Random cubic planar graphs revisited

Marc Noy, Clément Requilé, **Juanjo Rué**

Journées ALEA 2017
CIRM, Luminy

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONA
Departament de Matemàtiques

BGSMath
BARCELONA GRADUATE SCHOOL OF MATHEMATICS

SEVENTH FRAMEWORK PROGRAMME

The starting point

[redacted]

Random Cubic Planar Graphs

Manuel Bodirsky,[1,*] Mihyun Kang,[1,†] Mike Löffler,[1] Colin McDiarmid[2]
[1] *Humboldt-Universität zu Berlin, Institut für Informatik, Unter den Linden 6, D-10099 Berlin, Germany;*
e-mail: {bodirsky,kang,loeffler}@informatik.hu-berlin.de
[2] *University of Oxford, Department of Statistics, 1 South Parks Road, Oxford OX 1 3TG, United Kingdom; e-mail: cmcd@stats.ox.ac.uk*

Received 30 September 2005; accepted 7 February 2006; received in final form 18 July 2006
Published online 1 December 2006 in Wiley InterScience (www.interscience.wiley.com).
DOI 10.1002/rsa.20149

The model

Study of **simple labelled** cubic planar graphs

- **Enumeration** (exact and asymptotics).
- **Parameters** in a random cubic planar graph.

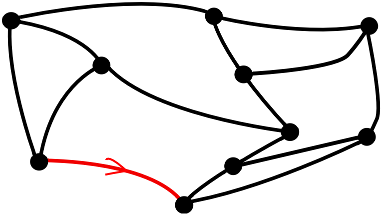
Ingredients:

generatingfunctiology, asymptotic analysis and probability.

Enumeration: rooting

In order to get counting formulas we use **Tutte's rooting**.

- We distinguish one edge+orientation: $R(x)$



$C(x)$: EGF for **connected** cubic planar graphs, then

$$R(x) = 3xC'(x) = 3C^{\bullet}$$

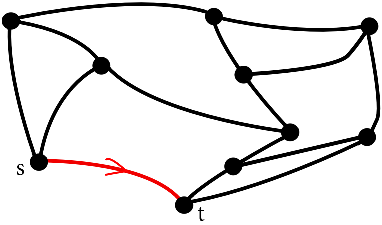
If we get $3C^{\bullet}$, then we have $C(x)$ (by integration).

Enumeration: networks

We describe $C^\bullet$ in terms of **networks**

Connected cubic multigraph with an ordered pair of adjacent vertices (s, t), such that the graph obtained by removing the edge st is simple.

- The edge st can be a *loop*.
- The edge st can belong to a *multiple edge*.
- The edge st is *simple*.



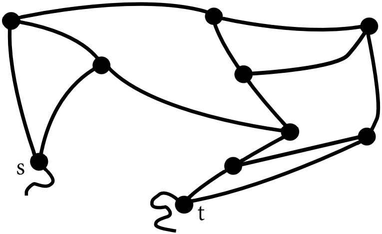
Strategy: decomposition *à la Tutte*

Enumeration: networks

We describe $C^\bullet$ in terms of **networks**

Connected cubic multigraph with an ordered pair of adjacent vertices (s, t), such that the graph obtained by removing the edge st is simple.

- The edge st can be a *loop*.
- The edge st can belong to a *multiple edge*.
- The edge st is *simple*.



Strategy: decomposition *à la Tutte*

All possible cases for networks

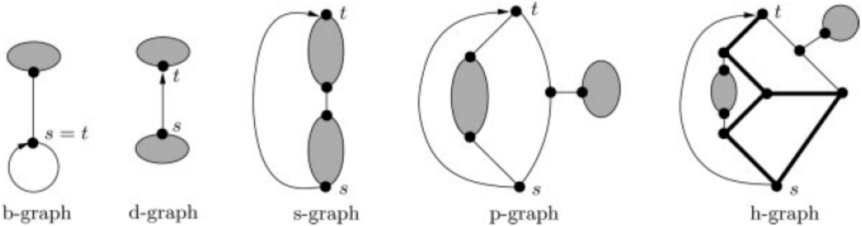


Fig. 1. The five types of rooted cubic graphs in Lemma 1.

