

A bijection between permutation tableaux and EW tableaux

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joint work with Jason Smith and Einar Steingrímsson

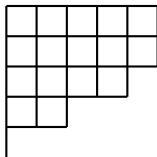
Journées ALEA, CIRM

22 March, 2017

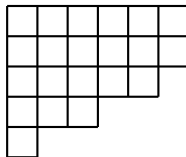
Ferrers diagram

Definition

A Ferrers diagram is a left-aligned collection of cells with a finite number of rows and columns such that the number of cells in each row is weakly decreasing.



(a) F



(b) F'

F' is the Ferrers diagram F with an extra column on the left-hand side.

Permutation tableaux

Definition (Postnikov 06)

A *permutation tableau* (PT) \mathcal{T} is a 0–1 filling of a Ferrers diagram, some of whose bottom-most rows may be empty, satisfying the following properties:

- 1 Every column of \mathcal{T} has a 1 in some cell.
- 2 If a cell has a 1 above it in the same column and a 1 to its left in the same row then it has a 1.

The *size* of a permutation tableau is the sum of its number of rows and number of columns.

0	0	0	1	0
0	1	1	1	1
0	0	1	1	
1	1			

(c) a PT

0	0	0	1	0
0	1	1	1	1
0	0	1	1	
1	0			

(d) not a PT

Definition (Ehrenborg, van Willigenburg 04)

An *EW-tableau* (EWT) \mathcal{T} is a 0–1 filling of a Ferrers diagram that satisfies the following properties:

- 1 The top row of \mathcal{T} has a 1 in every cell.
- 2 Every other row has at least one 0.
- 3 No four cells of \mathcal{T} that form the corners of a rectangle have 0s in two diagonally opposite corners and 1s in the other two.

The *size* of a EW-tableau is one less than the sum of its number of rows and number of columns.

1	1	1	1	1	1
1	1	0	0	0	1
0	1	0	0	0	
0	1	0			
0					

(e) an EWT

1	1	1	1	1	1
1	1	0	0	0	1
0	1	0	0	1	
0	1	0			
0					

(f) not an EWT

The main result

Theorem (Ehrenborg, van Willigenburg 04; S., Smith, Steingrímsson ++)

Let F be a Ferrers diagram (possibly with some empty rows). Then the number of PTs of shape F and the number of EWTs of shape F' are the same.

The main result

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- Formulation in different but equivalent form by EW (04).
Proof is recursive.

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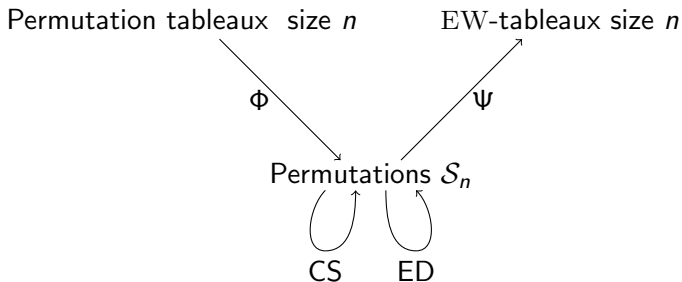
The main result

Theorem (Ehrenborg, van Willigenburg 04; S., Smith, Steingrímsson ++)

Let F be a Ferrers diagram (possibly with some empty rows). Then the number of PTs of shape F and the number of EWTs of shape F' are the same.

- Formulation in different but equivalent form by EW (04). Proof is recursive.
- A more bijective proof by Josuat-Vergès (10).
- We present a bijection through permutations.

Map of the proof



The map $\psi \circ ED \circ CS \circ \phi$ (restricted to permutation tableaux with a fixed shape F) is the desired bijection.

