2-adic valuations of coefficients of certain integer powers of formal power series with $\{-1, +1\}$ coefficients

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Short plan of the presentation

- Basic definitions and the main question

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- The Prouhet-Thue-Morse sequence and the binary partition function

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- Some general results
- Applications to the Rudin-Shapiro sequence and the Lafrance-Rampersad-Yee sequence
- Questions, problems and conjectures

In the sequel we will use the following notation:

- $\mathbb{N}$ - the set of non-negative integers,
- $\mathbb{N}_+$ - the set of positive integers,
- $\mathbb{P}$ - the set of prime numbers,
- $A_{\geq k}$ - the set $\{n \in A : \ n \geq k\}$.

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If $p \in \mathbb{P}$ and $n \in \mathbb{Z}$ we define the p-adic valuation of n as:

$$\nu_p(n) := \max\{k \in \mathbb{N} : \ p^k \mid n\}.$$

We also adopt the standard convention that $\nu_p(0) = +\infty$.

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From the definition we easily deduce that for each $n_1, n_2 \in \mathbb{Z}$ the following properties hold:

$$\nu_p(n_1 n_2) = \nu_p(n_1) + \nu_p(n_2) \quad \text{and} \quad \nu_p(n_1 + n_2) \geq \min\{\nu_p(n_1), \nu_p(n_2)\}.$$

If $\nu_p(n_1) \neq \nu_p(n_2)$ then the inequality can be replaced by the equality.

Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{Z}[[x]]$$

and

$$g(x) = \sum_{n=0}^{\infty} b_n x^n \in \mathbb{Z}[[x]]$$

be a formal power series with integer coefficients.

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Let $M\in\mathbb{N}_{\geq 2}$ be given. We say that f, g are *congruent modulo* M if and only if for all n the coefficients of x^n in both series are congruent modulo M.

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In other words

$$f \equiv g \pmod{M} \Longleftrightarrow \forall n \in \mathbb{N}: \ a_n \equiv b_n \pmod{M}.$$

One can prove that for any given $f, F, g, G \in \mathbb{Z}[[x]]$ satisfying

$$f \equiv g \pmod{M} \quad \text{and} \quad F \equiv G \pmod{M}$$

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Moreover, if $f(0), g(0) \in \{-1, 1\}$ then the series $1/f, 1/g$ have integer coefficients and we also have

$$\frac{1}{f} \equiv \frac{1}{g} \pmod{M}.$$

In consequence, in this case we have

$$f^k \equiv g^k \pmod{M}$$

for any $k \in \mathbb{Z}$.

We formulate the following general

Question 1

Let $f(x)=\sum_{n=0}^{\infty}\varepsilon_n x^n\in\mathbb{Z}[[x]]$ *with* $\varepsilon_0\in\{-1,1\}$ *and take* $m\in\mathbb{N}_+$*. What can be said about the sequences* $(\nu_p(a_m(n)))_{n\in\mathbb{N}}, (\nu_p(b_m(n)))_{n\in\mathbb{N}}$*, where*

$$f(x)^m=\left(\sum_{n=0}^{\infty}\varepsilon_n x^n\right)^m=\sum_{n=0}^{\infty}a_m(n)x^n,$$

$$\frac{1}{f(x)^m}=\left(\frac{1}{\sum_{n=0}^{\infty}\varepsilon_n x^n}\right)^m=\sum_{n=0}^{\infty}b_m(n)x^n,$$

i.e., $a_m(n)$ $(b_m(n))$ *is the n-th coefficient in the power series expansion of the series* $f^m(x)$ $(1/f(x)^m$ *respectively)?*

It is clear that in its full generality, the Question 1 is too difficult and we cannot expect that the sequences $(\nu_p(a_m(n)))_{n\in\mathbb{N}}$ $(\nu_p(b_m(n)))_{n\in\mathbb{N}}$ can be given in closed form or even that a reasonable description can be obtained. Indeed, in order to give an example let us consider the formal power series

$$f(x)=\prod_{n=1}^{\infty}(1-x^n)=1+\sum_{n=1}^{\infty}(-1)^n(x^{\frac{n(3n-1)}{2}}+x^{\frac{n(3n+1)}{2}}).$$

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In particular $a_1(n)\in\{-1,0,1\}$ and thus for any given $p\in\mathbb{P}$ we have $\nu_p(a_1(n))=0$ in case when n is of the form $n=\frac{m(3m\pm 1)}{2}$ for some $m\in\mathbb{N}_+$, and $\nu_p(a_1(n))=\infty$ in the remaining cases.

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However, the characterization of the 2-adic behaviour of the sequence $(p(n))_{n\in\mathbb{N}}$ given by

$$\frac{1}{f(x)}=\prod_{n=1}^{\infty}\frac{1}{1-x^n}=1+\sum_{n=1}^{\infty}p(n)x^n$$

is unknown. Let us note that the number $p(n)$ counts the integer partitions of n, i.e., the number of non-negative integer solutions of the equation $\sum_{i=1}^{n}x_i=n$. In fact, even the proof that $\nu_2(p(n))>0$ infinitely often is quite difficult (this was proved by Kolberg in 1959).

The Prouhet-Thue-Morse sequence and the binary partition function

Let $n \in \mathbb{N}$ and $n = \sum_{i=0}^{k} \varepsilon_i 2^i$ be the unique expansion of n in base 2 and define the sum of digits function

$$s_2(n) = \sum_{i=0}^{k} \varepsilon_i .$$

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Next, we define the Prouhet-Thue-Morse sequence $\mathbf{t} = (t_n)_{n \in \mathbb{N}}$ (on the alphabet $\{-1, +1\}$) in the following way

$$t_n = (-1)^{s_2(n)},$$

i.e., $t_n = 1$ if the number of 1's in the binary expansion of n is even and $t_n = -1$ in the opposite case. We will call the sequence $\mathbf{t}$ as the PTM sequence in the sequel.

