Prime Numbers and Automatic Sequences : Determinism and Randomness
Nombres premiers et suites automatiques : aléa et déterminisme

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Abstracts / Résumés

Christoph Aistleitner (TU Graz):
Extreme values of the Riemann zeta function via the resonance method

Abstract: In recent years there has been significant progress in proving lower bounds for large values of the Riemann zeta function in the critical strip. The so-called resonance method, which was initiated by Voronin, Soundararajan and Hilberdink, was refined such that it permits the use of extremely long Dirichlet polynomials as resonators. In our talk we describe the functional principle of this method, two ways of how to deal with extremely long Dirichlet polynomials in this context, and discuss the problems which arise when trying to implement the method for general L-functions.

Ramachandran Balasubramanian (IMSc Chennai):
Poisson distribution of a prime counting function corresponding to elliptic curves

Abstract: Let E be an elliptic curve over $\mathbb{Q}$ and N be a positive integer. Now, $M_E(N)$ denotes the number of primes p, such that the group $E_p(F_p)$ is of order N. In this talk, we shall prove that $M_E(N)$ follows Poisson distribution when an average is taken over a large class of curves.

Yann Bugeaud (Université de Strasbourg):
On the representation of real numbers to distinct bases

Abstract: Let b be an integer greater than or equal to 2. A real number is called simply normal to base b if each digit $0, \ldots, b-1$ occurs in its b-ary expansion with the same frequency $1/b$. It is called normal to base b if it is simply normal to every base b^k, where k is a positive integer (or, equivalently, if, for every positive integer k, each block of k digits from $0, \ldots, b-1$ occurs in its b-ary expansion with the same frequency $1/b^k$). This notion was introduced in by 1909 Émile Borel, who established that almost every real number (in the sense of the Lebesgue measure) is normal to every integer base. We start with the existence of uncountably many numbers normal to base 2 but not simply normal to base 3, a result proved independently by Cassels and Schmidt more than fifty years ago. More generally, under some necessary conditions on a (finite or infinite) set B of integers greater than or equal to 2, we discuss the existence of uncountably many numbers which are simply normal to any base in B and not simply normal to any base not in B. This is a joint work with Verónica Becher and Ted Slaman. Finally, we show that no irrational real numbers ξ have a 'too simple? expansion in base 2 and in base 3, in the following sense: if the sequence of binary digits of ξ is Sturmian, then the sequence of its ternary digits cannot be Sturmian. This is a joint work with Dong Han Kim.

Lucile Devin (Université Paris-Sud):
Généralisations des biais de Chebyshev

Résumé : Poursuivant les idées de Rubinstein, Sarnak et Fiorilli, on s'intéresse la généralisation de la notion de biais de Chebyshev dans les courses de nombres premiers. Pour une fonction $L(s) = \sum_{n\geq 1} a_n n^{-s}$ vérifiant certaines propriétés anaylitques généralisant celles vérifiées par les fonctions L

de Dirichlet, on étudie le signe de la fonction sommatoire $x \mapsto \sum_{p \leq x} a_p$. On peut montrer qu'une normalisation de cette fonction admet une distribution logarithmique limite. En supposant une certaine indépendance des zéros de la fonction L, on obtient une certaine régularité de la distribution. Cela permet de montrer sous une hypothèse assez faible l'existence de la densité logarithmique de l'ensemble $\{x \geq 2 : \sum_{p \leq x} a_p > 0\}$.

Sary Drappeau (Aix-Marseille Université):
Values of polynomials without large prime factors

Abstract: We will discuss means to bound the frequency with which a polynomial takes values free of large prime factors, and consequences on additive problems. A central ingredient is the equidistribution of certain sets of integers; we will talk about recent progress in the irreducible quadratic case, eg. $\{n^2 + 1, n \text{ integer}\}$. This is joint work with R. de la Bretèche.

Elie Goudout (IMJ - Paris Rive Gauche):
Majoration du nombre d'entiers n tels que $\omega(n) = k_1$ et $\omega(n + 2) = k_2$

Résumé : Suite aux travaux de Matomäki & Radziwill, on s'intéresse au comportement asymptotique de $\pi_k(x + h) - \pi_k(x)$ pour presque tout x, lorsque h est très petit. Lorsque $k \approx \log \log x$, on obtient l'estimation attendue condition de savoir majorer efficacement des cardinaux du type

$$\# \{n \leq x : \quad \omega(n) = k_1, \ \omega(n + 2) = k_2\}$$

pour certains k_1, k_2 proches de k. On présentera une telle majoration.