The map Φ from permutation tableaux to permutations

	9	8	6	5	3	
1	0	0	0	1	0	1
2	0	1	1	1	1	2
4	0	0	1	1	1	3
7	1	1	1	1	1	4
10	9	8	7	6	5	

- Label the rows and columns of the permutation tableau;

The map Φ from permutation tableaux to permutations

	9	8	6	5	3	
1	0	0	0	1	0	1
2	0	1	1	1	1	2
4	0	0	1	1		3
7	1	1				4
10						5
	9	8	7	6	5	4

- Label the rows and columns of the permutation tableau;
- For each $1 \leq i \leq n$, construct the path from i to σ_i : enter the row, resp. column, labelled i from the left, resp. top; traverse cells with 0; turn $S \rightarrow E$ or $E \rightarrow S$ when cell is 1; σ_i is the label of the edge through which the path exits on the South-East border;

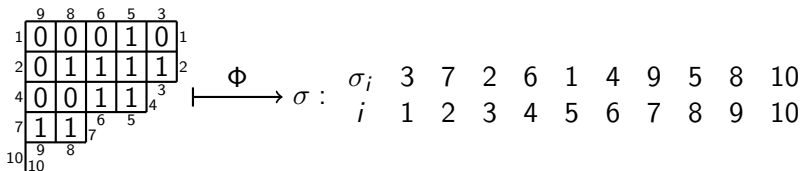
The map Φ from permutation tableaux to permutations

	9	8	6	5	3	
1	0	0	0	1	0	1
2	0	1	1	1	1	2
4	0	0	1	1		3
7	1	1				4
10						5
						6
						7
						8
						9
						10

- Label the rows and columns of the permutation tableau;
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- We get a map $\Phi(\mathcal{T}) := \sigma$

σ_i	3	7	2	6	1	4	9	5	8	10
i	1	2	3	4	5	6	7	8	9	10

The map Φ from permutation tableaux to permutations



For $\sigma \in \mathcal{S}_n$, $w_ex(\sigma) := \{1 \leq i \leq n; i \leq \sigma_i\}$ (weak excedences).

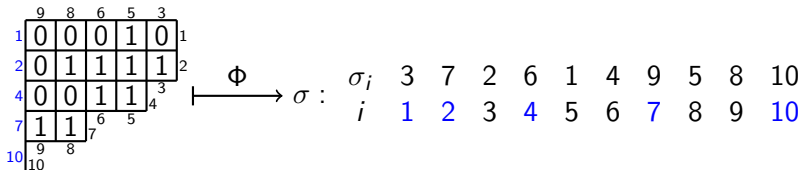
Theorem (Steingrímsson, Williams 07)

Let F be a Ferrers diagram of size n with row labels $RL(F)$, then

$$\Phi : PT(F) \longrightarrow \{\sigma \in \mathcal{S}_n; w_ex(\sigma) = RL(F)\}$$

is a bijection.

The map Φ is a bijection



$$\Phi : \text{PT}(F) \longrightarrow \{\sigma \in \mathcal{S}_n; w_{\text{ex}}(\sigma) = RL(F)\}$$

- $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ is a bijection (can construct σ^{-1}).
- $w_{\text{ex}}(\sigma) = RL(F)$.
- Φ is a bijection. We can construct Φ^{-1} .

Cyclic Shift CS

For $\sigma = \sigma_1 \cdots \sigma_n \in \mathcal{S}_n$, define $\tilde{\sigma} := CS(\sigma) = \sigma_2 \cdots \sigma_n \sigma_1$.

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For $\sigma = \sigma_1 \cdots \sigma_n \in \mathcal{S}_n$, define $\tilde{\sigma} := CS(\sigma) = \sigma_2 \cdots \sigma_n \sigma_1$.

