Bounded remainder sets for the discrete and continuous irrational rotation

Sigrid Grepstad

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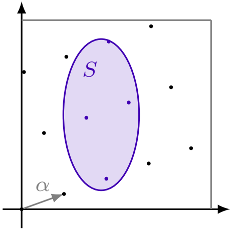
Irrational rotation on the torus

$\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$

$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$

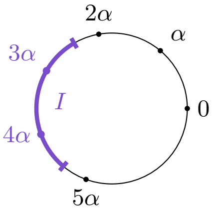
The sequence $\{n\alpha\}$ is equidistributed.



$$\frac{1}{n}\sum_{k=0}^{n-1}\chi_S(x+k\alpha) \to \operatorname{mes} S \quad (n\to\infty)$$

$$D_n(S,x) = \sum_{k=0}^{n-1}\chi_S(x+k\alpha) - n\operatorname{mes} S = o(n)$$

In one dimension



Hecke (1921) and Ostrowski (1927):
If $|I| \in \mathbb{Z}\alpha \pmod 1$, then $D_n(I, x) = \mathcal{O}(1)$ as $n \to \infty$.

The converse statement was confirmed by Kesten (1966).

Bounded remainder sets

Definition

A set S is a BRS if there is a constant $C = C(S, \alpha)$ such that

$$|D_n(S,x)| = \left| \sum_{k=0}^{n-1} \chi_S(x + k\alpha) - n \,\mathrm{mes}\, S \right| \leqslant C$$

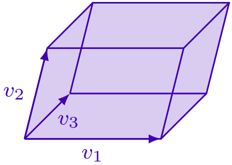
for all n and a.e. x.

An interval I is a BRS if and only if $|I| \in \mathbb{Z}\alpha + \mathbb{Z}$.

Bounded remainder parallelotopes

Theorem (G., Lev 2014)

Any parallelotope in $\mathbb{R}^d$ spanned by vectors $v_1, \ldots, v_d \in \mathbb{Z}\alpha + \mathbb{Z}^d$ is a BRS.



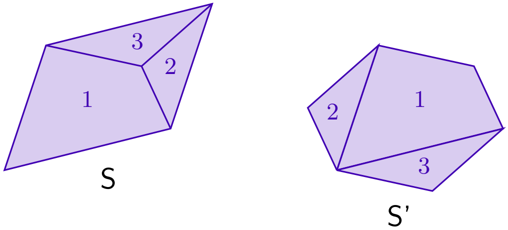
Measure of bounded remainder sets

Furstenberg, Keynes and Shapiro (1973):
If S is a BRS, then

$$\operatorname{mes} S = n_0 + n_1\alpha_1 + \cdots + n_d\alpha_d,$$

where $n_0, n_1, \ldots n_d$ are integers.

Equidecomposability



$S \sim S'$: The sets are equidecomposable (or scissors congruent).

Equidecomposability

$S \overset{\alpha}{\sim} S'$: The sets are equidecomposable using translations by vectors in $\mathbb{Z}\alpha + \mathbb{Z}^d$ only.

Claim: If $S \overset{\alpha}{\sim} S'$, and S is a BRS, then so is S'.

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Theorem (G., Lev 2014)

If S and S' are Jordan measurable BRS of equal measure, then

$$S \overset{\alpha}{\sim} S'.$$

Bounded remainder parallelotopes revisited

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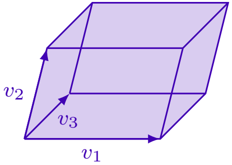
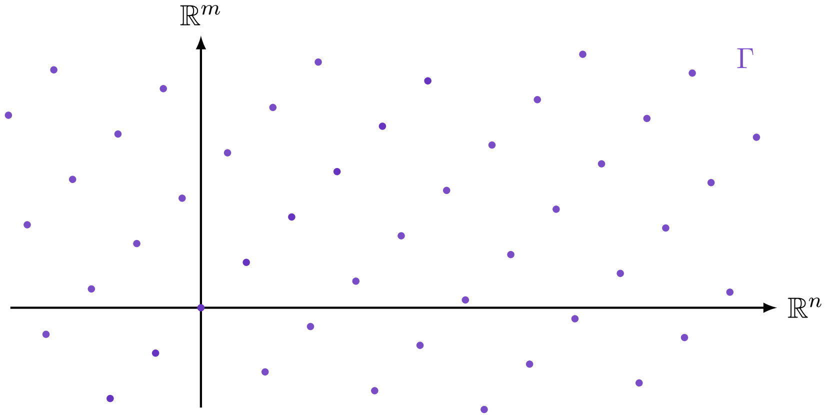