Sigrid Grepstad (University of Linz):
Bounded remainder sets for the discrete and continuous irrational rotation

Abstract: Let $\alpha \in \mathbb{R}^d$ be a vector whose entries $\alpha_1, \dots, \alpha_d$ and 1 are linearly independent over the rationals. We say that $S \subset \mathbb{T}^d$ is a *bounded remainder set* for the sequence of irrational rotations $\{n\alpha\}_{n \geqslant 1}$ if the discrepancy

$$\sum_{k=1}^{N} \mathbb{1}_S(\{k\alpha\}) - N \operatorname{mes}(S)$$

is bounded in absolute value as $N \to \infty$. In one dimension, Hecke, Ostrowski and Kesten characterized the intervals with this property.

We will discuss the bounded remainder property for sets in higher dimensions. In particular, we will see that parallelotopes spanned by vectors in $\mathbb{Z}\alpha + \mathbb{Z}^d$ have bounded remainder. Moreover, we show that this condition can be established by exploiting a connection between irrational rotation on $\mathbb{T}^d$ and certain cut-and-project sets. If time allows, we will discuss bounded remainder sets for the continuous irrational rotation $\{t\alpha : t \in \mathbb{R}^+\}$ in two dimensions.

Gautier Hanna (Aix-Marseille Université):
On the digits of primes

Abstract: In this talk, I will explain how we can extend the previous work of Mauduit and Rivat about the Rudin–Shapiro sequence for all sequences which count blocks of digits. At first I will consider the case when the length of the block is constant, and then the case when it is a non decreasing function. If time permits, I will present briefly new results (work in progress with Olivier Robert) about the orthogonality between the Möbius function, and some functions related to polynomials along binary digits (like the Rudin–Shapiro sequence).

Yining Hu (IMJ - Paris Rive Gauche):
Subword complexity and non-automaticity of certain completely multiplicative functions

Abstract: In this article, we prove that for a completely multiplicative function f from $\mathbb{N}^*$ to a field K such that the set

$$\{p \mid f(p) \neq 1_K \text{ and } p \text{ is prime}\}$$

is finite, the asymptotic subword complexity of f is $\Theta(n^t)$, where t is the number of primes p that $f(p) \neq 0_K, 1_K$. This proves in particular that sequences like $((-1)^{v_2(n)+v_3(n)})_n$ are not k-automatic for $k \geq 2$.

Teturo Kamae (Osaka City University):
Selection rules preserving normality

Abstract: Let $M = (\Sigma,\ \sigma_0,\ \sigma_0,\ F)$ be a **countable** automaton over $\{0,1\}$, which is also called a *selection rule*. That is, Σ is a countable set, $\sigma_0 \in \Sigma$, $F \subset \Sigma$ and $\psi : \Sigma \times \{0,1\} \to \Sigma$. Let $\alpha = \alpha(0)\alpha(1)\alpha(2)\cdots \in \{0,1\}^{\mathbb{N}}$ be a binary normal number. Define a subset $S = S(M,\alpha) = \{s_0 < s_1 < \cdots\} \subset \mathbb{N}$ by

$$S = \{i \in \mathbb{N};\ \psi(\sigma_0,\ \alpha(0)\alpha(1)\cdots\alpha(i-1)) \in F\},$$

where

$$\psi(\sigma_0,\ \alpha(0)\alpha(1)\cdots\alpha(i-1)) = \psi(\cdots\psi(\psi(\sigma_0,\alpha(0)),\alpha(1))\cdots,\alpha(i-1)).$$

If S is an infinite set, we define $\alpha[S] \in \{0,1\}^{\mathbb{N}}$ by $\alpha[S](n) = \alpha(s_n)$ $(\forall n \in \mathbb{N})$.

Theorem. *For a binary normal number $\alpha \in \{0,1\}^{\mathbb{N}}$ and a selection rule M, if F is a finite set such that the set*

$$\{i \in \mathbb{N};\ \psi(\sigma_0,\ \alpha(0)\alpha(1)\cdots\alpha(i-1)) \in F\}$$

has a positive lower density, then $\alpha[S(M,\alpha)]$ is a normal number. That is, the selection rule M preserves normality.

This is originally proved by Kamae and B. Weiss (1975) as a generalization of the case of finite automata (Y.N.Agafanov, 1968).

