max_{\substack{\frac{2J-2}{2J-1} < \alpha < 1 \\ 1-\alpha \leqslant \beta < \gamma < \frac{\alpha}{2(J-1)}}} \left(\beta \log \frac{\gamma}{\beta}\right)^{J-1} \log \frac{1}{\alpha} > 0.$$

Tools for the proofs

- Theorems 1, 3 and 4
  - (i). a well adapted system of weights
  - (ii). the Bombieri-Vinogradov theorem and the Bombieri-Vinogradov theorem for friable integers
- Theorem 2
  - (i). two theorems of Bombieri-Vinogradov type
  - (ii). a well adapted system of weights
  - (iii). the Rosser-Iwaniec linear sieve

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Theorem of Bombieri-Vinogradov type for $S(x; y, z)$

Proposition 1.

For any $A > 0$, we have

$$\sum_{q \leqslant \frac{x^{1/2}}{(\log x)^B}} \max_{t \leqslant x} \max_{(a,q)=1} \left| \sum_{\substack{n \in S(t; y, z) \\ n \equiv a (\bmod q)}} 1 - \frac{1}{\varphi(q)} \sum_{\substack{n \in S(t; y, z) \\ (n,q)=1}} 1 \right| \ll \frac{x}{(\log x)^A}$$

uniformly for

$$2 \leqslant z \leqslant y \leqslant x \quad \text{and} \quad \exp\{(\log x)^{2/5+\varepsilon}\} \leqslant y \leqslant x,$$

where B and the implied constant depend on A, ε alone and

$$S(x; y, z) := \{n \leqslant x : \ p | n \Rightarrow p \notin (z, y]\}.$$

Motohashi's works

Let f be a complex valued arithmetic function, and let introduce the following properties.

$(\mathcal{A})$: $f(n) \ll \tau(n)^C$, where $\tau(n)$ is the divisor function, C is a fixed constant.

$(\mathcal{B})$: If the conductor of a non-principal character χ is $O((\log x)^D)$, then

$$\sum_{n \leqslant x} f(n)\chi(n) \ll x(\log x)^{-3D}$$

where D is an arbitrarily large constant.

Motohashi's works

$(\mathcal{C})$: Let

$$E_f(y;\, q, a) := \sum_{\substack{n \leqslant y \\ n \equiv a (\mathrm{mod}\, q)}} f(n) - \frac{1}{\varphi(q)} \sum_{\substack{n \leqslant y \\ (n,q)=1}} f(n),$$

then for any $A > 0$, there exists $B = B(A) > 0$ such that

$$\sum_{q \leqslant x^{1/2}/(\log x)^B} \max_{y \leqslant x} \max_{(a,q)=1} \left| E_f(y;\, q, a) \right| \ll \frac{x}{(\log x)^A}.$$

The constants $A, B = B(A), C, D$ depend only on f.

Motohashi's works

Lemma 1. (Motohashi, 1976)

Let f and g have the properties $(\mathcal{A})$, $(\mathcal{B})$ and $(\mathcal{C})$. Then the multiplicative convolution $f * g$ does so.

Proof of Proposition 1

Let λ be the indicator function of the set $S(x; y, z)$ and v_z, u_y be two arithmetic functions defined by

$$v_z(n) := \begin{cases} 1 & \text{si} \quad P^+(n) \leqslant z, \\ 0 & \text{sinon}, \end{cases}$$

and

$$u_y(n) := \begin{cases} 1 & \text{si} \quad P^-(n) > y, \\ 0 & \text{sinon}. \end{cases}$$

Then it suffices to prove that v_z, u_y have the properties $(\mathcal{A})$, $(\mathcal{B})$ and $(\mathcal{C})$, considering $\lambda = v_z * u_y$.

Proof of Proposition 1

v_z : $(\mathcal{A}) \rightarrow$ trivial

$(\mathcal{B}) \rightarrow$ Théorème 4, Fouvry & Tenenbaum, 1991

$(\mathcal{C}) \rightarrow$ Théorème 6, Fouvry & Tenenbaum, 1991

and

u_y : $(\mathcal{A}) \rightarrow$ trivial

$(\mathcal{B}) \rightarrow$ Theorem 1, Xuan, 2000

$(\mathcal{C}) \rightarrow$ Theorem(Satz), Wolke, 1973

$\Rightarrow$ Proposition 1 is proved.