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From the relations

$$s_2(0) = 0, \quad s_2(2n) = s_2(n), \quad s_2(2n+1) = s_2(n) + 1$$

we deduce the recurrence relations for the PTM sequence: $t_0 = 1$ and

$$t_{2n} = t_n, \quad t_{2n+1} = -t_n.$$

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Let us also note that the (multiplicative) inverse of the series T, i.e.,

$$B(x) = \frac{1}{T(x)} = \prod_{n=0}^{\infty} \frac{1}{1-x^{2^n}} = \sum_{n=0}^{\infty} b_n x^n$$

is an interesting object.

Indeed, for $n \in \mathbb{N}$, the number b_n counts the number of binary partitions of n. The binary partition is the representation of the integer n in the form

$$n = \sum_{i=0}^{n} u_i 2^i,$$

where $u_i \in \mathbb{N}$ for $i = 0, \ldots, n$.

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The sequence $(b_n)_{n\in\mathbb{N}}$ was introduced by Euler. However, it seems that the first nontrivial result concerning its arithmetic properties was obtained by Churchhouse. He proved that $\nu_2(b_n) \in \{1, 2\}$ for $n \geq 2$.

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More precisely, $b_0 = 1, b_1 = 1$ and for $n \geq 2$ we have $\nu_2(b_n) = 2$ if and only if n or $n - 1$ can be written in the form $4^r(2u + 1)$ for some $r \in \mathbb{N}_+$ and $u \in \mathbb{N}$. In the remaining cases we have $\nu_2(b_n) = 1$.

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We can compactly write

$$\nu_2(b_n) = \begin{cases} \frac{1}{2}|t_n + 6t_{n-1} + t_{n-2}|, & \text{if } n \geq 2 \\ 0, & \text{if } n \in \{0, 1\}. \end{cases}$$

In other words we have simple characterization of the 2-adic valuation of the number b_n for all $n \in \mathbb{N}$.

Let $m \in \mathbb{N}_+$ and consider the series

$$B_m(x) := B(x)^m = \prod_{n=0}^{\infty} \frac{1}{(1-x^{2^n})^m} = \sum_{n=0}^{\infty} b_m(n)x^n.$$

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$$b_m(n) = \sum_{i_1+i_2+\ldots+i_m=n} \prod_{k=1}^{m} b_1(i_k),$$

i.e., $b_m(n)$ is the Cauchy convolution of m-copies of the sequence $(b_n)_{n\in\mathbb{N}}$. For $m \in \mathbb{N}_+$ we denote the sequence $(b_m(n))_{n\in\mathbb{N}}$ by $\mathbf{b}_m$.

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We see that the number $b_m(n)$ has a natural combinatorial interpretation. Indeed, $b_m(n)$ counts the number of representations

$$n = \sum_{i=0}^{n} u_i 2^i,$$

where each summand can have one of m colors.

Now we can formulate the natural

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Let $m \in \mathbb{N}_+$ be given. What can be said about the sequence $(\nu_2(b_m(n)))_{n\in\mathbb{N}}$?

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Lemma 1

Let $m \in \mathbb{N}_+$ be fixed and write $m = 2^k(2u+1)$ with $k \in \mathbb{N}$. Then:

1. *We have $b_m(n) \equiv \binom{m}{n} + 2^{k+1}\binom{m-2}{n-2} \pmod{2^{k+2}}$ for m even;*
2. *We have $b_m(n) \equiv \binom{m}{n} \pmod{2}$ for m odd;*
3. *For infinitely many n we have $b_m(n) \not\equiv 0 \pmod{4}$ for m odd.*

Lemma 2

Let m be a positive integer ≥ 2. *Then*

$$\binom{2^m-1}{k} \equiv 1 \pmod{2}, \quad \textit{for} \quad k = 0, 1, \ldots, 2^m - 1,$$

and

$$\binom{2^m}{k} \equiv \begin{cases} 1 & \textit{for } k = 0, 2^m \\ 4 & \textit{for } k = 2^{m-2}, 3 \cdot 2^{m-2} \\ 6 & \textit{for } k = 2^{m-1} \\ 0 & \textit{ in the remaining cases} \end{cases} \pmod{8}, \quad \textit{for} \quad k = 0, 1, \ldots, 2^m.$$

We are ready to prove the following

Theorem 3

Let $k \in \mathbb{N}_+$ be given. Then $\nu_2(b_{2^k-1}(n)) = 0$ for $n \leq 2^k - 1$ and

$$\nu_2(b_{2^k-1}(2^k n + i)) = \nu_2(b_1(2n))$$

for each $i \in \{0, \ldots, 2^k - 1\}$ and $n \in \mathbb{N}_+$.

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Proof: First of all, let us observe that the second part of Lemma 1 and the first part of Lemma 2 implies that $b_{2^k-1}(n)$ is odd for $n \leq 2^k - 1$ and thus $\nu_2(b_{2^k-1}(n)) = 0$ in this case.

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Let us observe that from the identity $B_{2^k-1}(x) = T(x)B_{2^k}(x)$ we get the relation

$$b_{2^k-1}(n) = \sum_{j=0}^{n} t_{n-j} b_{2^k}(j), \tag{1}$$

where t_n is the n-th term of the PTM sequence.

Now let us observe that from the first part of Lemma 1 and the second part of Lemma 2 we have

$$b_{2^k}(n) \equiv \binom{2^k}{n} \pmod{8}$$

for $n = 0, 1, \ldots, 2^k$ and $b_{2^k}(n) \equiv 0 \pmod{8}$ for $n > 2^k$, provided $k \geq 2$ or $n \neq 2$.