$CS :$	σ_i	3	7	2	6	1	4	9	5	8	10
	i	1	2	3	4	5	6	7	8	9	10

CS

$\tilde{\sigma}_i$	7	2	6	1	4	9	5	8	10	3
i	1	2	3	4	5	6	7	8	9	10

Cyclic Shift CS

For $\sigma = \sigma_1 \cdots \sigma_n \in \mathcal{S}_n$, define $\tilde{\sigma} := CS(\sigma) = \sigma_2 \cdots \sigma_n \sigma_1$.

$$CS : \begin{array}{cccccccccc} \sigma_i & 3 & 7 & 2 & 6 & 1 & 4 & 9 & 5 & 8 & 10 \\ i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}$$

↓ CS

$$\begin{array}{cccccccccc} \tilde{\sigma}_i & 7 & 2 & 6 & 1 & 4 & 9 & 5 & 8 & 10 & 3 \\ i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}$$

For $\tilde{\sigma} \in \mathcal{S}_n$, $\text{exc}(\tilde{\sigma}) := \{1 \leq i \leq n; i < \tilde{\sigma}_i\}$ ((strong) excedences).

Proposition

For any $2 \leq a_1 < \cdots < a_k$,

$$\begin{array}{l} CS : \quad \{\sigma \in \mathcal{S}_n; \text{w_ex}(\sigma) = \{1, a_1, \dots, a_k\}\} \\ \longrightarrow \quad \{\tilde{\sigma} \in \mathcal{S}_n; \text{exc}(\tilde{\sigma}) = \{a_1 - 1, \dots, a_k - 1\}\} \end{array}$$

is a bijection.

The map ED : $\tilde{\sigma} \mapsto \tau$

Algorithm: inputs $\tilde{\sigma}$, outputs τ .

0. Initialise $\tau = \emptyset$.
1. $j = \min\{1 \leq i \leq n; i \notin \tau\}$. If $j = +\infty$ return τ . Else $j' = j$ and proceed to 2.
2. Find k s.t. $\tilde{\sigma}_k = j'$. $\tau \leftarrow \tau * k$. If $k \neq j$ then $j' = k$ and repeat 2. Else return to 1.

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Example: $\tilde{\sigma} :$

$\tilde{\sigma}_i$	7	2	6	1	4	9	5	8	10	3
i	1	2	3	4	5	6	7	8	9	10

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Example: $\tilde{\sigma} :$

$\tilde{\sigma}_i$	7	2	6	1	4	9	5	8	10	3
i	1	2	3	4	5	6	7	8	9	10

$$\tau = 4$$

The map $ED : \tilde{\sigma} \mapsto \tau$

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Example: $\tilde{\sigma} :$

$\tilde{\sigma}_i$	7	2	6	1	4	9	5	8	10	3
i	1	2	3	4	5	6	7	8	9	10

$$\tau = 4\ 5$$

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Example: $\tilde{\sigma} :$

$\tilde{\sigma}_i$	7	2	6	1	4	9	5	8	10	3
i	1	2	3	4	5	6	7	8	9	10

$$\tau = 4\ 5\ 7$$

The map $ED : \tilde{\sigma} \mapsto \tau$

Algorithm: inputs $\tilde{\sigma}$, outputs τ .

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Example: $\tilde{\sigma} :$

$\tilde{\sigma}_i$	7	2	6	1	4	9	5	8	10	3
i	1	2	3	4	5	6	7	8	9	10

$$\tau = 4\ 5\ 7\ 1$$

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$\tau = 4\ 5\ 7\ 1\ 2$

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$\tau = 4\ 5\ 7\ 1\ 2\ 10$

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$\tau = 4\ 5\ 7\ 1\ 2\ 10\ 9$

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Example: $\tilde{\sigma} :$

$\tilde{\sigma}_i$	7	2	6	1	4	9	5	8	10	3
i	1	2	3	4	5	6	7	8	9	10

$\tau = 4\ 5\ 7\ 1\ 2\ 10\ 9\ 6$

The map $ED : \tilde{\sigma} \mapsto \tau$

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$\tau = 4\ 5\ 7\ 1\ 2\ 10\ 9\ 6\ 3$

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$\tilde{\sigma}_i$	7	2	6	1	4	9	5	8	10	3
i	1	2	3	4	5	6	7	8	9	10