Theorem of Bombieri-Vinogradov type with *well factorable* function

An arithmetic fonction λ is called to be of level Q and of finite order k if

$$\lambda(q) = 0 \quad (q > Q) \quad \text{et} \quad |\lambda(q)| \leqslant \tau_k(q) \quad (q \leqslant Q).$$

λ is called well factorable of level Q if for any $Q_1, Q_2 \geqslant 1$, $Q = Q_1 Q_2$, there exist two functions λ_1, λ_2 of levels Q_1, Q_2 and orders k respectively such that

$$\lambda = \lambda_1 * \lambda_2.$$

In addition, we define

$$\pi(x; \ell, a, q) := \sum_{\substack{\ell p \leqslant x \\ \ell p \equiv a (\mathrm{mod}\, q)}} 1.$$

Theorem of Bombieri-Vinogradov type with *well factorable* function

Proposition 2.

Let $a \in \mathbb{Z}^*$ and $A > 0$, then for any well factorable function $\lambda(q)$ of level Q, we have

$$\sum_{(a,q)=1} \lambda(q) \sum_{\substack{L_1 \leqslant \ell \leqslant L_2 \\ (\ell,q)=1}} \left(\pi(x;\, \ell, a, q) - \frac{\mathrm{li}(x/\ell)}{\varphi(q)} \right) \ll \frac{x}{(\log x)^A}$$

for

$$Q = x^{4/7-\varepsilon}, \quad 1 \leqslant L_1 \leqslant L_2 \leqslant x^{1-\varepsilon}.$$

The implied constant depend only on a, A and ε.

$L_1 = L_2 = 1 \Rightarrow$ Theorem of Bombieri-Friedlander-Iwaniec.

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Ideas of Theorem 1

- Theorem 1 (i) : $P^+(n-1) > P^+(n) < P^+(n+1)$

  Let $y = x^\alpha$, $\alpha > 0$

$$n \rightarrow P^+(n) \leqslant y$$
$$n-1, n+1 \rightarrow P^+(n-1) > y,\ P^+(n+1) > y$$

- Theorem 1 (ii) : $P^+(n-1) < P^+(n) > P^+(n+1)$

$$n \rightarrow n = mp,\ p \text{ is sufficiently large}$$
$$n-1, n+1 \rightarrow n-1 = p_1 n_1,\ n+1 = p_2 n_2,$$
$$x/p < p_1, p_2 < p$$

Theorem 1 (i) : $P^+(n-1) > P^+(n) < P^+(n+1)$

Let

$$S(x,y) := \{n \leqslant x : P^+(n) \leqslant y\}, \quad P(y,z) := \prod_{z<p\leqslant y} p,$$

and

$$\Psi(x,y;a,q) := \sum_{\substack{n\in S(x,y)\\ n\equiv a (\bmod q)}} 1, \qquad \Psi_q(x,y) := \sum_{\substack{n\in S(x,y)\\ (n,q)=1}} 1.$$

Then we have

$$\sum_{\substack{n\leqslant x\\ P^+(n-1)>P^+(n)<P^+(n+1)}} 1 \geqslant \sum_{\substack{n\in S(x,y)\\ (n\pm 1,\ P(x,y))>1}} 1.$$

Theorem 1 (i) : $P^+(n-1) > P^+(n) < P^+(n+1)$

In order to detect the condition $(n \pm 1,\ P(x,y)) > 1$, we introduce **a well adapted system of weights** :

$$\omega(n;\ y, z) := \sum_{\substack{z < p \leqslant y \\ p \mid n}} 1 \leqslant \frac{\log x}{\log z} \quad (n \leqslant x,\ \ z < y),$$

which implies

$$\left(\frac{\log x}{\log z}\right)^{-1} \omega(n;\ y, z) \begin{cases} \leqslant 1 & \text{if } \ (n,\ \ P(z,y)) > 1, \\ = 0 & \text{otherwise.} \end{cases}$$