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Moreover,

$$b_2(2) \equiv \binom{2}{2} + 4\binom{0}{0} = 5 \pmod 8.$$

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Moreover,

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Summing up this discussion we have the following expression for $b_{2^k-1}(n) \pmod{8}$, where $k \geq 2$ and $n \geq 2^k$:

$$\begin{aligned} b_{2^k-1}(n) &= \sum_{j=0}^{n} t_{n-j} b_{2^k}(j) = \sum_{j=0}^{2^k} t_{n-j} b_{2^k}(j) + \sum_{j=2^k+1}^{n} t_{n-j} b_{2^k}(j) \\ &\equiv \sum_{j=0}^{2^k} t_{n-j} b_{2^k}(j) \equiv \sum_{j=0}^{2^k} t_{n-j} \binom{2^k}{j} \pmod{8} \\ &\equiv t_n + t_{n-2^k} + 4t_{n-2^{k-2}} + 4t_{n-3\cdot 2^{k-2}} + 6t_{n-2^{k-1}} \pmod{8}. \end{aligned}$$

However, it is clear that $t_{n-2^{k-2}} + t_{n-3\cdot 2^{k-2}} \equiv 0 \pmod 2$ and thus we can simplify the above expression and get

$$b_{2^k-1}(n) \equiv t_n + t_{n-2^k} + 6t_{n-2^{k-1}} \pmod 8$$

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If $k = 1$ and $n \geq 2$ then, analogously, we get

$$b_1(n) \equiv \sum_{j=0}^{2^k} t_{n-j} b_{2^k}(j) \pmod 8 \equiv t_n + 5t_{n-2} + 2t_{n-1} \pmod 8$$

and since $t_{n-1} \equiv t_{n-2} \pmod 2$, we thus conclude that

$$b_1(n) \equiv t_n + 6t_{n-1} + t_{n-2} \pmod 8.$$

Let us put

$$R_k(n) = t_n + t_{n-2^k} + 6t_{n-2^{k-1}}.$$

Using now the recurrence relations for t_n, i.e., $t_{2n} = t_n, t_{2n+1} = -t_n$, we easily deduce the identities

$$R_k(2n) = R_{k-1}(n), \quad R_k(2n+1) = -R_{k-1}(n)$$

for $k \geq 2$.

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for $k \geq 2$.

Using a simple induction argument, one can easily obtain the following identities:

$$|R_k(2^k m + j)| = |R_1(2m)| \tag{2}$$

for $k \geq 2, m \in \mathbb{N}$ and $j \in \{0, \ldots, 2^k - 1\}$.

From the above identity we easily deduce that

$$R_k(n) \not\equiv 0 \pmod{8}$$

for each $n \in \mathbb{N}$ and each $k \geq 1$. If $k = 1$ then $R_1(n) = t_n + 6t_{n-1} + t_{n-2}$ and $R_1(n) \equiv 0 \pmod{8}$ if and only if $t_n = t_{n-1} = t_{n-2}$. However, a well known property of the PTM sequence is that there are no three consecutive terms which are equal.

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If $k \geq 2$ then our statement about $R_k(n)$ is clearly true for $n \leq 2^k$. If $n > 2^k$ then we can write $n = 2^k m + j$ for some $m \in \mathbb{N}$ and $j \in \{0, 1, \ldots, 2^k - 1\}$. Using the reduction (2) and the property obtained for $k = 1$, we get the result.

From the above identity we easily deduce that

$$R_k(n) \not\equiv 0 \pmod{8}$$

for each $n \in \mathbb{N}$ and each $k \geq 1$. If $k = 1$ then $R_1(n) = t_n + 6t_{n-1} + t_{n-2}$ and $R_1(n) \equiv 0 \pmod{8}$ if and only if $t_n = t_{n-1} = t_{n-2}$. However, a well known property of the PTM sequence is that there are no three consecutive terms which are equal.

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Summing up: we have proved that $\nu_2(b_{2^k-1}(n)) \leq 2$ for each $n \in \mathbb{N}$, since $\nu_2(b_1(n)) \in \{0, 1, 2\}$. Moreover, as an immediate consequence of our reasoning we get the equality

$$\nu_2(b_{2^k-1}(2^k n + j)) = \nu_2(b_1(2n))$$

for $j \in \{0, ..., 2^k - 1\}$ and our theorem is proved.

Some general results

Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of integers and write

$$f(x)=\sum_{n=0}^{\infty}\varepsilon_n x^n\in\mathbb{Z}[[x]].$$

Moreover, for $m\in\mathbb{N}_+$ we define the sequence $\mathbf{b}_m=(b_m(n))_{n\in\mathbb{N}}$, where

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Theorem 4

Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of integers and suppose that $\varepsilon_n\equiv 1 \pmod{2}$ for each $n\in\mathbb{N}$. Then for any $m\in\mathbb{N}_+$ and $n\geq m$ we have the congruence

$$b_{m-1}(n)\equiv\sum_{i=0}^{m}\binom{m}{i}\varepsilon_{n-i}\pmod{2^{\nu_2(m)+1}}. \tag{3}$$

Proof: Let $f(x)=\sum_{n=0}^{\infty}\varepsilon_n x^n\in\mathbb{Z}[[x]]$. From the assumption on sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ we get that

$$f(x)\equiv\frac{1}{1+x}\pmod{2}.$$

In consequence, writing $m=2^{\nu_2(m)}k$ with k odd, and using the well known property saying that $U\equiv V\pmod{2^k}$ implies $U^2\equiv V^2\pmod{2^{k+1}}$, we get the congruence

$$\frac{1}{f(x)^m}\equiv(1+x)^m\pmod{2^{\nu_2(m)+1}}.$$

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Thus, multiplying both sides of the above congruence by $f(x)$ we get

$$\frac{1}{f(x)^{m-1}}\equiv f(x)(1+x)^m\pmod{2^{\nu_2(m)+1}}.$$

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From the power series expansion of $f(x)(1+x)^m$ by comparing coefficients on the both sides of the above congruence we get that

$$b_{m-1}(n)\equiv\sum_{i=0}^{\min\{m,n\}}\binom{m}{i}\varepsilon_{n-i}\pmod{2^{\nu_2(m)+1}},$$

i.e., for $n\geq m$ we get the congruence (3). Our theorem is proved.