Cut-and-project sets



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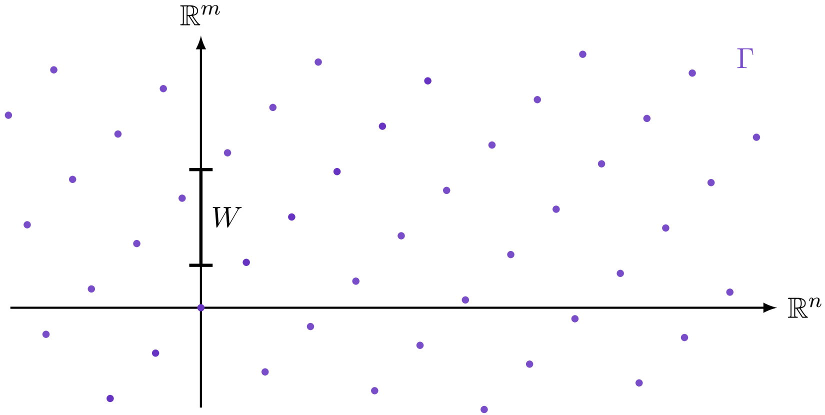

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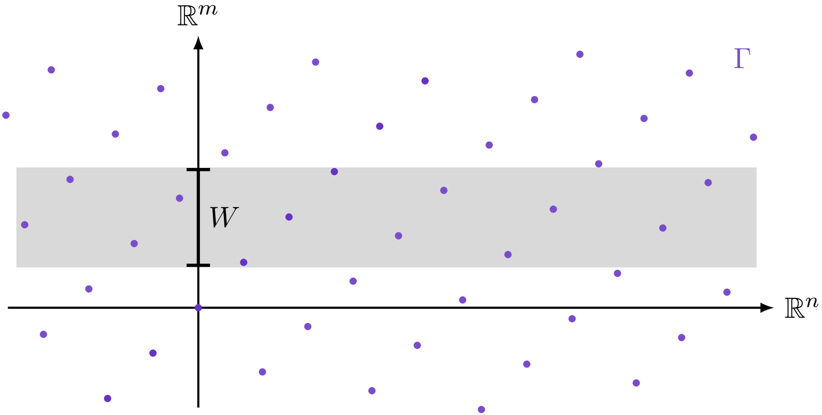

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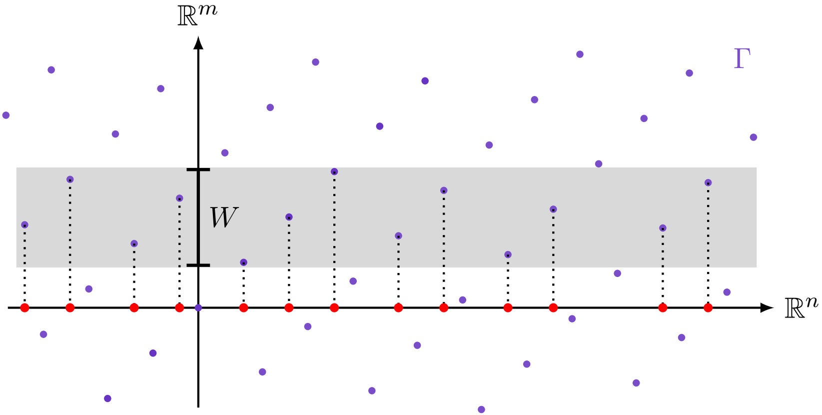