$\tau = 4\ 5\ 7\ 1\ 2\ 10\ 9\ 6\ 3\ 8.$

The map ED

$$\begin{array}{r} \tilde{\sigma} : \\ \tilde{\sigma}_i \quad 7 \quad 2 \quad 6 \quad 1 \quad 4 \quad 9 \quad 5 \quad 8 \quad 10 \quad 3 \\ i \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \end{array}$$
$$\tau = 4 \ 5 \ 7 \ 1 \ 2 \ 10 \ 9 \ 6 \ 3 \ 8.$$

For $\tau \in \mathcal{S}_n$, $\text{des_bot}(\tau) := \{\tau_i; 2 \leq i \leq n \text{ and } \tau_i < \tau_{i-1}\}$.

Theorem (Steingrímsson 92)

For any $A \subseteq \{2, \dots, n\}$,

$$ED : \{\tilde{\sigma} \in \mathcal{S}_n; \text{exc}(\tilde{\sigma}) = A\} \rightarrow \{\tau \in \mathcal{S}_n; \text{des_bot}(\tau) = A\}$$

is a bijection.

ED is a bijection : proof

$$\begin{array}{r} \tilde{\sigma} : \\ \tilde{\sigma}_i \quad 7 \quad 2 \quad 6 \quad 1 \quad 4 \quad 9 \quad 5 \quad 8 \quad 10 \quad 3 \\ i \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \end{array}$$
$$\tau = 4 \ 5 \ 7 \ 1 \ 2 \ 10 \ 9 \ 6 \ 3 \ 8.$$

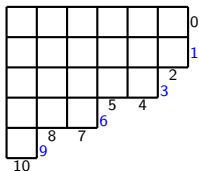
- ED : $\mathcal{S}_n \longrightarrow \mathcal{S}_n$ is a bijection.
- To see that $\text{exc}(\tilde{\sigma}) = \text{des_bot}(\tau)$, notice that

$$\cdots \begin{array}{r} \tilde{\sigma}_i \\ i \end{array} \cdots \longleftrightarrow \cdots \tilde{\sigma}_i i \cdots = \tau$$

The map Ψ

$$\tau = 4\ 5\ 7\ 1\ 2\ 10\ 9\ 6\ 3\ 8.$$

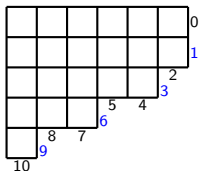
Label edges of a Ferrers diagram F' (with no empty rows),
starting at 0: rows and columns labelled $0, \dots, n$ if $n = \text{size}(F')$.



The map Ψ

$$\tau = 4\ 5\ 7\ 1\ 2\ 10\ 9\ 6\ 3\ 8.$$

Label edges of a Ferrers diagram F' (with no empty rows), **starting at 0**: rows and columns labelled $0, \dots, n$ if $n = \text{size}(F')$.



We have $RL'(F') = \text{des_bot}(\tau)$.

The map Ψ

$\tau = 4\ 5\ 7\ 1\ 2\ 10\ 9\ 6\ 3\ 8$. Fill the cells of F' as follows.

- Fill the cells of the top row with 1.
- Read τ from left to right. For a letter τ_i :
 - If τ_i is a column, fill remaining cells of that column with 0.
 - If τ_i is a row, fill remaining cells of that row with 1.
- $\mathcal{T}' := \Psi(\tau)$ is the filling of F' we obtain.

The map Ψ

$\tau = 4\ 5\ 7\ 1\ 2\ 10\ 9\ 6\ 3\ 8$. Fill the cells of F' as follows.