Theorem 1 (i) : $P^+(n-1) > P^+(n) < P^+(n+1)$

The inequality of weights :

$$\sum_{\substack{n\leqslant x \\ P^+(n-1)>P^+(n)<P^+(n+1)}} 1 \geqslant \sum_{n\in S(x,y)} \frac{\omega(n-1;\,x,y)}{\left(\frac{\log x}{\log y}\right)} \cdot \frac{\omega(n+1;\,x,y)}{\left(\frac{\log x}{\log y}\right)}$$

$$\geqslant \alpha^2 \sum_{\substack{y<p_1\leqslant x \\ p_1p_2\leqslant x^{1/2}/(\log x)^B \\ p_1\neq p_2}} \sum_{y<p_2\leqslant x} \sum_{\substack{n\in S(x,y) \\ n\equiv 1\,(\mathrm{mod}\,p_1) \\ n\equiv -1\,(\mathrm{mod}\,p_2)}} 1$$

for $y = x^{\alpha}$, $\alpha < 1/4$. By Chinese remainder theorem, there exists $a < p_1p_2$ such that

$$n \equiv a\,(\mathrm{mod}\,p_1p_2).$$

Theorem 1 (i) : $P^+(n-1) > P^+(n) < P^+(n+1)$

So we have

$$\sum_{\substack{n\leqslant x \\ P^+(n-1)>P^+(n)<P^+(n+1)}} 1 \geqslant \alpha^2 \sum_{\substack{y<p_1\leqslant x \\ p_1p_2\leqslant x^{1/2}/(\log x)^B \\ p_1\neq p_2}} \sum_{y<p_2\leqslant x} \sum_{\substack{n\in S(x,y) \\ n\equiv a (\bmod p_1p_2)}} 1$$

$$= \alpha^2(\mathcal{S}_1 + \mathcal{S}_2),$$

where

$$\mathcal{S}_1 := \sum_{\substack{y<p_1\leqslant x \\ p_1p_2\leqslant x^{1/2}/(\log x)^B,\ p_1\neq p_2}} \sum_{y<p_2\leqslant x} \frac{\Psi_{p_1p_2}(x,\ y)}{\varphi(p_1p_2)},$$

$$\mathcal{S}_2 := \sum_{\substack{y<p_1\leqslant x \\ p_1p_2\leqslant x^{1/2}/(\log x)^B,\ p_1\neq p_2}} \sum_{y<p_2\leqslant x} \left(\Psi(x,y;\ a,p_1p_2) - \frac{\Psi_{p_1p_2}(x,y)}{\varphi(p_1p_2)}\right).$$

Theorem 1 (i) : $P^+(n-1) > P^+(n) < P^+(n+1)$

For the error term $\mathcal{S}_2$ and any $A > 0$, by the following lemma 2 of Wolke or Fouvry & Tenenbaum for the friable integers, combined with the Cauchy-Schwarz inequality

Lemma 2.

For $x \geqslant y \geqslant 2$

$$\sum_{q \leqslant x^{1/2}/(\log x)^B} \max_{z \leqslant x} \max_{(a,q)=1} \left| \Psi(z, y;\, a, q) - \frac{\Psi_q(z, y)}{\varphi(q)} \right| \ll_A \frac{x}{(\log x)^A}$$

we have

$$\mathcal{S}_2 \ll x(\log x)^{-A}.$$

Theorem 1 (i) : $P^+(n-1) > P^+(n) < P^+(n+1)$

For the principal term $\mathcal{S}_1$, by a result of Hildebrand for the friable integers we obtain

$$\mathcal{S}_1 = x\rho\Big(\frac{1}{\alpha}\Big)\alpha\int_1^{\frac{1}{2\alpha}-1}\frac{\log t}{\frac{1}{2}-\alpha t}\,\mathrm{d}t + o(x) \qquad (\alpha < 1/4).$$

So Theorem 1 (i) is proved :

$$\sum_{\substack{n\leqslant x\\ P^+(n-1)>P^+(n)<P^+(n+1)}} 1 \geqslant x\rho\Big(\frac{1}{\alpha}\Big)\alpha^3\int_1^{\frac{1}{2\alpha}-1}\frac{\log t}{\frac{1}{2}-\alpha t}\,\mathrm{d}t + o(x)$$

$$> 1.063\times 10^{-7}x$$

by taking $\alpha \approx \frac{1}{4.6}$, with the help of *Mathematica 9.0*.