From our result we can deduce the following

Corollary 5

Let $(\varepsilon_n)_{n\in\mathbb{N}}$ *be a non-eventually constant sequence,* $\varepsilon_n \in \{-1, 1\}$ *for each* $n \in \mathbb{N}$*, and suppose that for each* $N \in \mathbb{N}_+$ *there are infinitely many* $n \in \mathbb{N}$ *such that* $\varepsilon_n = \varepsilon_{n+1} = \ldots = \varepsilon_{n+N}$*. Then, for each even* $m \in \mathbb{N}_+$*, there are infinitely many* $n \in \mathbb{N}$ *such that*

$$\nu_2(b_{m-1}(n)) \geq \nu_2(m) + 1 \quad \text{and} \quad \nu_2(b_{m-1}(n+1)) = 1.$$

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$$\nu_2(b_{m-1}(n))\geq \nu_2(m)+1 \quad \text{and} \quad \nu_2(b_{m-1}(n+1))=1.$$

Proof: From our assumption on the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ we can find infinitely many $(m+1)$-tuples such that $\varepsilon_{n+1}=\varepsilon, \varepsilon_n=\ldots=\varepsilon_{n-m}=-\varepsilon$, where ε is a fixed element of $\{-1,1\}$. We apply (3) and get

$$b_{m-1}(n)\equiv\sum_{i=0}^{m}\binom{m}{i}\varepsilon_{n-i}\equiv-\sum_{i=0}^{m}\binom{m}{i}\varepsilon\equiv-\varepsilon 2^m\equiv 0 \pmod{2^{\nu_2(m)+1}},$$

$$b_{m-1}(n+1)\equiv\sum_{i=0}^{m}\binom{m}{i}\varepsilon_{n+1-i}\equiv 2\varepsilon-\sum_{i=0}^{m}\binom{m}{i}\varepsilon\equiv\varepsilon(2-2^m)\equiv 2\varepsilon \pmod{2^{\nu_2(m)+1}}.$$

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In consequence $\nu_2(b_{m-1}(n))\geq\nu_2(m)+1$ and $\nu_2(b_{m-1}(n+1))=1$.

Example: Let $F : \mathbb{N} \to \mathbb{N}$ satisfy the condition

$$\limsup_{n \to +\infty} (F(n+1) - F(n)) = +\infty$$

and define the sequence

$$\varepsilon_n(F) = \begin{cases} 1 & n = F(m) \text{ for some } m \in \mathbb{N} \\ -1 & \text{otherwise} \end{cases} .$$

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It is clear that the sequence $(\varepsilon_n(F))_{n \in \mathbb{N}}$ satisfies the conditions from Theorem 5 and thus for any even $m \in \mathbb{N}_+$ there are infinitely many $n \geq m$ such that $\nu_2(b_{m-1}(n)) \geq \nu_2(m) + 1$ and $\nu_2(b_{m-1}(n+1)) = 1$.

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A particular examples of F's satisfying required properties include:

- positive polynomials of degree ≥ 2;
- the functions which for given $n \in \mathbb{N}_+$ take as value the n-th prime number of the form $ak + b$, where $a \in \mathbb{N}_+$, $b \in \mathbb{Z}$ and $\gcd(a, b) = 1$;
- and many others.

Lemma 6

Let $s \in \mathbb{N}_{\geq 3}$. *Then*

$$\binom{2^s}{i} \pmod{16} \equiv \begin{cases} 1 & \text{for } i = 0, 2^s \\ 6 & \text{for } i = 2^{s-1} \\ 8 & \text{for } i = (2j+1)2^{s-3}, j \in \{0,1,2,3\} \\ 12 & \text{for } i = 2^{s-2}, 3 \cdot 2^{s-2} \\ 0 & \text{in the remaining cases} \end{cases} .$$

Theorem 7

Let $s \in \mathbb{N}_+$ and $(\varepsilon_n)_{n\in\mathbb{N}}$ be an integer sequence and suppose that $\varepsilon_n \equiv 1 \pmod{2}$ for $n \in \mathbb{N}$.

(A) *For $n \geq 2^s$ we have*

$$b_{2^s-1}(n) \equiv \varepsilon_n + 2\varepsilon_{n-2^{s-1}} + \varepsilon_{n-2^s} \pmod{4}. \tag{4}$$

In particular, if $\varepsilon_n \in \{-1, 1\}$ for all $n \in \mathbb{N}$ then:

$$\nu_2(b_{2^s-1}(n)) > 1 \iff \varepsilon_n = \varepsilon_{n-2^{s-1}} = \varepsilon_{n-2^s} \text{ or } \varepsilon_n = -\varepsilon_{n-2^{s-1}} = \varepsilon_{n-2^s}$$
$$\nu_2(b_{2^s-1}(n)) = 1 \iff \varepsilon_n = -\varepsilon_{n-2^s}.$$

(B) *For $s \geq 2$ and $n \geq 2^s$ we have*

$$b_{2^s-1}(n) \equiv \varepsilon_n + 6\varepsilon_{n-2^{s-1}} + \varepsilon_{n-2^s} \pmod{8}. \tag{5}$$