Shanta Laishram (Indian Statistical Institute):
On the sums of the digits in bases 2 and 3

Abstract: Let $b \geq 2$ be an integer and let $s_b(n)$ denote the sum of the digits of the representation of an integer n in base b. It is an interesting problem to show that there are infinitely many positive integers n for which $s_{b_1}(n) = s_{b_2}(n)$ for multiplicatively independent bases b_1 and b_2. In fact it is an open problem even to show that $|s_3(n) - s_2(n)|$ is *small* infinitely often. Recently, in a joint work with Deshouillers, Habsieger and Landreau, we show that $|s_3(n) - s_2(n)|$ is significantly smaller than $(\frac{1}{\log 3} - \frac{1}{\log 4}) \log n$ for infinitely many n. More precisely, we show that for sufficiently large N, one has

$$|\{n \leq N : |s_3(n) - s_2(n)| \leq 0.1457205 \log n\}| \ > \ N^{0.970359}.$$

In this talk, I will give a brief overview of the problem and prove this result which is based on separate distributions of the values of $s_2(n)$ and $s_3(n)$.

Youness Lamzouri (York University): Large character sums

Abstract: For a non-principal Dirichlet character χ modulo q, the classical Pólya-Vinogradov inequality asserts that

$$M(\chi) := \max_x |\sum_{n \leq x} \chi(n)| = O\left(\sqrt{q} \log q\right).$$

This was improved to $\sqrt{q} \log \log q$ by Montgomery and Vaughan, assuming the Generalized Riemann hypothesis GRH. For quadratic characters, this is known to be optimal, owing to an unconditional omega result due to Paley. In this talk, we shall present recent results on higher order character sums. In the first part, we discuss even order characters, in which case we obtain optimal omega results for $M(\chi)$, extending and refining Paley's construction. The second part, joint with Alexander Mangerel, will be devoted to the more interesting case of odd order characters, where we build on previous works of Granville and Soundararajan and of Goldmakher to provide further improvements of the Pólya-Vinogradov and Montgomery-Vaughan bounds in this case. In particular, assuming GRH, we are able to determine the order of magnitude of the maximum of $M(\chi)$, when χ has odd order $g \geq 3$ and conductor q, up to a power of $\log_4 q$ (where $\log_4$ is the fourth iterated logarithm).

Manfred Madritsch (Université de Lorraine):
The sum-of-digits function of linearly recurrent number systems and almost primes

Abstract: This is joint work with Jörg Thuswaldner from University of Leoben and Mario Weitzer from Graz University of Technology.

A linear recurrent number system is a generalization of the q-adic number system. In particular, we replace the sequence of powers of q by a linear recurrent sequence $G_{k+d} = a_1 G_{k+d-1} + \cdots + a_d G_k$ for $k \geq 0$. Under some mild conditions for every positive integer n we have a representation of the form

$$n = \sum_{j=0}^{k} \varepsilon_j(n) G_j.$$

The q-adic number system corresponds to the linear recursion $G_{k+1} = qG_k$ and $G_0 = 1$. The first example of a real generalization is due to Zeckendorf who showed that the Fibonacci sequence $G_0 = 1$, $G_1 = 2$, $G_{k+2} = G_{k+1} + G_k$ for $k \geq 0$ yields a representation for each positive integer. This is unique if we additionally suppose that no two consecutive ones exist in the representation.

In the present talk we investigate the representation of primes and almost primes in linear recurrent number systems. We start by showing the different results due to Fouvry, Mauduit and Rivat in the case of q-adic number systems. Then we shed some light on their main tools and techniques. Generalizing their methods to linear recurrent number systems we show a Bombieri-Vinogradov type result for the distribution in residue classes of the sum of digits function along arithmetic progressions if the trace $a_1 \geq 30$. This lower bound for the trace is due to numerical estimation. An improvement using graphic card based algorithms is under current development.

Alexander Mangerel (University of Toronto):
Some rigidity theorems for multiplicative functions

Abstract: We discuss several "rigidity" problems for multiplicative functions, all solved using the methods of "Pretentious" analytic number theory. This is joint work with O. Klurman.

Firstly, we shall sketch a proof of the folklore conjecture that any unimodular *completely* multiplicative function f satisfies the property $\liminf_{n\to\infty} |f(n+1) - f(n)| = 0$, and classify those multiplicative functions for which this fails.