Theorem 1 (i) : $P^+(n-1) > P^+(n) < P^+(n+1)$

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So Theorem 1 (i) is proved :

$$\sum_{\substack{n\leqslant x\\ P^+(n-1)>P^+(n)<P^+(n+1)}} 1 \geqslant x\rho\Big(\frac{1}{\alpha}\Big)\alpha^3 \int_1^{\frac{1}{2\alpha}-1} \frac{\log t}{\frac{1}{2}-\alpha t}\,\mathrm{d}t + o(x)$$

$$> 1.063 \times 10^{-7} x$$

by taking $\alpha \approx \frac{1}{4.6}$, with the help of *Mathematica 9.0*.

Theorem 1 (ii) : $P^+(n-1) < P^+(n) > P^+(n+1)$

Let α, β, γ be three parameters with

$$4/5 < \alpha \leqslant 1, \quad 1-\alpha \leqslant \beta < \gamma < \alpha/4.$$

Given an integer m and $p_1 \neq p_2$ satisfying

$$1 \leqslant m \leqslant x^{1-\alpha}, \quad x^\beta < p_1,\ p_2 \leqslant x^\gamma,$$

we consider the congruence system :

$$mp - 1 \equiv 0 \pmod{p_1}, \quad mp + 1 \equiv 0 \pmod{p_2}.$$

Chinese remainder theorem implies that $p \equiv b \pmod{p_1 p_2}$.

Then for these m and $p > x^\alpha$, we have

$$P^+(mp-1) < P^+(mp) > P^+(mp+1).$$

Theorem 1 (ii) : $P^+(n-1) < P^+(n) > P^+(n+1)$

So we also get an inequality of weights

$$\sum_{\substack{n\leqslant x \\ P^+(n)>P^+(n-1) \\ P^+(n)>P^+(n+1) \\ x^\alpha<P^+(n)\leqslant x}} 1 \geqslant \sum_{m\leqslant x^{1-\alpha}} \sum_{x^\alpha<p\leqslant \frac{x}{m}} \prod_{i=\{1,-1\}} \frac{\omega(mp+i;\, x^\gamma, x^\beta)}{\left(\frac{\log x}{\log x^\beta}\right)}$$

$$= \beta^2 \sum_{m\leqslant x^{1-\alpha}} \sum_{\substack{x^\beta<p_1,\, p_2\leqslant x^\gamma \\ p_1\neq p_2}} \sum_{\substack{x^\alpha<p\leqslant \frac{x}{m} \\ p\equiv b \pmod{p_1 p_2}}} 1$$

$$= \beta^2(\mathbb{S}_1 + \mathbb{S}_2),$$

Theorem 1 (ii) : $P^+(n-1) < P^+(n) > P^+(n+1)$

where

$$\mathbb{S}_1 := \sum_{m \leqslant x^{1-\alpha}} \sum_{x^\beta < p_1,\, p_2 \leqslant x^\gamma,} \sum_{p_1 \neq p_2} \frac{\pi(x/m) - \pi(x^\alpha)}{\varphi(p_1 p_2)}$$

and

$$\mathbb{S}_2 := \sum_{m \leqslant x^{1-\alpha}} \sum\sum_{\substack{x^\beta < p_1,\, p_2 \leqslant x^\gamma \\ p_1 \neq p_2}} \Big\{ \pi(x/m;\, b, p_1 p_2) - \frac{\pi(x/m)}{\varphi(p_1 p_2)} - \Big(\pi(x^\alpha;\, b, p_1 p_2) - \frac{\pi(x^\alpha)}{\varphi(p_1 p_2)} \Big) \Big\}.$$

For $\mathbb{S}_2$, by the theorem of Bombieri-Vinogradov we have

$$\mathbb{S}_2 \ll x(\log x)^{-A} \qquad (\gamma < \alpha/4).$$

Theorem 1 (ii) : $P^+(n-1) < P^+(n) > P^+(n+1)$

where

$$\mathbb{S}_1 := \sum_{m \leqslant x^{1-\alpha}} \sum_{x^\beta < p_1, p_2 \leqslant x^\gamma,} \sum_{p_1 \neq p_2} \frac{\pi(x/m) - \pi(x^\alpha)}{\varphi(p_1 p_2)}$$

and

$$\mathbb{S}_2 := \sum_{m \leqslant x^{1-\alpha}} \sum\sum_{\substack{x^\beta < p_1, p_2 \leqslant x^\gamma \\ p_1 \neq p_2}} \bigg\{ \pi(x/m;\, b, p_1 p_2) - \frac{\pi(x/m)}{\varphi(p_1 p_2)} - \bigg(\pi(x^\alpha;\, b, p_1 p_2) - \frac{\pi(x^\alpha)}{\varphi(p_1 p_2)} \bigg) \bigg\}.$$