(From Bodirsky et al.)

Using the Symbolic Method Dictionary

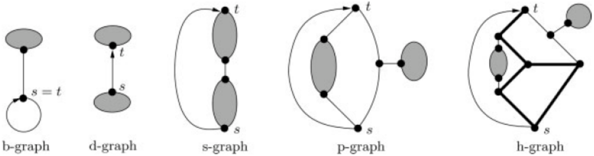


Fig. 1. The five types of rooted cubic graphs in Lemma 1.

$$D = L + S + P + H$$

$$L = \frac{x^2}{2}(I + D - L)$$
$$I = \frac{L^2}{x^2}$$
$$S = D(D - S)$$
$$P = x^2 D + x^2 \frac{D^2}{2}$$

What about h-graphs?

Visiting the map reialm

- h-graph: obtained from a **3-con. cubic graph** by pasting a network on each edge, except the root.
- Whitney+ Duality $\Rightarrow$ rooted triangulations

Tutte: triangulations (parameter: vertices minus 2)

$$T(z) = u(z)(1 - 2u(z)),\ z = u(z)(1 - u(z))^3$$

Labelled 3-connected cubic graphs, rooted at a directed edge:

$$M(x, y) = \frac{1}{2}(T(x^2y^3) - x^2y^3)$$

Properties of T

- Unique dominant singularity at $\tau = 27/256$, $T(\tau) = 1/8$.
- $T(z)$ has singular expansion at $z = \tau$

$$\frac{1}{8} - \frac{3}{16}Z^2 + \frac{\sqrt{6}}{24}Z^3 - \frac{13}{192}Z^4 + \frac{35\sqrt{6}}{1728}Z^5 + O(Z^6).$$

Using the Symbolic Method Dictionary

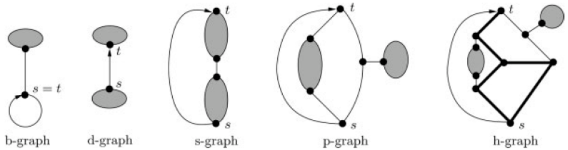


Fig. 1. The five types of rooted cubic graphs in Lemma 1.

$$D = L + S + P + H$$

$$L = \frac{x^2}{2}(I + D - L)$$
$$I = \frac{L^2}{x^2}$$
$$S = D(D - S)$$
$$P = x^2 D + x^2 \frac{D^2}{2}$$
$$H = \frac{M(x, 1+D)}{1+D} = \frac{1}{2(1+D)} \left(T(x^2(1+D)^3) - x^2(1 + D^3)\right)$$

$$3C^\bullet = D - L + I - x^2 D - L^2$$
$$G = \exp(C)$$

Asymptotic enumeration (I)

We have equations, then we can analyze them

$$P(x) = x^2C(x) + x^2C(x)^2/2. \tag{10}$$

We can also describe the substitution in Eq. (1) for $H(x)$ algebraically, using Eqs. (4) and (5).

$$2(C(x) + 1)H(x) = u(1 - 2u) - u(1 - u)^3 \tag{11}$$

$$x^2(C(x) + 1)^3 = u(1 - u)^3. \tag{12}$$

Using algorithms for computing resultants and factorizations (these are standard procedures in e.g., Maple or Mathematica), we obtain a single algebraic equation $Q(C(x), x) = 0$ from Eqs. (6)–(12) that describes the generating function $C(x)$ uniquely, given sufficiently many initial terms of c_n. This is in principle also possible for all other generating functions involved in the above equations; however, the computations turn out to be more tedious, whereas the computations to compute the algebraic equation for $C(x)$ are manageable.