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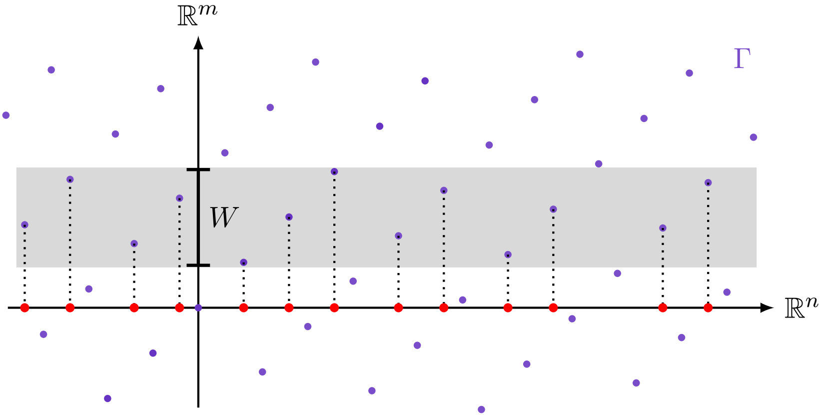
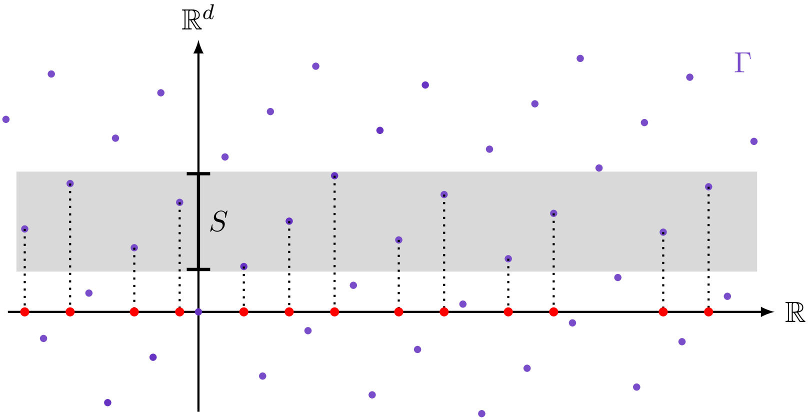


We define the Meyer cut-and-project set:

$$X = \{p_1(\gamma) \,:\, \gamma \in \Gamma,\, p_2(\gamma) \in W\}\,.$$



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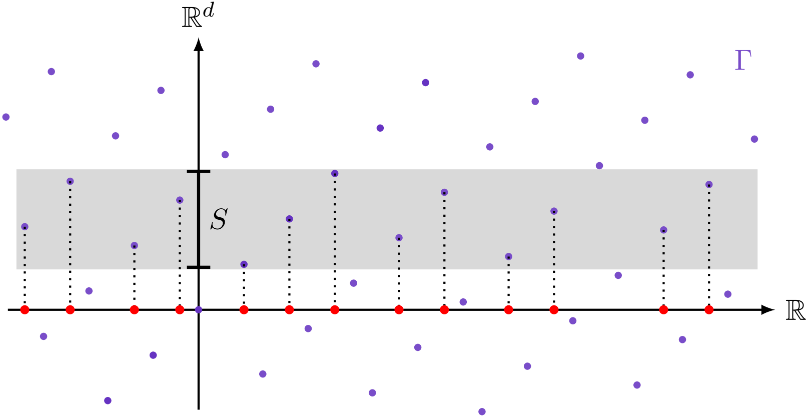
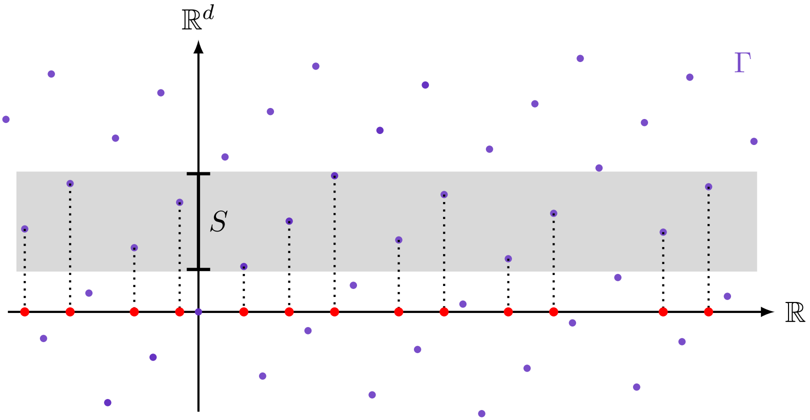