In particular, if $\varepsilon_n \in \{-1, 1\}$ for all $n \in \mathbb{N}$, then:

$$\nu_2(b_{2^s-1}(n)) > 2 \iff \varepsilon_n = \varepsilon_{n-2^{s-1}} = \varepsilon_{n-2^s}$$
$$\nu_2(b_{2^s-1}(n)) = 2 \iff \varepsilon_n = -\varepsilon_{n-2^{s-1}} = \varepsilon_{n-2^s}$$
$$\nu_2(b_{2^s-1}(n)) = 1 \iff \varepsilon_n = -\varepsilon_{n-2^s}.$$

Theorem 7 (continuation)

(C) *For $s \geq 3$ and $n \geq 2^s$ we have*

$$b_{2^s-1}(n) \equiv \varepsilon_n + \varepsilon_{n-2^s} + 6\varepsilon_{n-2^{s-1}} + 12(\varepsilon_{n-2^{s-2}} + \varepsilon_{n-3\cdot 2^{s-2}}) \pmod{16} \quad (6)$$

In particular, if $\varepsilon_n \in \{-1, 1\}$ for all $n \in \mathbb{N}$, then:

$$\begin{aligned}
\nu_2(b_{2^s-1}(n)) > 3 \quad &\iff \quad \varepsilon_n = \varepsilon_{n-2^{s-2}} = \varepsilon_{n-2^{s-1}} = \varepsilon_{n-3\cdot 2^{s-2}} = \varepsilon_{n-2^s} \quad \text{or} \\
&\qquad\quad \varepsilon_n = -\varepsilon_{n-2^{s-2}} = \varepsilon_{n-2^{s-1}} = -\varepsilon_{n-3\cdot 2^{s-2}} = \varepsilon_{n-2^s}; \\
\nu_2(b_{2^s-1}(n)) = 3 \quad &\iff \quad \varepsilon_n = \varepsilon_{n-2^{s-2}} = \varepsilon_{n-2^{s-1}} = -\varepsilon_{n-3\cdot 2^{s-2}} = \varepsilon_{n-2^s} \quad \text{or} \\
&\qquad\quad \varepsilon_n = -\varepsilon_{n-2^{s-2}} = \varepsilon_{n-2^{s-1}} = \varepsilon_{n-3\cdot 2^{s-2}} = \varepsilon_{n-2^s} \\
&\iff \quad \varepsilon_n \equiv -\varepsilon_{n-2^s} + 2\varepsilon_{n-2^{s-1}} + 8 \pmod{16}
\end{aligned} \quad (7)$$

As a first application of Theorem 7 we get the following:

Corollary 8

Let $s \in \mathbb{N}_{\geq 2}$ *and* $(\varepsilon_n)_{n\in\mathbb{N}}$ *with* $\varepsilon_n \in \{-1, 1\}$ *for all* $n \in \mathbb{N}$. *If there is no* $n \in \mathbb{N}_{\geq 2^s}$ *such that* $\varepsilon_n = \varepsilon_{n-2^{s-1}} = \varepsilon_{n-2^s}$, *then*

$$\nu_2(b_{2^s-1}(n)) = \nu_2(\varepsilon_n + 6\varepsilon_{n-2^{s-1}} + \varepsilon_{n-2^s}).$$

In particular, for each $n \in \mathbb{N}_{\geq 2^s}$ *we have* $\nu_2(b_{2^s-1}(n)) \in \{1, 2, 3\}$.

Let $\mathbf{r} = (r_n)_{n\in\mathbb{N}}$ be the Rudin-Shapiro sequence (the RS sequence for short), i.e., the sequence defined as

$$r_n = (-1)^{u_n},$$

where u_n is the number of occurrences of the word "11" in the binary expansion of the number n.

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One can easily check that the sequence $\mathbf{r}$ satisfies the following recurrence relations: $r_0 = 1$ and

$$r_{2n} = r_n, \quad r_{2n+1} = (-1)^n r_n$$

for $n \in \mathbb{N}$.

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It is well known that the formal power series $R(x) = \sum_{n=0}^{\infty} r_n x^n$ associated with the sequence $\mathbf{r}$ satisfies the following functional equation:

$$R(x) = (1-x)R(x^2) + 2xR(x^4).$$

Let $m \in \mathbb{N}_+$ and write

$$\frac{1}{R(x)^m} = \sum_{n=0}^{\infty} b_m(n)x^n.$$

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We prove boundedness of the 2-adic valuation of $b_m(n)$ for $m = 2$ and $m = 2^s - 1$ with $s \in \mathbb{N}_{\geq 2}$. The first step needed in the proof is the following:

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Lemma 9

The following congruence holds:

$$\frac{1}{R(x)} \equiv \frac{\sqrt{(1+x)(1-x-x^2-3x^3)}}{1+x} \pmod{4}.$$

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To get the above result it is enough to write $1/R(x) = 1 + x + 2T(x)$ and observe that T satisfies the congruence

$$(1+x)^4 T(x)^2 + (1+x)^5 T(x) + x(1+x)^3(1+x+x^2) \equiv 0 \pmod{2}.$$

As a consequence of the above result we get: As a simple consequence of the above result we get:

Corollary 10

Let $1/R(x)^2 = \sum_{n=0}^{\infty} b_2(n)x^n$. *Then*

$$\nu_2(b_2(n)) = \begin{cases} 0 & n = 0, 2 \\ 1 & n = 1 \\ 2 & n \geq 3 \end{cases}.$$

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We note the congruence

$$\sum_{n=0}^{\infty} b_2(n)x^n = \frac{1}{R(x)^2} \equiv \frac{1 - x - x^2 - 3x^3}{1 + x} \equiv 1 - 2x + x^2 + \sum_{n=3}^{\infty} 4(-1)^n x^n \pmod{8}.$$

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Lemma 11

Let $(r_n)_{n \in \mathbb{N}}$ *be the Rudin-Shapiro sequence. Then there is no* $n \in \mathbb{N}_{\geq 4}$ *such that*

$$r_n = r_{n-1} = r_{n-2} = r_{n-3} = r_{n-4} \quad \text{or} \quad r_n = -r_{n-1} = r_{n-2} = -r_{n-3} = r_{n-4}.$$