Asymptotic enumeration (and II)

Theorem 2. *The asymptotic number of cubic planar graphs, connected cubic planar graphs, 2-connected cubic planar graphs, and 3-connected cubic planar graphs is given by the following. For large even n*

$$g_n^{(0)} \sim \alpha_0\, n^{-7/2}\, \rho^{-n}\, n!$$
$$g_n^{(1)} \sim \alpha_1\, n^{-7/2}\, \rho^{-n}\, n!$$
$$g_n^{(2)} \sim \alpha_2\, n^{-7/2}\, \eta^{-n}\, n!$$
$$g_n^{(3)} \sim \alpha_3\, n^{-7/2}\, \theta^{-n}\, n!.$$

All constants are analytically given. Also $\alpha_1/\alpha_0 = e^{-\lambda}$ *where* $\lambda = G^{(1)}(\rho)$*. The first digits of* ρ^{-1}*,* η^{-1}*, and* θ^{-1} *are* 3.132595 , 3.129684, *and* 3.079201, *respectively.*

Q: complete arguments, and free of resultants?
Q: which are the values of α_i? And λ?

1.- Asymptotic analysis

Direct analysis

Easy expression for D:

$$F(x, D) := (1+D)\sqrt{\frac{x^4}{4} + 1 - x^2(D-1)} - \frac{T(x^2(1+D)^3)}{2} - 1 = 0.$$

Possible sources of singularities for D:

- A **branch point** from equation $F(x, D) = 0$.
- A **critical composition scheme**: $T(x^2(1+D)^3)$.

Study the system $F(x, D) = 0$, $F_D(x, D) = 0$

$$\Downarrow$$

NO branch point

Singularity analysis: networks

We have a critical composition scheme:

$$\rho^2(1+D(\rho))^3 = 27/256, \quad F(\rho, D(\rho)) = 0.$$

Use that $T(\tau) = 1/8$, we solve the system:

$$\rho \approx \pm 0.3192246062, \; D_0 = D(\rho) = 0.0115259444.$$

We get the singular expansion for D from the singular expansion for T + equation $F(x, D) = 0$.

$$D(x) = D_0 + D_2 X^2 + D_3 X^3 + O(X^4), \; X = (1 - x/\rho)^{1/2},$$

$$D_1 = 0, \, D_2 \approx -0.1182076128, \, D_3 \approx 0.2542672141$$

Singularity analysis: graphs

From the singular expansion of D we get the singular expansion of the rest of the networks.

$$\Downarrow\Downarrow$$

We get the singular expansion for $C^{\bullet}$

$$C^{\bullet} = C_0^{\bullet} + C_2^{\bullet} X^2 + C_3^{\bullet} X^3 + O(X^4),\ C_3^{\bullet} \approx 0.2256967553.$$

By singularity analysis:

$$nC_n = n![x^n]C^{\bullet} \sim n!(\rho^{-n} + (-\rho)^{-n}) \cdot \frac{2C_3^{\bullet}}{\Gamma(-3/2)} n^{-5/2}.$$

dividing by n we get the result for the connected level.

2-connected cubic planar graphs

Similar arguments, different type of networks:

$$E = S + P + H$$

$$\begin{aligned} S &= E(E - S) \\ P &= x^2 E + x^2 \tfrac{E^2}{2} \\ H &= \tfrac{M(x,1+E)}{1+E} \end{aligned}$$

$$3B^\bullet = E - x^2 E.$$

By a similar analysis we get the asymptotics

$$\rho_b \approx \pm 0.3195228840; |\rho_b| > 0.3192246062 = |\rho|$$

Exponentially less!

2.- Connectivity constant

Asymptotic for general cubic planar

$$C^\bullet = C_0^\bullet + C_2^\bullet X^2 + C_3^\bullet X^3 + O(X^4)$$

$$\Downarrow$$

$$C(x) = C_0 + C_2 X^2 + C_4 X^4 + C_5 X^5 + O(X^6), \ C_0 = C(\rho).$$

$$\Downarrow$$

$$G = \exp(C) = G_0 + G_2 X^2 + G_4 X^4 + G_5 X^5$$
$$= \exp(C_0)(1 + \cdots + C_5 X^5 + O(X^6)).$$