$$\Gamma = \left\{ \left(n + \beta^\top (n\alpha + m), n\alpha + m \right) \,:\, n \in \mathbb{Z},\, m \in \mathbb{Z}^d \right\} \subset \mathbb{R} \times \mathbb{R}^d,$$



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$$X = \left\{ n + \beta^\top (n\alpha + m) \,:\, n \in \mathbb{Z},\ m \in \mathbb{Z}^d,\ n\alpha + m \in S \right\} \subset \mathbb{R}$$

Notice that $p_2(\Gamma) = \mathbb{Z}\alpha + \mathbb{Z}^d$

Bounded displacement of a lattice

A cut-and-project set $X = X(\Gamma, W) \subset \mathbb{R}^n$ is at bounded distance from a lattice $L \subset \mathbb{R}^n$ if there exists a bijective map $\varphi : X \mapsto L$ such that

$$\sup_{x \in X} \|\varphi(x) - x\| < \infty.$$

Duneau and Oguey (1990):
If W is a fundamental domain of a lattice in $p_2(\Gamma)$, then $X(\Gamma, W)$ is at bounded distance from a lattice.

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If S is a fundamental domain of a lattice in $p_2(\Gamma) = \mathbb{Z}\alpha + \mathbb{Z}^d$ (e.g. a parallelotope spanned by $v_1, \ldots, v_d \in \mathbb{Z}\alpha + \mathbb{Z}^d$), then $X = X(\Gamma, S)$ is at bounded distance from the arithmetical progression $\{j / \operatorname{mes} S\}_{j \in \mathbb{Z}}$.

Claim: If $X(\Gamma, S)$ is at bounded distance from $\{j/\operatorname{mes} S\}_{j\in\mathbb{Z}}$, then S is a BRS (with respect to α).

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$$X = \Big\{ n + \beta^\top(n\alpha + m) \,:\, n \in \mathbb{Z},\, m \in \mathbb{Z}^d,\, n\alpha + m \in S \Big\}$$

$$= \bigcup_n X_n\,, \quad X_n = \Big\{ n + \beta^\top s \,:\, s = n\alpha + m \in S \Big\}$$

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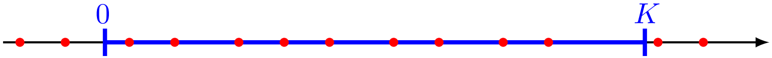

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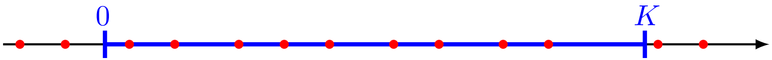

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$$N = |X \cap [0, K)| = \sum_{k=0}^{K-1} |X_k| + const = \sum_{k=0}^{K-1} \chi_S(k\alpha) + const$$

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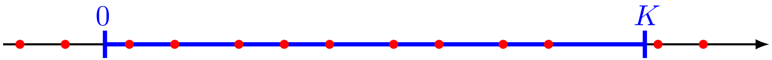

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$$= |\mathbb{Z}/\operatorname{mes} S \cap [0, K)| + const = K \operatorname{mes} S + const$$

Open problem

Consider the sequence

$$\{x_n\}_{n=1}^{\infty} = \left\{ \sum_{k=0}^{n-1} \chi_S(k\alpha) - n \operatorname{mes} S \right\}_{n=1}^{\infty}$$

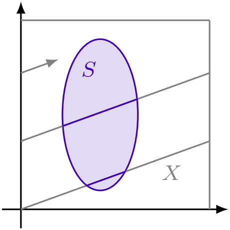
Do there exist sets $S \subset \mathbb{R}^d$ for which this sequence is unbounded, but in BMO, i.e. for which

$$\frac{1}{m-n} \sum_{k=n}^{m-1} \left| x_k - \frac{x_n + \cdots + x_{m-1}}{m-n} \right| \leqslant M$$

for all $n < m$?