Theorem 12

Let $s \in \mathbb{N}_{\geq 2}$, $R(x) = \sum_{n=0}^{\infty} r_n x^n$ *and write*

$$\frac{1}{R(x)^m} = \sum_{n=0}^{\infty} b_m(n) x^n.$$

Then for $n \geq 2^s$ *we have* $\nu_2(b_{2^s-1}(n)) \in \{1, 2, 3\}$*. Moreover, the following formula holds*

$$\nu_2(b_{2^s-1}(n)) = \nu_2(r_n + 6r_{n-2^{s-1}} + r_{n-2^s}).$$

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Sketch of the proof: First we consider the case $s = 2$, i.e. $m = 3$. From Lemma 9 we have

$$\frac{1}{R(x)^4} \equiv 1 - 4x + 6x^2 + 4x^3 + 9x^4 \pmod{16}.$$

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$$\frac{1}{R(x)^4} \equiv 1 - 4x + 6x^2 + 4x^3 + 9x^4 \pmod{16}.$$

In consequence

$$\frac{1}{R(x)^3} \equiv R(x)(1 - 4x + 6x^2 + 4x^3 + 9x^4)$$

$$\equiv 1 + 13x + 3x^2 + 5x^3 + 8x^4 + \sum_{n=5}^{\infty} h_n x^n \pmod{16},$$

where $h_n := r_n - 4r_{n-1} + 6r_{n-2} + 4r_{n-3} + 9r_{n-4}$.

One can check that

$$h_n \equiv 0 \pmod{16} \iff r_n = r_{n-1} = r_{n-2} = r_{n-3} = r_{n-4}$$
$$\text{or} \quad r_n = -r_{n-1} = r_{n-2} = -r_{n-3} = r_{n-4},$$

which according to Lemma 11 is impossible.

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which according to Lemma 11 is impossible.

Thus h_n does not vanish modulo 16 and

$$h_n = r_n - 4(r_{n-1} - r_{n-3}) + 6r_{n-2} + 9r_{n-4} \equiv r_n + 6r_{n-2} + r_{n-4} \pmod{8}.$$

In consequence, due to non-vanishing of the integer $r_n + 6r_{n-2} + r_{n-4}$ we get that $\nu_2(b_3(n)) = \nu_2(r_n + 6r_{n-2} + r_{n-4})$.

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We proceed by induction on s and $n \geq 2^s$. For $s = 3$ and $n \geq 8$ we have

$$b_7(n) \equiv r_n + r_{n-8} + 6r_{n-4} + 12(r_{n-2} + r_{n-6}) \pmod{16}$$

and careful analysis shows that the right side doesn't vanish modulo 16.

One can check that

$$h_n \equiv 0 \pmod{16} \iff r_n = r_{n-1} = r_{n-2} = r_{n-3} = r_{n-4}$$
$$\text{or} \quad r_n = -r_{n-1} = r_{n-2} = -r_{n-3} = r_{n-4},$$

which according to Lemma 11 is impossible.

Thus h_n does not vanish modulo 16 and

$$h_n = r_n - 4(r_{n-1} - r_{n-3}) + 6r_{n-2} + 9r_{n-4} \equiv r_n + 6r_{n-2} + r_{n-4} \pmod{8}.$$

In consequence, due to non-vanishing of the integer $r_n + 6r_{n-2} + r_{n-4}$ we get that $\nu_2(b_3(n)) = \nu_2(r_n + 6r_{n-2} + r_{n-4})$.

We proceed by induction on s and $n \geq 2^s$. For $s = 3$ and $n \geq 8$ we have

$$b_7(n) \equiv r_n + r_{n-8} + 6r_{n-4} + 12(r_{n-2} + r_{n-6}) \pmod{16}$$

and careful analysis shows that the right side doesn't vanish modulo 16.

Similarly, for $s \geq 4, n \geq 2^s$, one can show impossibility of the conditions

$$\begin{array}{lccccccccc} C_1(n,s): & r_n & = & r_{n-2^{s-2}} & = & r_{n-2^{s-1}} & = & r_{n-3\cdot 2^{s-2}} & = & r_{n-2^s}, \\ C_2(n,s): & r_n & = & -r_{n-2^{s-2}} & = & r_{n-2^{s-1}} & = & -r_{n-3\cdot 2^{s-2}} & = & r_{n-2^s} \end{array}$$

and get the result.

Theorem 13

Let $s \in \mathbb{N}_{\geq 2}$ *and write* $\mathcal{H}_s(x) = \sum_{n=2^s}^{\infty} R_s(n)x^n$*, where*

$$R_s(n) = \nu_2(r_n + 6r_{n-2^{s-1}} + r_{n-2^s}).$$

Then $\mathcal{H}_2$ *satisfies the following Mahler type functional equation*

$$P(x) + Q(x)\mathcal{H}_2(x) + R(x)\mathcal{H}_2(x^2) = 0,$$

where

$$\begin{aligned} P(x) = x^4(&3 + 5x + 9x^2 + 12x^3 + 9x^4 + 13x^5 + 12x^6 + 12x^7 + 8x^8 + 4x^9 + 7x^{10} + 12x^{11} \\ &+ 11x^{12} + 13x^{13} + 13x^{14} + 12x^{15} + 12x^{16} + 13x^{17} + 12x^{18} + 12x^{19} + 6x^{20} + 4x^{21} \\ &+ 9x^{22} + 12x^{23} + 11x^{24} + 13x^{25} + 10x^{26} + 12x^{27} + 11x^{28} + 13x^{29} + 13x^{30} + 12x^{31} \\ &+ 9x^{32} + 8x^{33} + 3x^{34}) \\ Q(x) = (&x-1)(x+1)^2(x^4+1)(x^8+1)(x^{16}+1)(x^4+3x^2+1) \\ R(x) = (&x-1)(x+1)(x^4+1)(x^8+1)(x^{16}+1)(x^2+3x+1). \end{aligned}$$

Moreover, for $s \geq 3$ *we have*

$$\mathcal{H}_s(x) = \frac{1 - x^{2^{s-2}}}{1 - x}\mathcal{H}_2(x^{2^{s-2}}).$$

Corollary 14

Let $s \in \mathbb{N}$. Then

$$R_{s+2}(2^s n) = R_2(n) \quad \text{and} \quad R_{s+2}(2^s n - i) = R_2(n-1) \quad \text{for} \quad i \in \{1, \ldots, 2^s - 1\}$$

for $n \geq 5$.