The prob. of connectivity of a random cubic planar graph is

$$p = \exp(-C_0) = \exp(-C(\rho)) = \exp(-\lambda).$$

Connectivity Constant

The value for λ: connectivity constant

7.1. Connectedness

Theorem 3. *Let* $\lambda = G^{(1)}(\rho)$*. As* $n \to \infty$ *with* n *even,* $\Pr(G_n^{(0)}$ *is connected*$) \to e^{-\lambda}$*, whereas each of* $\Pr(G_n^{(0)}$ *is 2-connected*$)$*,* $\Pr(G_n^{(1)}$ *is 2-connected*$)$ *and* $\Pr(G_n^{(2)}$ *is 3-connected*$)$ *tends to 0.*

Proof. From Theorem 2, we see that as $n \to \infty$ with n even

$$\Pr\left(G_n^{(0)} \text{ is connected}\right) = g_n^{(1)}/g_n^{(0)} \to \alpha_1/\alpha_0 = e^{-\lambda}.$$

Also,

$$\Pr\left(G_n^{(0)} \text{ is 2-connected}\right) = g_n^{(2)}/g_n^{(0)} \sim \alpha_2/\alpha_0(\eta/\rho)^{-n} \to 0,$$

with a similar proof in the other cases. ∎

Using the numbers in Table 1 we compute the probability that $G_n^{(0)}$ is connected, up to $n = 20$, in Table 2.

TABLE 2. The Probability that a Random Cubic Planar Graph is Connected

n	4	6	8	10	12	14	16	18	20
$g_n^{(1)}/g_n^{(0)}$	1	1	0.997403	0.997837	0.997982	0.998117	0.998249	0.998368	0.998472

Random Structures and Algorithms DOI 10.1002/rsa

Q: Which is the value of λ?

Some difficulties...

We can try to integrate $C^\bullet$ but...

- The genus of the equation is high.
- For planar graphs is 0 (Giménez, Noy).

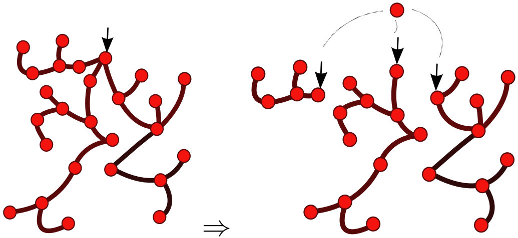
$$\Downarrow$$

We use an indirect way to 'integrate' the equation:

Dissymmetry theorem

A toy example: trees

ROOTED trees



$$T(x) = xe^{T(x)}$$

To forget the root, we just integrate: $(xU'(x) = T(x))$

$$\int_0^x \frac{T(s)}{s} ds = \left\{ \begin{array}{c} T(s) = u \\ T'(s)\, ds = du \end{array} \right\} = \int_{T(0)}^{T(x)} 1 - u\, du = T(x) - \frac{1}{2} T(x)^2$$

The Dissymmetry Theorem for trees

That can be explained only by combinatorial means:

$$\mathcal{T} \cup \mathcal{T}_{\bullet\to\bullet} \simeq \mathcal{T}_{\bullet-\bullet} \cup \mathcal{T}_{\bullet}$$

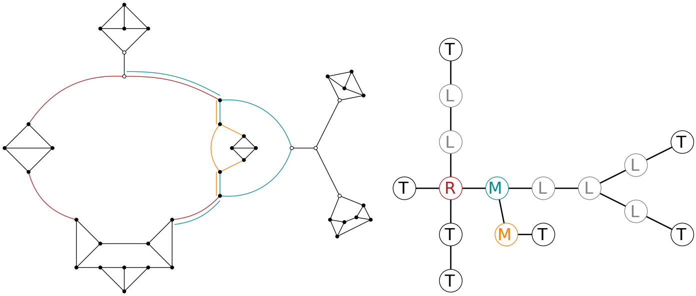
where $\mathcal{T}_{*}$ is the family of trees $\mathcal{T}$ with an extra rooted structure.

$$U(x) = U_{\bullet-\bullet}(x) + U_{\bullet}(x) - U_{\bullet\to\bullet}(x) = \frac{1}{2}T(x)^2 + T(x) - T(x)^2$$

We use an extension of this result to tree-like families

A tree-like family

Based on [Chapuy, Fusy, Kang, Shoilekova]



4 types of vertices in the tree:

- Vertex L: cut-vertex
- Vertex R: Ring (Series operation)
- Vertex M: multiedge (Parallel operation)
- Vertex T: edge-substitution on a 3-con. cubic graph

Several combinatorial restrictions...