For $\mathbb{S}_2$, by the theorem of Bombieri-Vinogradov we have

$$\mathbb{S}_2 \ll x(\log x)^{-A} \qquad (\gamma < \alpha/4).$$

Theorem 1 (ii) : $P^+(n-1) < P^+(n) > P^+(n+1)$

For $\mathbb{S}_1$, we can calculate that

$$\mathbb{S}_1 = x\Big(\beta \log \frac{\gamma}{\beta}\Big)^2 \log \frac{1}{\alpha} + o(x)$$
$$> 7 \times 10^{-4} x$$

with

$$\alpha \approx 0.895, \quad \beta \approx 0.105, \quad \gamma \approx 0.22375.$$

So we have the following lower bound

$$\sum_{\substack{n \leqslant x \\ P^+(n-1) < P^+(n) > P^+(n+1) \\ x^{0.895} < P^+(n) \leqslant x}} 1 > 7 \times 10^{-4} x.$$

Theorem 1 (ii) : $P^+(n-1) < P^+(n) > P^+(n+1)$

With the same method we have

$$\sum_{\substack{n\leqslant x \\ P^+(n-1)<P^+(n)>P^+(n+1) \\ x^{0.835}<P^+(n)\leqslant x^{0.895}}} 1 > 1.84\times 10^{-4}x,$$

and so Theorem 1 (ii) is proved

$$\sum_{\substack{n\leqslant x \\ P^+(n-1)<P^+(n)>P^+(n+1) \\ x^{0.835}<P^+(n)\leqslant x}} 1 > 8.84\times 10^{-4}x.$$

Corollary 1 : Upper bounds of four patterns

We note for $a_1(x) + a_2(x) + a_3(x) + a_4(x) = [x]$

$$a_1(x) := \left|\left\{n \leqslant x : \ P^+(n-1) > P^+(n) < P^+(n+1)\right\}\right|,$$
$$a_2(x) := \left|\left\{n \leqslant x : \ P^+(n-1) < P^+(n) > P^+(n+1)\right\}\right|,$$
$$a_3(x) := \left|\left\{n \leqslant x : \ P^+(n-1) < P^+(n) < P^+(n+1)\right\}\right|,$$
$$a_4(x) := \left|\left\{n \leqslant x : \ P^+(n-1) > P^+(n) > P^+(n+1)\right\}\right|.$$

Corollary 2 implies that

$$[x] - a_2(x) - a_3(x) > 0.1356x, \ [x] - a_2(x) - a_4(x) > 0.1356x.$$

In view of the lower bound of $a_2(x)$ in Theorem 1 (ii), we get

$$a_3(x), \ a_4(x) < (0.8644 - 8.84 \times 10^{-4})x.$$

Corollary 1 : Upper bounds of four patterns

We note for $a_1(x) + a_2(x) + a_3(x) + a_4(x) = [x]$

$$a_1(x) := \left|\left\{n \leqslant x : \ P^+(n-1) > P^+(n) < P^+(n+1)\right\}\right|,$$

$$a_2(x) := \left|\left\{n \leqslant x : \ P^+(n-1) < P^+(n) > P^+(n+1)\right\}\right|,$$

$$a_3(x) := \left|\left\{n \leqslant x : \ P^+(n-1) < P^+(n) < P^+(n+1)\right\}\right|,$$

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In view of the lower bound of $a_2(x)$ in Theorem 1 (ii), we get

$$a_3(x), \ a_4(x) < (0.8644 - 8.84 \times 10^{-4})x.$$

Corollary 1 : Upper bounds of four patterns

Lemma 3. (De Koninck & Doyon, 2011)

We have

$$\sum_{n \leqslant x} \delta(n)^{-1} > 2x/3 + o(x)$$

where

$$\delta(n) := \min_{\substack{m \neq n \\ P^+(m) \leqslant P^+(n)}} |m - n|.$$

By Lemma 3,

$$\frac{2}{3}x < \sum_{n \leqslant x} \delta(n)^{-1} \leqslant a_2(x) + a_3(x) + a_4(x) + \frac{a_1(x)}{2}.$$

So we have $a_1(x) < 2x/3$.