Secondly, we will outline a resolution in the affirmative of a sixty-year-old conjecture due to N.G. Chudakov, stating that

any completely multiplicative function $f : \mathbb{N} \to \mathbb{C}$ that: i) takes only finitely many values; ii) vanishes on a finite, non-empty set of primes; iii) has *bounded* partial sums, is a Dirichlet character.

Time permitting, we will discuss the following result: if "many" of the binary correlations of f are asymptotically equal to those of a Dirichlet character χ mod q then $f(n) = \chi'(n)n^{it}$ for all n, where χ' is a Dirichlet character modulo q and $t \in \mathbb{R}$. This establishes a variant of a conjecture of H. Cohn for multiplicative arithmetic functions.

James Maynard (Magdalen College, Oxford):
Large gaps between primes in subsets

Abstract: All previous methods of showing the existence of large gaps between primes have relied on the fact that smooth numbers are unusually sparse. This feature of the argument does not seem to generalise to showing large gaps between primes in subsets, such as values of a polynomial. We will talk about recent work which allows us to show large gaps between primes without relying on smooth number estimates. This then generalizes naturally to show long strings of consecutive composite values of a polynomial. This is joint work with Ford, Konyagin, Pomerance and Tao.

Clemens Müllner (TU Wien):
Möbius orthogonality for the Zeckendorf sum-of-digits function

Abstract: This is joint work with Michael Drmota and Lukas Spiegelhofer.

We show that the sequence $(-1)^{s_\varphi(n)}$ is asymptotically orthogonal to all bounded multiplicative functions, where s_φ denotes the Zeckendorf sum-of-digits function. In particular we have

$$\sum_{n<N} (-1)^{s_\varphi(n)} \mu(n) = o(N),$$

that is, this sequence satisfies the Sarnak conjecture. The sequence $(-1)^{s_\varphi(n)}$ is a morphic sequence, i.e., it is obtained by a coding of a fixed point of a substitution. This is one of the first examples of a non-constant length substitution that satisfies the Sarnak conjecture. The case of constant length substitutions, i.e. automatic sequences, has recently been handled by the author.

We use the Katai - Bourgain - Sarnak - Ziegler criterion to reduce the problem to estimates of $\sum_{n<N}(-1)^{s_\varphi(pn)+s_\varphi(qn)}$. To analyze such sums we use the concept of quasi-additivity with respect to the Zeckendorf expansion which allows a generating function approach, which was introduced by Kropf and Wagner for integer bases.

Janos Pintz (Hungarian Academy of Sciences):
The distribution of Zeta-zeros and the remainder term of the prime number theorem

Abstract: An 84-year-old classical result of Ingham states that a rather general zero-free region of the Riemann zeta function implies an upper bound for the absolute value of the remainder term of the prime number theorem. In 1950 Turán proved a partial conversion of the mentioned theorem of Ingham. Later the author proved sharper forms of both Ingham's theorem and its conversion by Turán. The present work shows a very general theorem which describes the average and the maximal order of the error terms by a relatively simple function of the distribution of the zeta zeros. It is proved that the maximal term in the explicit formula of the remainder term coincides with high accuracy with the average and maximal order of the error term.

Anne de Roton (Université de Lorraine):
Small sumsets in continuous and discrete settings

Abstract: Given a subset A of an additive group, how small can the sumset $A+A = \{a+a' : a, a' \in A\}$ be ? And what can be said about the structure of A when $A+A$ is very close to the smallest possible size ? The aim of this talk is to partially answer these two questions when A is either a subset of $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{R}$ or $\mathbb{T}$ and to explain how in this problem discrete and continuous setting are linked. This should also illustrate two important principles in additive combinatorics : reduction and rectification.
This talk is partially based on some joint work with Pablo Candela and some other work with Paul Péringuey.

Zeev Rudnick (Tel-Aviv University):
Angles of Gaussian primes

Abstract: Fermat showed that every prime $p = 1 \bmod 4$ is a sum of two squares: $p = a^2 + b^2$, and hence such a prime gives rise to an angle whose tangent is the ratio b/a. Hecke showed, in 1919, that these angles are uniformly distributed, and uniform distribution in somewhat short arcs was given in by Kubilius in 1950 and refined since then. I will discuss the statistics of these angles on fine scales and present a conjecture, motivated by a random matrix model and by function field considerations.