Putting all together

We get an **explicit** expression of $C(x)$ in terms of networks:

$$\begin{aligned} C = \; & C_{\mathcal{R}} + C_{\mathcal{M}} + C_{\mathcal{T}} + C_{\mathcal{L}} + C_{\mathcal{M}-\mathcal{M}} + C_{\mathcal{T}-\mathcal{T}} + C_{\mathcal{L}-\mathcal{L}} \\ & + C_{\mathcal{R}-\mathcal{M}} + C_{\mathcal{R}-\mathcal{T}} + C_{\mathcal{R}-\mathcal{L}} + C_{\mathcal{M}-\mathcal{T}} + C_{\mathcal{M}-\mathcal{L}} + C_{\mathcal{T}-\mathcal{L}} \\ & -2(C_{\mathcal{R}\to\mathcal{M}} + C_{\mathcal{R}\to\mathcal{T}} + C_{\mathcal{R}\to\mathcal{L}} + C_{\mathcal{M}\to\mathcal{T}} + C_{\mathcal{M}\to\mathcal{L}} + C_{\mathcal{T}\to\mathcal{L}}) \\ & -(C_{\mathcal{M}\to\mathcal{M}} + C_{\mathcal{T}\to\mathcal{T}} + C_{\mathcal{L}\to\mathcal{L}}). \end{aligned}$$

$C(x)$ is equal to ($\overline{M} = \frac{1}{2}\int \frac{1}{y} M(x,y)dy$)

$$\begin{aligned} & \frac{x^2}{2}\left(\frac{D^2}{2} + \frac{D^3}{6}\right) + \overline{M}(x, 1+D) + \frac{L^3}{6x^2} \\ & -\frac{1}{2}\left(\log(1-D+S) + (D-S) + \frac{(D-S)^2}{2} + P(S+H) + HS\right) \\ & -\frac{1}{2}\left(\frac{P^2+H^2}{2} + \frac{L^2}{x^2}\right). \end{aligned}$$

We get $\lambda \simeq 0.0006035047 \to \exp(-\lambda) \simeq 0.9993966774.$

An extension: random cubic multigraphs

We cannot get a $C^\bullet$-type equation for multigraphs:

- If we want, then we need to distinguish the edge type.
- No way to do it with a single variable.

After getting a similar system for networks + dissymmetry theorem we get

$$\frac{x^2}{2}\left(1 + D + \frac{D^2}{2} + \frac{D^3}{6}\right) + \overline{M}(x, 1+D) + \frac{L^3}{6x^2}$$
$$-\frac{1}{2}\left(\log(1 - D + S) + D - S + \frac{(D-S)^2}{2} + P(S+H)\right)$$
$$-\frac{1}{2}\left(HS + \frac{P^2+H^2}{2} + \frac{L^2}{x^2} + L^2\right).$$

and the probability of connectivity is $\simeq 0.9029052067$.

3.- Subgraphs. Triangles

Some results in subgraphs

7.3. Triangles and Other Subgraphs

In order to discuss coloring later we also need to know about triangles, in particular the unsurprising result that $G_n^{(k)}$ usually contains at least one triangle. In fact, far more is true.

Lemma 5. *Let $Y_n^{(k)}$ be the number of triangles in $G_n^{(k)}$. Then there exists $\delta > 0$ such that for even n*

$$\Pr\left(Y_n^{(k)} \geq \delta n\right) = 1 - e^{-\Omega(n)}.$$

Theorem 5. *Let H be a fixed connected planar graph with one vertex of degree 1 and each other vertex of degree 3. Let k be 0 or 1. Then there exists $\delta > 0$ such that for even n*

$$\Pr\left(G_n^{(k)} \text{ contains } < \delta n \text{ copies of } H\right) = e^{-\Omega(n)}.$$

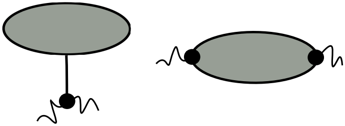
Note that each copy of H contributes at least one cut-edge to the graph, and each such edge is counted at most twice, so we see that $G_n^{(0)}$ and $G_n^{(1)}$ are very far from being 2-edge-connected; see Theorem 3 above.