Corollary 1 : Upper bounds of four patterns

Very similar to the proof of Lemma 3, we can deduce that

$$\sum_{n \leqslant x} \delta_*(n)^{-1} > 2x/3 + o(x)$$

where

$$\delta_*(n) := \min_{\substack{m \neq n \\ P^+(m) \geqslant P^+(n)}} |m - n|.$$

Just like $a_1(x)$, we have

$$a_2(x) < 2x/3.$$

Proofs of Theorems 3 and 4

The proof of Theorem 3, for the n such

$$P^+(n+j_0) = \min_{0 \leqslant j \leqslant J-1} P^+(n+j)$$

is similar to that of Theorem 1 (i).

The proof of Theorem 4, for the n such

$$P^+(n+j_0) = \max_{0 \leqslant j \leqslant J-1} P^+(n+j)$$

is similar to that of Theorem 1 (ii).

Proof of Theorem 2 : $P_y^+(n+1) > P_y^+(n)$

- Theorem 2 : $y = x^{\alpha},\ \alpha \in (0,\ 1/2]$
  - (i). a well adapted system of weights
  - (ii). Proposition 1 for $S(x;\ y, z)$
- Theorem 2 : $y = x^{\alpha},\ \alpha \in (1/2,\ 1]$
  - (i). a well adapted system of weights
  - (ii). Proposition 2 for *well factorable* function
  - (iii). the Rosser-Iwaniec linear sieve

Proof of Theorem 2 : $\alpha \in (0, 1/2]$

For $\alpha \in (0, 1/2]$, the proof is very similar to that of Theorem 1 (i). The difference is the following error term :

$$\mathrm{TE} := \sum_{z<p\leqslant y} \Big(\sum_{\substack{n\in S(x;y,z)\\ n\equiv -1 \pmod p}} 1 - \frac{1}{\varphi(p)} \sum_{\substack{n\in S(x;y,z)\\ (n,p)=1}} 1 \Big),$$

where

$$y = x^{\alpha},\ z = x^{\beta} \quad \text{avec} \quad 0 < \beta < \alpha \leqslant 1/2.$$

By Proposition 1, the error term is admissible.

Proof of Theorem 2 : $\alpha \in (1/2, 1]$

For $\alpha \in (1/2, 1]$ and a parameter $c \in [1-\alpha, \frac{1}{2}]$, we have

$$\sum_{\substack{n\leqslant x \\ P_y^+(n)<P_y^+(n+1)}} 1 = \sum_{\substack{n\leqslant x \\ y\geqslant P^+(n+1)>x^{1-c}}} 1 - \sum_{\substack{n\leqslant x \\ y\geqslant P^+(n)>P^+(n+1)>x^{1-c}}} 1$$

$$+ \sum_{\substack{n\leqslant x \\ P^+(n)<P^+(n+1)\leqslant x^{1-c}}} 1$$

$$=: \mathscr{S}_A - \mathscr{S}_B + \mathscr{S}_C.$$

Next we shall evaluate $\mathscr{S}_A$, $\mathscr{S}_B$ and $\mathscr{S}_C$ separately.

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Proof of Theorem 2 : estimation of $\mathscr{S}_A$

By the following formula of Hildebrand

$$\Psi(x,y) := \Psi_1(x,y) = x\rho(u)\left\{1 + O_\varepsilon\left(\frac{\log(u+1)}{\log y}\right)\right\}$$

for

$$\exp\{(\log_2 x)^{5/3+\varepsilon}\} \leqslant y \leqslant x, \quad u = \log x / \log y,$$

and

$$\rho(u) = 1 - \log u \quad (1 \leqslant u \leqslant 2),$$

we have

$$\mathscr{S}_A = x\left\{\rho\left(\frac{\log x}{\log y}\right) - \rho\left(\frac{\log x}{\log x^{1-c}}\right)\right\} + o(x)$$

$$= x\log\left(\frac{\alpha}{1-c}\right) + o(x).$$

Proof of Theorem 2 : estimation of $\mathscr{S}_B$

For $\mathscr{S}_B$, we use the Rosser-Iwaniec sieve.

$$\mathscr{S}_B \leqslant \left|\left\{n \leqslant x : \ n = ap = bp' - 1, \ x^{1-\alpha} < a < b \leqslant x^c\right\}\right| + o(x)$$

$$\leqslant \sum_{x^{1-\alpha} < b \leqslant x^c} \left|\left\{n \in \mathscr{A}(b) : n \text{ is prime}\right\}\right| + o(x),$$

where

$$\mathscr{A}(b) := \left\{\frac{ap+1}{b} : \ ap \leqslant x, \ x^{1-\alpha} < a < b, \ ap \equiv -1 \pmod{b}\right\}.$$

Then we shall sieve the sequence $\mathscr{A}(b)$.