Continuous irrational rotation

$\alpha > 0$ irrational and $x = (x_1, x_2) \in I^2 = [0,1)^2$
$X(t) = (\{x_1 + t\}, \{x_2 + \alpha t\})$



$$D_T(S,x) = \int_0^T \chi_S\left(\{x_1 + t\}, \{x_2 + \alpha t\}\right) dt - T \,\mathrm{mes}\, S$$

Super-uniformity

Theorem (Beck)

Let $S \subset I^2$ be an arbitrary Lebesgue measurable set with positive measure. Then for every $\varepsilon > 0$ and almost all α, we have

$$\int_0^T \chi_S\left(\{t\}, \{\alpha t\}\right)\, dt - T \operatorname{mes} S = o\left((\log T)^{3+\varepsilon}\right).$$

Theorem 1 (G., Larcher 2016)

Let $S \subset I^2$ be a polygon. Then the discrepancy $D_T(S, \alpha)$ is bounded (in absolute value) as $T \to \infty$ for almost every $\alpha > 0$ and every starting point $x \in I^2$.

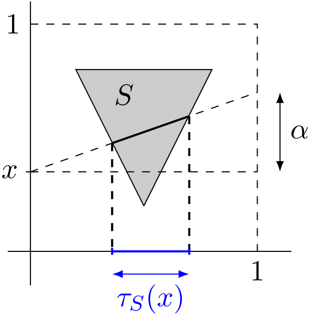
Theorem 2 (G., Larcher 2016)

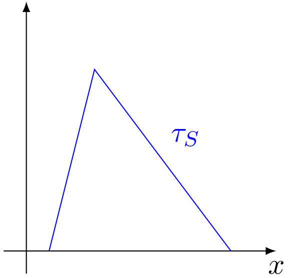
Let $S \subset I^2$ be a convex set whose boundary ∂S is a twice continuously differentiable curve with positive curvature at every point. Then the discrepancy $D_T(S, \alpha)$ is bounded (in absolute value) as $T \to \infty$ for almost every $\alpha > 0$ and every starting point $x \in I^2$.

Proof Outline

Let

$$\tau_S(x) = \int_0^1 \chi_S(t, \{t\alpha + x\})\, dt.$$





$$|D_T(S,x)| \leqslant C_1 \quad \Leftrightarrow \quad \left| \sum_{k=0}^{N-1} \tau_S(\{k\alpha\}) - N \int_0^1 \tau_S(x)\, dx \right| \leqslant C_2$$

Ostrowski expansion to base α: $N = b_s q_s + \cdots + b_1 q_1 + b_0 q_0$

Condition on $\alpha = [0; a_1, a_2, \ldots]$:

$$\sum_{l=0}^{s} \frac{a_{l+1}}{\sqrt{q_l}} \sum_{k=1}^{l+1} a_k \leqslant C$$

We then have:

1. $$\left| \sum_{k=0}^{N-1} \tau_S\left(\{k\alpha\}\right) - \sum_{l=0}^{s} \sum_{b=0}^{b_l} \sum_{k=0}^{q_l} \tau_S\left(\frac{k}{q_l}\right) \right| \leqslant C_1,$$

2. $$\left| \sum_{l=0}^{s} \sum_{b=0}^{b_l} \sum_{k=0}^{q_l} \tau_S\left(\frac{k}{q_l}\right) - N \int_0^1 \tau_S(x)\, dx \right| \leqslant C_2$$

Open problems

In two dimensions: Is every convex set $S \subset I^2$ a bounded remainder set with respect to almost every continuous irrational rotation?

In higher dimensions: Can we establish any conditions on $S \subset I^d$ sufficient for bounding the discrepancy $|D_T(S, x)|$ as $T \to \infty$ for a given $\alpha > 0$?

Thank you!