Q: Precise distribution for these r.v.?

Some normal distributions

By refining the equations for networks we can encode:

- Cut-vertices, bridges, blocks.
- Cherries and bricks.



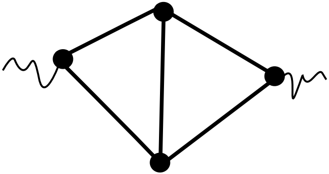
Normal distributions with linear expectation and variance (Quasi Powers).

Parameter	μ	σ^2
Cut-vertices	0.0018774448	0.0037934519
Bridges	0.0009389848	0.0009496835
Blocks	0.0018777072	0.0037958302

For cherries and bricks these depend on the object.

Triangles (I)

Except for K_4, triangles in cubic graphs are disjoint



We can refine the previous equations in order to encode the number of triangles:

- Encode networks whose root is adjacent with triangles.
- When forgetting about the parameter for triangles, we get the initial system.

We need to encode triangles in the 3-conn. level

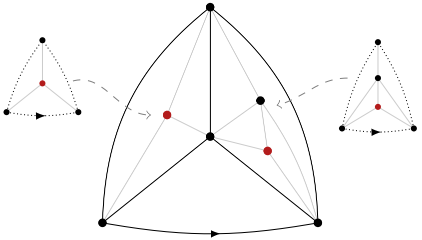
Triangles (II)

KEY observation: in a 3-con. triangulation, vertices of degree 3 transform (duality) in triangles.

We start from 4-con. triangulations: no separating triangle (Tutte)

$$T_4(z) = z + v(z)(v(z)-1)(v(z)+1)^{-2} - z^2, \; z = v(z)(1-v(z))^2$$

- We create vertices of degree 3 when pasting copies of K_4



We can get $T(z,u)$, where u encodes triangles

The result

Combining everything:

- 4-connected triangulations
- Refined grammar for networks to encode triangles

By Hwang's Quasi-Powers Theorem we get normal distribution for triangles:

$$\mathbb{E}[Y_n^{(1)}] \sim 0.1219742813n, \ \mathbb{V}[Y_n^{(1)}] \sim 0.0649847862n$$

By putting the parameter to 0, we encode **triangle-free** cubic planar graphs, and we get :

$$t_n \sim t \cdot n^{-7/2} \gamma_t^n n!, \, \gamma_t \approx 2.6466859711.$$

Other random variables

1. Number of components: $1 + Po(\lambda)$.

2. Size of the largest 3-connected component: K_n

$$\mathbf{P}(K_n = \lfloor \alpha n + x n^{2/3} \rfloor) \sim n^{-2/3} c \mathcal{A}(cx),$$

where $\alpha \approx 0.42543$ and $\mathcal{A}$ is the Airy function [Banderier, Flajolet, Schaeffer, Soria]

$$\Downarrow$$

Planar-like family: picture similar to maps

Things to do

Things to do

Q: Counting 4-cycles: 5-con. triangulations [Gao, Wanless, Wormald].

- [Chapuy, Fusy, Giménez, Noy] Diameter for **planar graphs** is $n^{1/4+o(1)}$
- [Panagiotou, Stufler, Weller] Diameter for **subcritical graphs** is $\sqrt{n}$, and scaling limit converge to the **Brownian Continuum Random Tree**

Q: can one get diameter $n^{1/4}$? Scaling limits on this setting?

Thank you

Random cubic planar graphs revisited

Marc Noy, Clément Requilé, **Juanjo Rué**

Journées ALEA 2017
CIRM, Luminy

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONA
Departament de Matemàtiques

BGSMath
BARCELONA GRADUATE SCHOOL OF MATHEMATICS

SEVENTH FRAMEWORK PROGRAMME