January 24, 2017

Differential Tropical Geometry

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Definition

Let (R, d) be a differential ring and let $R[x_i^{(j)} \mid 1 \leq i \leq n, j \geq 0]$ be the set of polynomials with coefficients in R in variables $\{x_i^{(j)} \mid 1 \leq i \leq n, j \geq 0\}$, the derivation d in R can be extended to a derivation D of $R[x_i^{(j)} \mid 1 \leq i \leq n, j \geq 0]$ such that

1. $D(x_i^{(j)}) = x_i^{(j+1)}$ *for* $1 \leq i \leq n$ *y* $j \geq 0$
2. $D(a) = d(a), \forall a \in \mathbb{K}.$

The pair $(R[x_i^{(j)} \mid 1 \leq i \leq n, j \geq 0], D)$ is a differential ring is called the **differential ring of polynomials** *in n variables with coefficients in R.*

Notation

We consider the tropical semiring $\mathbb{L} = \mathbb{Z}_{\geq 0} \cup \{\infty\}$, with $a \oplus b =$ min$\{a, b\}$ and $a \odot b = a + b$, $\forall a, b \in L$.

Definition

A tropical differential polynomial in the variables $x_1, \ldots, x_n$ of order less or equal than r is :

$$\begin{aligned} \phi(x) &= \bigoplus_{M \in \Lambda \subset \mathcal{M}_{n\times(r+1)}(\mathbb{Z}_{\geq 0})} \left(a_M \bigodot_{\substack{1\leq i \leq n \\ 0 \leq j \leq r}} {x_i^{(j)}}^{\odot M_{ij}} \right) \\ &= \min_{M\in\Lambda} \left\{ a_M + \sum_{i,j} M_{ij} x_i^{(j)} \right\} \quad . \end{aligned}$$

where $a_M \in \mathbb{L}$ and Λ is a finite set, $M = (M_{ij})_{\substack{1\leq i\leq n \\ 0\leq j\leq r}}$ is a matrix in $\mathcal{M}_{n\times(r+1)}(\mathbb{Z}_{\geq 0})$.

Example

An example of tropical differential linear polynomial is :

$$\phi = (x^{(1)} \odot 1) \oplus (x^{(3)} \odot 2) \oplus 3.$$

Notation

We denote the power set of $\mathbb{Z}_{\geq 0}$ *by* $\mathcal{P}(\mathbb{Z}_{\geq 0})$.

Definition

Let $S \subseteq \mathbb{Z}_{\geq 0}$*, we define* $Val_S : \mathbb{Z}_{\geq 0} \longrightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$ *by*

$$Val_S(j) := \begin{cases} s - j, & \text{with } s = \min\{\alpha \in S \ : \ \alpha \geq j\}, \\ \infty, & \text{when } S \cap \mathbb{Z}_{\geq j} = \emptyset. \end{cases} \tag{1}$$

Example

Consider the set $S := \{1, 3, 4\}$*. We have*

1. $Val_S(2) = \min\{s \in S \mid s \geq 2\} - 2 = 3 - 2 = 1$
2. $Val_S(5) = \infty$.

- *A tropical differential polynomial*

$$\phi = \bigoplus_{M \in \Lambda} (a_M \bigodot_{\substack{1 \le i \le n \\ 0 \le j \le r}} x_i^{(j)\,\odot M_{ij}})$$

induces a mapping

$$\begin{array}{rccl} \phi : & (\mathcal{P}(\mathbb{Z}_{\ge 0}))^n & \longrightarrow & \mathbb{Z}_{\ge 0} \\ & S = (S_1, \ldots, S_n) & \longmapsto & \bigoplus_{M \in \Lambda} \left(a_M \bigodot_{\substack{1 \le i \le n \\ 0 \le j \le r}} Val_{S_i}(j)^{\odot M_{ij}} \right). \end{array}$$