Corollary 14

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for $n \geq 5$.

The form of the functional equation for $\mathcal{H}_2$ allows to deduce the following:

Corollary 15

For $s \in \mathbb{N}_{\geq 2}$ *the series* $\mathcal{H}_s(x)$ *is transcendental over* $\mathbb{Q}(x)$. *In particular, the sequence* $(R_s(n))_{n \in \mathbb{N}_{\geq 2^s}}$ *is not periodic.*

Application to the Lafrance-Rampersad-Yee sequence

Let $n \in \mathbb{N}$ and denote by $\mathrm{inv}_2(n)$ the number of occurrences of the word "10" as a scattered subsequence of the representation of n in base 2. For example $13 = 2^3 + 2^2 + 2^0 = (1101)_2$ and thus $\mathrm{inv}_2(13) = 2$. Recently, Lafrance, Rampersad and Yee introduced the sequence $\mathbf{j} = (j_n)_{n \in \mathbb{N}}$, where

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In the sequel the sequence $(j_n)_{n\in\mathbb{N}}$ will be called the LRY sequence for short. We have the following recurrence relation:

$$j_0 = 1, \quad j_{2n} = t_n j_n, \quad j_{2n+1} = j_n,$$

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Defining now $J(x) := \sum_{n=0}^{\infty} j_n x^n$ it is possible to prove that J satisfies the following functional equation

$$J(x) + x(x-1)J(x^2) - (1+x^4)J(x^4) = 0.$$

Lemma 16

Let $(j_n)_{n\in\mathbb{N}}$ *be the LRY sequence. Then there is no* $n \in \mathbb{N}$ *such that* $j_n = j_{n-1} = j_{n-2}$.

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For $m \in \mathbb{N}_+$ let us write

$$\frac{1}{J(x)^m} = \sum_{n=0}^{\infty} b_m(n)x^n.$$

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Theorem 17

Let $s \in \mathbb{N}_{\geq 2}$. Then for $n \geq 2^s$ we have $\nu_2(b_{2^s-1}(n)) \in \{1, 2\}$ and

$$\nu_2(b_{2^s-1}(n)) = \nu_2(L_s(n)), \text{ where } \quad L_s(n) := j_n + 6j_{n-2^{s-1}} + j_{n-2^s}.$$

Theorem 18

Let $s \in \mathbb{N}$ *and write* $\mathcal{J}_s(x) = \sum_{n=0}^{\infty} J_s(n)x^n$*, where*

$$J_s(n) = \nu_2(j_n + 6j_{n-2^s-1} + j_{n-2^s}).$$

Then $\mathcal{J}_i, i = 2, 3$*, satisfies the following Mahler type functional equations*

$$P_i(x) + Q_i(x)\mathcal{J}_i(x) + R_i(x)\mathcal{J}_i(x^2) + S_i(x)\mathcal{J}_i(x^4) = 0,$$

where

$$P_1(x) = x^6(x+1)(2x^{12} - 2x^{11} + 3x^{10} - 3x^9 + 3x^8 - 2x^7 + 2x^6 - 2x^5 + 3x^4 - 2x^3 + 3x^2 - 3x + 2),$$

$$P_2(x) = x^9(x+1)(2x^{24} - 2x^{23} + 3x^{22} - 3x^{21} + 3x^{20} - 3x^{19} + 2x^{18} - 2x^{17} + 3x^{16} - 2x^{15} + 2x^{14} - 2x^{13} + x^{12} + 2x^8 - 2x^7 + 3x^6 - 3x^5 + 2x^4 - x^3 + 1),$$

and

$$Q_1(x) = x^2(x-1)(x^4+1)(x^8+1), \quad Q_2(x) = x(x-1)(x^8+1)(x^{16}+1),$$
$$R_1(x) = x^3(x^2-1)(x^8+1), \quad R_2(x) = x^2(x^2-1)(x^{16}+1),$$
$$S_1(x) = x^2(x-1)(x^4-1), \quad S_2(x) = x^2(x-1)(x^4-1).$$

Moreover, the following relation is true:

$$\mathcal{J}_{s+2}(x) = (1+x^3)\mathcal{J}_s(x^4) + \frac{1}{2}x(x+1)(\mathcal{J}_{s+1}(x^2) + \mathcal{J}_{s+1}(-x^2)).$$

Conjecture 1

Let $(\varepsilon_n)_{n\in\mathbb{N}} \in \{-1,1\}^{\mathbb{N}}, f(x) = \sum_{n=0}^{\infty} \varepsilon_n x^n$ *and write* $f(x)^{-m} = \sum_{n=0}^{\infty} b_m(n)x^n$ *for* $m \in \mathbb{N}_+$. *Let us suppose that for each* $N \in \mathbb{N}_+$ *there are infinitely many* $n \in \mathbb{N}$ *such that* $\varepsilon_n = \varepsilon_{n+1} = \ldots = \varepsilon_{n+N}$. *Then for each* $m \in \mathbb{N}_+$ *we have*

$$\limsup_{n\to+\infty} \nu_2(b_m(n)) = +\infty.$$

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$$\limsup_{n\to+\infty}\nu_2(b_m(n))=+\infty.$$