Proof of Theorem 2 : estimation of $\mathscr{S}_B$

Some definitions of Rosser-Iwaniec sieve :

$$\mathcal{A} : \text{ finite sequence of integers}$$

$$\mathcal{P} : \text{ set of primes}$$

$$z : z \geqslant 2$$

$$\mathcal{A}_d := \{a \in \mathcal{A} : d \mid a\}$$

$$P_{\mathcal{P}}(z) := \prod_{p<z,\, p\in\mathcal{P}} p$$

$$S(\mathcal{A}; \mathcal{P}, z) := |\{a \in \mathcal{A} : (a, P_{\mathcal{P}}(z)) = 1\}|$$

Proof of Theorem 2 : estimation of $\mathscr{S}_B$

For $d \mid P_{\mathcal{P}}(z)$, we suppose

$$|\mathcal{A}_d| = \frac{w(d)}{d}X + r(\mathcal{A}, d)$$

where $X \sim |\mathcal{A}|$ and $w(d)$ is multiplicative verifying

$$\begin{cases} 0 < w(p) < p \quad (p \in \mathcal{P}) \\ \prod_{u<p\leqslant v} \left(1 - \frac{w(p)}{p}\right)^{-1} \leqslant \frac{\log v}{\log u}\left(1 + \frac{K}{\log u}\right) \end{cases}$$

In addition, we define

$$V(z) := \prod_{p<z,\, p\in\mathcal{P}} \left(1 - \frac{w(p)}{p}\right).$$

Proof of Theorem 2 : estimation of $\mathscr{S}_B$

Lemma 4. (Rosser-Iwaniec sieve)

For $D^{1/2} \geqslant z \geqslant 2$, we have

$$S(\mathcal{A}; \mathcal{P}, z) \leqslant XV(z)\Big\{F\Big(\frac{\log D}{\log z}\Big) + E\Big\} + \sum_{\ell < \exp(8/\varepsilon^3)} \sum_{d | P_{\mathcal{P}}(z)} \lambda_\ell^+(d) r(\mathcal{A}, d),$$

where

$$F(s) = \frac{2\mathrm{e}^\gamma}{s} \ (0 < s \leqslant 3), \quad E = O\Big(\varepsilon + \frac{\mathrm{e}^K (\log D)^{-\frac{1}{3}}}{\varepsilon^8}\Big).$$

$\lambda_\ell^+(d)$, the Rosser-Iwaniec weights with $|\lambda_\ell^+(d)| \leqslant 1$, denote a well factorable coefficient of level D and ordre 1.

Proof of Theorem 2 : estimation of $\mathscr{S}_B$

Take $D = z^2 = x^{4/7-\varepsilon}/b$ in Lemma 4, then we have

$$S(\mathscr{A}(b); \mathcal{P}, z) \leqslant \{1 + o(1)\} \frac{2X}{\log(x^{4/7-\varepsilon}/b)} + \sum_{\ell < \exp(8/\varepsilon^3)} \sum_{d<D,\, d|P(z)} \lambda_\ell^+(d)\, r\big(\mathscr{A}(b), d\big),$$

and so that

$$\mathscr{S}_B \leqslant \sum_{x^{1-\alpha}<b\leqslant x^c} \big(S(\mathscr{A}(b); \mathcal{P}, z) + z\big) \leqslant \big\{1 + o(1)\big\}\mathscr{S}_{B1} + \mathscr{S}_{B2} + O\big(x(\log x)^{-1}\big),$$