Asaki Saito (Future University Hakodate):
Pseudorandom number generator based on the binary expansion of algebraic integers and its p-adic analogue

Abstract: We introduce two generators of pseudorandom binary sequences. The first one exactly computes true orbits of the Bernoulli map (the $2x$ modulo 1 map) on quadratic algebraic integers. The second one is a p-adic analogue of the first one. For each generator, we develop a method which can select initial points (seeds) without bias and which can avoid overlaps in latter parts of the pseudorandom sequences derived from them. We also demonstrate through statistical testing that the generated sequences possess good statistical properties.

Adrian Scheerer (TU Graz):
Computable absolutely normal numbers and discrepancies

Abstract: A real number is called normal to base b, $b \geq 2$ an integer, if in its base b expansion all finite blocks of digits occur with the expected asymptotic frequency. A real number is called absolutely normal if it is normal to every integer base $b \geq 2$. It is unknown whether natural constants such as e

or π are normal to some base and there is not a single explicit example of an absolutely normal number although almost all real numbers have this property.

This talk surveys recent results on the efficient construction of absolutely normal numbers via recursive algorithms that output the digits of such a number one after the other. We focus on the question whether good convergence to normality necessarily implies large computational complexity. We will show how to extend some of these algorithms to produce absolutely normal numbers to other bases, such as Pisot numbers and continued fractions.

Cathy Swaenepoel (Aix-Marseille Université):
Digits in finite fields

Abstract: The study of the connection between the arithmetic properties of an integer and the properties of its digits in a given basis produces a lot of interesting questions and a lot of papers have been devoted to this topic. In the context of finite fields, the algebraic structure permits to formulate and study new problems of interest which might be out of reach in $\mathbb{N}$. Dartyge and Sárközy [3] initiated the study of the concept of digits in $\mathbb{F}_q$, establishing estimates for the number of squares whose sum of digits is fixed. They also obtained results for polynomial values, resp. polynomial values with primitive element arguments whose sum of digits is fixed. Further results can be found in [2], [4].

We will study new questions in this spirit: 1) give (more precise) estimates for the number of elements of $\mathbb{F}_q$ that belong to a special sequence and whose sum of digits is fixed; 2) given subsets $\mathcal{C}$ and $\mathcal{D}$ of $\mathbb{F}_q$, find conditions on $|\mathcal{C}|$ and $|\mathcal{D}|$ to ensure that there exists $(c, d) \in \mathcal{C} \times \mathcal{D}$ such that the sum of digits of cd belongs to a predefined subset of $\mathbb{F}_p$; 3) given a special sequence $\mathcal{Q}$ in $\mathbb{F}_q$, estimate the number of elements of $\mathcal{Q}$ with preassigned digits. For this last problem, we show that we can preassign a positive proportion of digits, in the spirit of a recent result of Bourgain [1] who studied the number of prime numbers with a positive proportion of preassigned digits.

[1] J. Bourgain, *Prescribing the binary digits of primes, II*, Israel J. Math., 206 (2015), pp. 165–182.
[2] C. Dartyge, C. Mauduit, and A. Sárközy, *Polynomial values and generators with missing digits in finite fields*, Funct. Approx. Comment. Math., 52 (2015), pp. 65–74.
[3] C. Dartyge and A. Sárközy, *The sum of digits function in finite fields*, Proc. Amer. Math. Soc., 141 (2013), pp. 4119–4124.
[4] R. Dietmann, C. Elsholtz, and I. E. Shparlinski, *Prescribing the binary digits of squarefree numbers and quadratic residues*, arXiv:1601.04754v1, (2016).

Jun-Ichi Tamura (Tsuda College):
Convergence theorems of substitutions and Rauzy fractals in the p-adic world

Abstract: We introduce p-adic substitutions, i.e., substitutions over Ω_p-powered symbols. Here, Ω_p is the spherically complete extension of $\mathbb{R}_p$, which is the algebraic closure of $\mathbb{Q}_p$, and $\mathbb{Q}_p$ is the closure of the rational number field $\mathbb{Q}$ with respect to the p-adic topology. We give a norm theorem which completely describes the distribution of p-adic absolute value of the zeros of a given polynomial in Ω_p[x] in terms of the coefficients of the polynomial. As an application of the norm theorem, we can show some convergence theorems for certain p-adic substitutions. Under convergence results, we can define the Rauzy fractal for a p-adic substitution in the vector space ${\Omega_p}^d$. Making some visualizations of a p-adic Rauzy fractal in the space $\mathbb{R}^d$, or $\mathbb{C}^d$, we show some examples of p-adic Rauzy fractals.

