Some Rigidity Theorems for Multiplicative Functions (joint with O. Klurman)

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CIRM Workshop

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Proven by Wirsing and independently by Shao and Tang.

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e.g., $f(n) := (-1)^{n-1}$ is multiplicative, with $f(2^k) = -1$ and $f(p^k) = 1$ for all $k \geq 1$ and $p \geq 3$.

What is "Pretentiousness"?

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$$\mathbb{D}(f, g; x) := \left(\sum_{p \leq x} \frac{1 - \mathrm{Re}\left(f(p)\overline{g(p)} \right)}{p} \right)^{\frac{1}{2}}.$$

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- *there is a* χ *mod* q *and a completely multiplicative function* g *such that* $g(n)^k = \chi(n)$ *for all* $(n, q) = 1$, $|g| = 1$, *and* $\mathbb{D}(f, gn^{it}; x) \ll_\varepsilon 1$;

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Take f comp. mult. with $f(p) := e(1/q_j)$ whenever $p \in A_j$ for all j, and $f(p) = 1$ otherwise.

Equidistribution and the Weyl Criterion

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A sequence $\{y_n\}_n \subset \mathbb{T}$ is equidistributed if, given any $0 \leq a < b \leq 1$,

$$\lim_{N\to\infty} \left| \frac{1}{N} |\{n \leq N : \frac{1}{2\pi} arg(y_n) \in [a, b]\}| - (b - a) \right| = 0.$$

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Theorem (Weyl's Criterion)

$\{y_n\}_n$ is equidistributed iff for each $l \in \mathbb{N}$,

$$\sum_{n \leq N} y_n^l = o(N).$$

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$\{f(n)\}_n$ is not equidistributed iff there is a multiplicative g, and $l \in \mathbb{N}$ such that $g^l = 1$ and $t \in \mathbb{R}$ such that $\mathbb{D}(f, gn^{it}; x) \ll 1$.

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A multiplicative function $f : \mathbb{N} \to \mathbb{U}$ *is highly non-pretentious if for all* $N \in \mathbb{N}$ *and* $g : \mathbb{N} \to \mathbb{C}$ *completely multiplicative such that* $g^N = 1$ *and any* t*, we have* $\mathbb{D}(f, gn^{it}; x) \to \infty$ *as* $x \to \infty$.

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Why is this interesting?

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If $\{y_n\}_n \subset \mathbb{T}$ *is such that* $\{y_n\overline{y_{n+h}}\}_n$ *is equidistributed for all* $h \in \mathbb{N}$ *then* $\{y_n\}_n$ *is equidistributed.*

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Suppose $\{y_n\}_n \subset \mathbb{T}$ is equidistributed and for all $\alpha \in \mathbb{R}$,

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Local-to-Global principle for correlations, as is known for mean values!

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If log-average of binary correlation is large then g is pretentious, and we can compute its binary correlations!

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Since $n^{it/k}$ varies slowly, can assume that $t = 0$ for the rest of the proof.

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Main Ideas and Condition 2

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$$\varepsilon^2 \sum_{n \in S} \frac{1}{n} \leq \sum_{n \in S} \frac{1}{n} |f(an+1) - f(an)|^2 = 2 \left(\sum_{n \in S} \frac{1}{n} - \text{Re} \left(f(a) \sum_{n \in S} \frac{f(n)\overline{f(an+1)}}{n} \right) \right).$$

Klurman $\Rightarrow$ if $\mathbb{D}(f, 1; x) \ll 1$ and $f(p) = 0$ for $p \leq N$ then $\frac{1}{\log x} \sum_{n \leq x} \frac{f(bn+c)\overline{f(dn+e)}}{n} = 1 + o_{N \to \infty}(1)$; $f\overline{g}$ is 1-pretentious here

Upshot: if i) $(a, n) = 1$, ii) $g(n) = g(an+1)$ and iii) $n(an+1)$ has no small prime factors on S then correlation sum with f will be too large as long as iv) $f(a)$ is close to 1! This is what forces f to satisfy Condition 2.

Theorem (Szemerédi, Gowers)

If $A \subset [1, x]$ has size $|A| = \delta x$ then A contains an arithmetic progression S of length $\log_2 (\log_3 x / \log(1/\delta))$.

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Strong **pointwise** algebraic condition!

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The $\alpha \neq 0$ case is due to Glazkov in the '60's using Delange's theorem.

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For $\alpha = 0$ this conjecture remained open.

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The modulus of f, assuming the conjecture is true, must be the product of the primes in this non-empty set.

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with G is a constant times a strongly multiplicative function $\tilde{G}$. Moreover, f is a character iff $\tilde{G}(d) = 0$ for all $d > 1$.

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- If $\tilde{G}(p) \neq 0$ then $\tilde{G}(p^k) = \tilde{G}(p) \neq 0 \Rightarrow$ Contradiction!

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$$x^{-1} \sum_{n \leq x} f(n)\overline{f(n+h)} \sim x^{-1} \sum_{n \leq x} g(n)\overline{g(n+h)}.$$

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Conjecture (Chowla)

Let $h \in \mathbb{N}$. Then

$$\sum_{n \leq x} \mu(n)\mu(n+h) = o(x).$$

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Let $p \geq 3$ be prime. Let $f : \mathbb{F}_p \to \mathbb{C}$ satisfy $f(0) = 0$, $f(1) = 1$ and $|f(a)| = 1$ on $\mathbb{F}_p^\times$. If, for all $h \in \mathbb{F}_p$,

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then f is a multiplicative character on $\mathbb{F}_p$.

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Let $1 \leq H \leq x$ with $H \to \infty$ as $x \to \infty$. Let q be odd. If there is a primitive Dirichlet character χ modulo q such that for all $1 \leq h \leq H$,

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then there is a $t \in \mathbb{R}$ and a Dirichlet character χ' modulo q such that $f(n) = \chi'(n)n^{it}$ for all n.

Remarks: a) the perturbation by $o(1)$ corresponds to the smooth and slowly-varying perturbation n^{it}. If $o(1)$ is deleted then $t = 0$;

Theorem (Klurman-M., 2017)

Let $1 \leq H \leq x$ with $H \to \infty$ as $x \to \infty$. Let q be odd. If there is a primitive Dirichlet character χ modulo q such that for all $1 \leq h \leq H$,

$$\sum_{n \leq x} f(n)\overline{f(n+h)} = (1+o(1)) \sum_{n \leq x} \chi(n)\overline{\chi(n+h)},$$

then there is a $t \in \mathbb{R}$ and a Dirichlet character χ' modulo q such that $f(n) = \chi'(n)n^{it}$ for all n.

Remarks: a) the perturbation by $o(1)$ corresponds to the smooth and slowly-varying perturbation n^{it}. If $o(1)$ is deleted then $t = 0$;
b) all primitive Dirichlet characters modulo q have the same binary correlations, up to $O(1)$, so $\chi' \neq \chi$ in general.

Thank you for listening!