Example

In this example

$$\phi = (x^{(1)} \odot 1) \oplus (x^{(3)} \odot 2) \oplus 3,$$

for any $S \subset \mathcal{P}(\mathbb{Z}_{\geq 0})$*, we have*

$$\phi(S) = (Val_S(1) \odot 1) \oplus (Val_S(3) \odot 2) \oplus 3.$$

Definition

A n-tuple $S = (S_1, \ldots, S_n) \in \mathcal{P}(\mathbb{Z}_{\geq 0})^n$ is a solution of ϕ if either

1. *There exists $M, N \in \Lambda$, $M \neq N$, such that* $\phi(S) = a_M \bigodot_{\substack{1\leq i\leq n\\ 0\leq j\leq r}} Val_{S_i}(j)^{\odot M_{ij}} = a_N \bigodot_{\substack{1\leq i\leq n\\ 0\leq j\leq r}} Val_{S_i}(j)^{\odot N_{ij}}$, *or*
2. $\phi(S) = \infty$.

We denote solutions of ϕ by $Sol(\phi)$.
If T is a set of differential tropical polynomial, the set of solutions of T is :

$$Sol(T) := \{S \subset (\mathcal{P}(\mathbb{Z}_{\geq 0}))^n / S \in Sol(\phi), \forall \phi \in T\}.$$

Example

We consider $\phi(S) = (Val_S(1) \odot 1) \oplus (Val_S(1) \odot 2) \oplus 3.$
A set $S \subset \mathcal{P}(\mathbb{Z}_{\geq 0})$ *is a solution of* ϕ *if we have one of the following conditions:*

1. $1 \odot Val_S(1) = 3 \leq 2 \odot Val_S(3)$
2. $1 \odot Val_S(1) = 2 \odot Val_S(3) \leq 3$
3. $2 \odot Val_S(3) = 3 \leq 1 \odot Val_S(1)$

Definition

Let $\varphi \in \mathbb{K}[[t]]$ *with* $\varphi = \sum_{i=0}^{\infty} a_i t^i$, *the* **tropicalization** *of* φ *has this form*

$$trop(\varphi) := \{i \in \mathbb{Z}_{\geq 0} / a_i \neq 0\}$$

let $\varphi \in \mathbb{K}[[t]]^n$, $\varphi = (\varphi_1, ..., \varphi_n)$, *the* **tropicalization** *of* φ *is :*

$$trop(\varphi) := (trop(\varphi_1), ..., trop(\varphi_n)) \in (\mathcal{P}(\mathbb{Z}_{\geq 0}))^n.$$

Let $T \subset \mathbb{K}[[t]]^n$, *the* **tropicalization** *of* T *has this form*

$$trop(T) := \{trop(\varphi), \varphi \in T\}.$$

- *Let* I *be a differential ideal in* $\mathbb{K}[[t]][x_i^{(j)} \mid 1 \leq i \leq n, j \geq 0]$ *then* $Sol(I)$ *is in* $(\mathbb{K}[[t]])^n$.

Definition

Let

$$P = \sum_{M\in\Lambda} \varphi_M \prod_{\substack{1\le i\le n\\ 0\le j\le r}} (x_i^{(j)})^{M_{ij}} \in \mathbb{K}[[t]][x_i^{(j)} \mid 1 \le i \le n, j \ge 0]$$

be a differential polynomial, the **tropicalization** *of P is tropical differential polynomial*

$$trop(P) := \bigoplus_{M\in\Lambda} \left(\nu(\varphi_M) \bigodot_{\substack{1\le i\le n\\ 0\le j\le r}} (x_i^{(j)})^{\odot M_{ij}} \right).$$

Let I a differential ideal, the **tropicalization** *of I is*

$$trop(I) := \{trop(P), P \in I\}.$$

Proposition

(Extension of the fundamental theorem of tropical geometry) Let $\mathbb{K}$ *be an uncountable algebraically closed field of characteristic zero, let* $I \subset \mathbb{K}[[t]][x_i^{(j)} \mid 1 \leq i \leq n, j \geq 0]$ *be a differential ideal, then*

$$trop(Sol(I)) = Sol(trop(I))$$