In fact we expect that the following strong statement is true:

Conjecture 2

Let $(\varepsilon_n)_{n\in\mathbb{N}} \in \{-1,1\}^{\mathbb{N}}$, $f(x)=\sum_{n=0}^{\infty}\varepsilon_n x^n$ and write $f(x)^{m}=\sum_{n=0}^{\infty} c_m(n)x^n$ for $m\in\mathbb{Z}$. Then there are infinitely many $m\in\mathbb{Z}$ (both positive and negative) such that

$$\limsup_{n\to+\infty}\nu_2(c_m(n))=+\infty.$$

We proved the boundedness of the 2-adic valuation of the coefficients of power series expansion of $R(x)^m$, where $m = 2, -2, 1 - 2^s, s \in \mathbb{N}_{\geq 2}$ and $R(x)$ is the generating fucntion for the RS sequence. Moreover, we also proved that the corresponding expressions for 2-adic valuations satisfy certain recurrence relations.

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In the remaining cases we expect that the following is true:

Conjecture 3

Let $m \in \mathbb{Z}$ and write $R(x)^m = \sum_{n=0}^{\infty} a_m(n)x^n$. If $m \neq 2, -2, 1-2^s, s \in \mathbb{N}_+$ then

$$\limsup_{n\to+\infty} \nu_2(a_m(n)) = +\infty.$$

In case when $m = 2^k$ then we expect the more precise:

Conjecture 4

Let $k \in \mathbb{N}_{\geq 2}$ *and write* $g_k(n) = \nu_2(a_{2^k}(n))$, $G_k(x) = \sum_{n=0}^{\infty} g_k(n)x^n$. *Then*

$$P_k(x) + Q_k(x)G_k(x) + R_k(x)G_k(x^2) = 0,$$

where

$$P_2(x) = x(2 - x + x^2), \qquad P_{k+1}(x) = (1 + x^{2^k})P_k(x) + x(1 - kx^{2^k-1}),$$
$$Q_2(x) = (x^2 - 1)(x^2 - x + 1), \qquad Q_{k+1}(x) = (1 + x^{2^k})Q_k(x) + (1 - x^2)x^{2^k-1},$$
$$R_k(x) = (1 - x^2)x^{2^k-1}.$$

We proved boundedness of 2-adic valuations of the sequences $(b_{2^s-1}(n))_{n\in\mathbb{N}}$ corresponding to the RS sequence and the LRY sequence. We also know that a similar property holds for the PTM sequence. All these sequences are 2-automatic and come from some kinds of binary patterns.

We proved boundedness of 2-adic valuations of the sequences $(b_{2^s-1}(n))_{n\in\mathbb{N}}$ corresponding to the RS sequence and the LRY sequence. We also know that a similar property holds for the PTM sequence. All these sequences are 2-automatic and come from some kinds of binary patterns.

This suggest the following general:

Problem 1

Let τ be a finite word on $\{0,1\}$ alphabet and $P_\tau(n)$ denotes the number of occurrences of the word τ (the scattered word τ) in the binary expansion of n. We define $\varepsilon_\tau(n)=(-1)^{P_\tau(n)}$ for $n\in\mathbb{N}$ and $f_\tau(x)=\sum_{n=0}^{\infty}\varepsilon_\tau(n)x^n$ and for $m\in\mathbb{Z}$ we put

$$f_\tau(x)^m=\sum_{n=0}^{\infty}c_{\tau,m}(n)x^n.$$

1. *What conditions need τ to satisfy in order to get boundedness of the sequence $(\nu_2(c_{\tau,m}(n))_{n\in\mathbb{N}}$ for some $m\in\mathbb{Z}$?*
2. *What conditions need τ to satisfy in order to get boundedness of the sequence $(\nu_2(c_{\tau,1-2^s}(n))_{n\in\mathbb{N}}$ for all but finitely many $s\in\mathbb{N}$?*

We performed some numerical experiments and noted that for the patterns $\tau = 0, 00, 10, 01$ it should be possible to prove similar results as in the case of the RS sequence, i.e., the sequence $(\nu_2(c_{\tau,1-2^s}(n)))_{n\in\mathbb{N}}$ is bounded. The bound seems to be: 2 for $\tau = 0$; 3 for $\tau = 00, 10$; and 4 for $\tau = 01$.

We performed some numerical experiments and noted that for the patterns $\tau = 0, 00, 10, 01$ it should be possible to prove similar results as in the case of the RS sequence, i.e., the sequence $(\nu_2(c_{\tau,1-2^s}(n)))_{n\in\mathbb{N}}$ is bounded. The bound seems to be: 2 for $\tau = 0$; 3 for $\tau = 00, 10$; and 4 for $\tau = 01$.

In case of patterns τ of length 3 the situation seems to be more complicated and we expect that for most $m \in \mathbb{Z}$ the sequence $(\nu_2(c_{\tau,m}(n)))_{n\in\mathbb{N}}$ is unbounded.

Problem 2

Generalize the above results for $p \in \mathbb{P}_{\geq 3}$.

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Theorem 19

Let $p \in \mathbb{P}_{\geq 3}$ *and write*

$$F_p(x) = \prod_{n=0}^{\infty} \frac{1}{1 - x^{p^n}}$$

and for $m \in \mathbb{N}_+$

$$F_p(x)^m = \sum_{n=0}^{\infty} b_{m,p}(n) x^n.$$

Then for $s \in \mathbb{N}_+$ *we have*

$$\nu_p(b_{(p-1)(p^s-1),p}(n)) = 1$$

for $n \geq p^s$.

Thank you for your attention!