Proof of Theorem 2 : estimation of $\mathscr{S}_B$

where

$$\mathscr{S}_{B1} := \sum_{x^{1-\alpha} < b \leqslant x^c} \frac{2x}{b \log(x^{4/7-\varepsilon}/b)} \log\left(\frac{\alpha \log x}{\log(x/b)}\right),$$

$$\mathscr{S}_{B2} := \sum_{\ell < \exp(8/\varepsilon^3)} \sum_{x^{1-\alpha} < b \leqslant x^c} \sum_{\substack{d < D \\ d \mid P(z)}} \lambda_\ell^+(d)\, r(\mathscr{A}(b), d).$$

Proof of Theorem 2 : estimation of $\mathscr{S}_B$

To evaluate the error term $\mathscr{S}_{B2}$, we define λ_ℓ by

$$\lambda_\ell(q) := \sum_{\substack{\mathcal{B}/2<b\leqslant\mathcal{B} \\ bd=q}} \sum_{d<D,\, d|P(z)} \mathbb{1}_{]\mathcal{B}/2,\,\mathcal{B}]}(b)\,\lambda_\ell^+(d),$$

where

$$\mathbb{1}_{]\mathcal{B}/2,\,\mathcal{B}]}(b) = \begin{cases} 1 & \text{if } \mathcal{B}/2 < b \leqslant \mathcal{B}, \\ 0 & \text{otherwise.} \end{cases}$$

We can deduce that λ_ℓ is well factorable of level $x^{4/7-\varepsilon}$ if we impose the condition $c \leqslant 2/7 - \varepsilon$.

Proof of Theorem 2 : estimation of $\mathscr{S}_B$

$\mathscr{S}_{B2}$ is admissible

$$\mathscr{S}_{B2} \ll x(\log x)^{-B}$$

for

$$\begin{cases} D = \frac{x^{4/7-\varepsilon}}{b}, & 0 < c \leqslant \frac{2}{7} - \varepsilon \text{ (Proposition 2)}, \\ D = \frac{x^{1/2}}{b(\log x)^B}, & \frac{2}{7} - \varepsilon < c < \frac{1}{2} \text{ (Pan-Ding-Wang)}. \end{cases}$$

$\mathscr{S}_{B1}$: partial summation

Proof of Theorem 2 : estimation of $\mathscr{S}_B$

So for $y = x^\alpha, 1 - c \leqslant \alpha \leqslant 1$, $\mathscr{S}_B$ is majorized by

$$\mathscr{S}_B \leqslant \begin{cases} 2x \int_{1-\alpha}^{c} \log\left(\frac{\alpha}{1-t}\right) \frac{\mathrm{d}t}{4/7-t} + o(x) & 0 < c \leqslant \frac{2}{7} - \varepsilon, \\ 2x \int_{1-\alpha}^{c} \log\left(\frac{\alpha}{1-t}\right) \frac{\mathrm{d}t}{1/2-t} + o(x) & \frac{2}{7} - \varepsilon < c < \frac{1}{2}. \end{cases}$$

Proof of Theorem 2 : estimation of $\mathscr{S}_C$

For $\mathscr{S}_C$, by the same method as Theorem 1 (i), we have

$$\mathscr{S}_C \geqslant x\vartheta_0\left(\frac{\delta}{\alpha}, \frac{1}{\alpha}\right)\delta \log \frac{1}{2\delta} + o(x),$$

where δ is a parameter with $c \leqslant \delta \leqslant 1/2$.

Proof of Theorem 2 : $1/2 < \alpha \leqslant 1$

So for $y = x^\alpha$, $\alpha \in (1/2, 1]$, combine $\mathscr{S}_A, \mathscr{S}_B, \mathscr{S}_C$ and we have

$$\left|\left\{n \leqslant x : \ P_y^+(n+1) > P_y^+(n)\right\}\right| \geqslant C_2(\alpha)x$$

where $C_2(\alpha) > 0$ is a constant.

Proof of Theorem 2

Finally, we get Theorem 2 :

$$\left|\left\{n \leqslant x : \; P_y^+(n+1) > P_y^+(n)\right\}\right| \geqslant C(\alpha)x$$

for $y = x^\alpha$, $0 < \alpha \leqslant 1$, where

$$C(\alpha) := \begin{cases} C_1(\alpha) & \text{if } \; 0 < \alpha \leqslant 1/2, \\ C_2(\alpha) & \text{if } \; 1/2 < \alpha \leqslant 1. \end{cases}$$

Thank you !