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									0
									1
									2
									3
									4
									5
									6
									7
									8
									9
									10

The map Ψ

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1	1	1	1	1	1	0
						1
						2
						3
						4
						5
						6
						7
						8
						9
						10

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1	1	1	1	1	1	0
				0		1
				0		2
						3
						4
						5
						6
						7
						8
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						10

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1	1	1	1	1	1	0
			0	0		1
			0	0		2
						3
						4
						5
						6
						7
						8
						9
						10

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1	1	1	1	1	1	0
		0	0	0		1
		0	0	0		2
		0				3
		0				4
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1	1	1	1	1	1	0
1	1	0	0	0	1	1
		0	0	0		2
		0				3
		0				4
						5
						6
						7
						8
						9
						10

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1	1	1	1	1	1	0
1	1	0	0	0	1	1
0		0	0	0		2
0		0				3
0		0				4
0						5
						6
						7
						8
						9
						10

The map Ψ

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 - If τ_j is a row, fill remaining cells of that row with 1.
- $\mathcal{T}' := \Psi(\tau)$ is the filling of F' we obtain.

1	1	1	1	1	1	0
1	1	0	0	0	1	1
0		0	0	0		2
0	1	0				3
0						4
						5
						6
						7
						8
						9
						10

The map Ψ

$\tau = 4\ 5\ 7\ 1\ 2\ 10\ 9\ 6\ 3\ 8$. Fill the cells of F' as follows.

- Fill the cells of the top row with 1.
- Read τ from left to right. For a letter τ_i :
 - If τ_i is a column, fill remaining cells of that column with 0.
 - If τ_i is a row, fill remaining cells of that row with 1.
- $\mathcal{T}' := \Psi(\tau)$ is the filling of F' we obtain.

1	1	1	1	1	1	0
1	1	0	0	0	1	1
0	1	0	0	0	0	2
0	1	0	5	4	3	
0	8	7	6			
9						10

The map Ψ

$$\tau = 4\ 5\ 7\ 1\ 2\ 10\ 9\ 6\ 3\ 8 \xrightarrow{\Psi} \begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & & & \\ 0 & & & & & \end{array}$$

₀
₁
₂
₃
₄
₅
₆
₇
₈
₉
₁₀

Theorem (S., Smith, Steingrímsson ++)

Let $A \subseteq \{2, \dots, n\}$ and F' be the Ferrers diagram such that $RL'(F') = A$. Then:

$$\Psi : \{\tau \in \mathcal{S}_n, \text{des_bot}(\tau) = A\} \longrightarrow EWT(F')$$

is a bijection.

- 1 $\Psi(\tau)$ is an EWT by construction.
- 2 Construct Ψ^{-1} . First erase all entries in the top row. Key lemma: the resulting tableau has at least one all-0 column. Erasing entries as we go, we:

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 - record all-0 columns in increasing order,
 - record all-1 rows in decreasing order,and iterate.

Some properties

- A column labelled i of \mathcal{T}' is all-1 iff i is a fixed point of $\tilde{\sigma}$.
- The number of almost-all-0 columns of \mathcal{T}' is the place of first descent in τ .
- The top row of \mathcal{T} is all-0 iff the leftmost column of \mathcal{T}' is its only almost-all-0 column.
- A non top row labelled i of \mathcal{T} is all-0 iff (conditions on row $i - 1$ in \mathcal{T}').
- There is a bijection between (0-minimal EWTs of size n with $k + 1$ columns) and (the set of binary strings of length $2n - 3$ with k 1s and no 11). In particular, the number of 0-minimal EWTs of size n is $\text{Fib}(2n - 2)$.
- τ avoids 213 iff (rows of \mathcal{T}' are $0 \cdots 01 \cdots 1$ and any leftmost 1 in a row has no 1 directly beneath it).
- ...

- Other statistics? Number of 1s in \mathcal{T} ?

- Other statistics? Number of 1s in \mathcal{T} ?
- PTs are linked to the PASEP model (Corteel and Williams 07). Link between PASEP and EWTs? In particular, what does cell deletion in \mathcal{T} correspond to in \mathcal{T}' ?
- Generalisations of PTs and EWTs. Link to Tree-like tableaux (Aval, Boussicault, Nadeau 13). Spanning trees? Sandpile model?

Thank you!