Maciej Ulas (Jagiellonian University):
2-adic valuations of coefficients of certain integer powers of formal power series with $\{-1, +1\}$ coefficients

Abstract: Let $F(x) = \prod_{n=0}^{\infty}(1-x^{2^n})$ be the generating function for the Prouhet-Thue-Morse sequence $((-1)^{s_2(n)})_{n\in\mathbb{N}}$. For each $m \in \mathbb{N}_+$, the n-th coefficient in the power series expansion

$$F(x)^{-m} = \sum_{n=0}^{\infty} b_m(n)x^n$$

counts the number of representations of the integer n as a sum of powers of two, where each summand can have one among m colors. In the first part of the talk, we present the exact value of the 2-adic

valuation of the number $b_m(n)$, with $m = 2^s - 1$, - a result which generalizes the well known expression concerning the 2-adic valuation of the values of the binary partition function introduced by Euler and studied by Churchhouse and others.

In the second part of the lecture, we will study integer powers of series $F(x) = \sum_{n=0}^{\infty} \epsilon_n x^n$, where $\epsilon_n \in \{-1, +1\}$. More precisely, we give some results concerning the behaviour of 2-adic valuations of the sequence $(c_m(n))_{n\in\mathbb{N}}$, where $m \in \mathbb{Z}$ is fixed and

$$f(x)^m = \sum_{n=0}^{\infty} c_m(n) x^n.$$

In fact, we propose a method which under suitable assumptions on the sequence $(\epsilon_n)_{n\in\mathbb{N}}$ allows to prove boundednees of the sequence $(\nu_2(c_m(n))_{n\in\mathbb{N}}$ for ceratin values of $m \in \mathbb{Z}$. In particular, if $(\epsilon_n)_{n\in\mathbb{N}}$ is the classical Rudin-Shapiro sequence then we prove that $\nu_2(c_{1-2^s}(n)) \in \{1, 2, 3\}$ for given $s \in \mathbb{N}_{\geq 2}$ and all $n \geq 2^s$. Similar result is proved for a relative of the Rudin-Shapiro sequence recently introduced by Lafrance, Rampersad and Yee.

Shin-Ichi Yasutomi (Toho University):
Multidimensional p-adic continued fraction algorithms

Abstract: We give a new class of multidimensional p-adic continued fraction algorithms. We propose an algorithm in the class for which we can expect that multidimensional p-adic version of Lagrange's Theorem holds.

Zhiwei Wang (Université de Lorraine):
On the largest prime factors of consecutive integers

Abstract: Let $P^+(n)$ denote the largest prime factor of the integer n and $P_y^+(n)$ denote the largest prime factor p of n which satisfies $p \leqslant y$. Firstly we show that the triple consecutive integers with the two patterns $P^+(n-1) > P^+(n) < P^+(n+1)$ and $P^+(n-1) < P^+(n) > P^+(n+1)$ have a positive proportion respectively. More generally, with the same methods we can prove that for any $J \in \mathbb{Z}, J \geqslant 3$, the $J-$tuple consecutive integers with the two patterns $P^+(n+j_0) = \min_{0\leqslant j\leqslant J-1} P^+(n+j)$ and $P^+(n+j_0) = \max_{0\leqslant j\leqslant J-1} P^+(n+j)$ also have a positive proportion respectively. Secondly for $y = x^\theta$ with $0 < \theta \leqslant 1$ we show that there exists a positive proportion of integers n such that $P_y^+(n) < P_y^+(n+1)$. Specially, we can prove that the proportion of integers n such that $P^+(n) < P^+(n+1)$ is larger than 0.1356.

Máté Wierdl (University of Memphis):
Random differences for arithmetic progressions in the primes

Abstract: The question that can be considered the beginning of this topic is this: flip a fair coin infinitely many times. Is it true that the resulting 1/2 density set R intersects the difference set of the primes, that is, is there an r in R which is the difference of two primes?

The next question is if there is an r in R which the difference of a three or more term arithmetic progression of primes?

What happens if instead of coin flipping, we consider a thinner randomly generated set of integers, such as when we choose the integer n into the random set with with probability $n^{-1/2}$? How small can this probability get? Can we take it to be n^{-1}?

Can we make this random set R work simultaneously for all positive density subsets of the primes?