Counting planar Eulerian orientations

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LaBRI, Bordeaux

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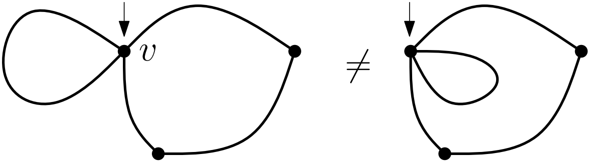
Some definitions

We consider:

- **planar maps** , rooted in a corner
- with loops and multiple edges



n: number of edges $(= 4)$
v is the root-vertex
Δ: root-degree $(= 4)$

Adding structure

Statistical physics and combinatorics: maps equipped with a structure

- proper q-colouring [Tutte 73-84...]
- spanning tree [Mullin 67...]
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In this talk → **Eulerian orientations**

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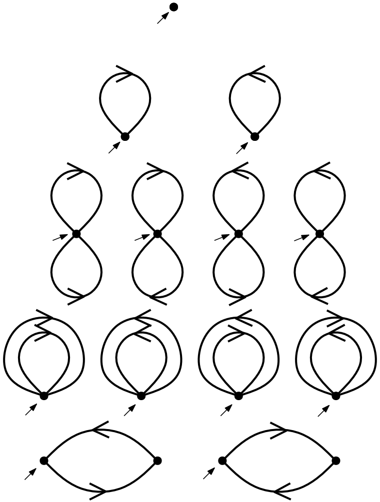
n = 1

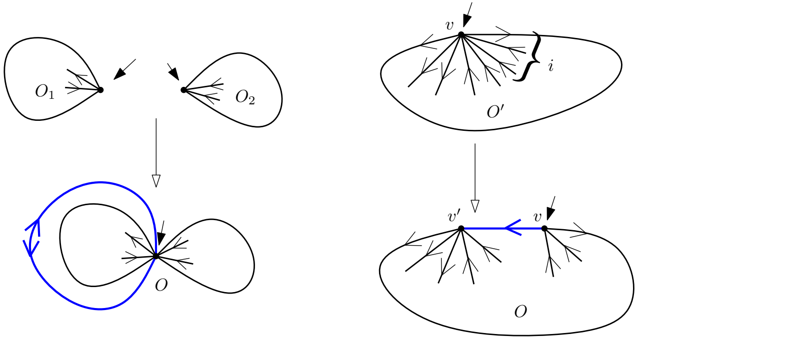
n = 2



Decomposition of PEO

Two ways of creating a PEO:

- merge two PEOs O_1, O_2 and orient the new edge
- split the root-vertex at index i iff the resulting map is still a PEO



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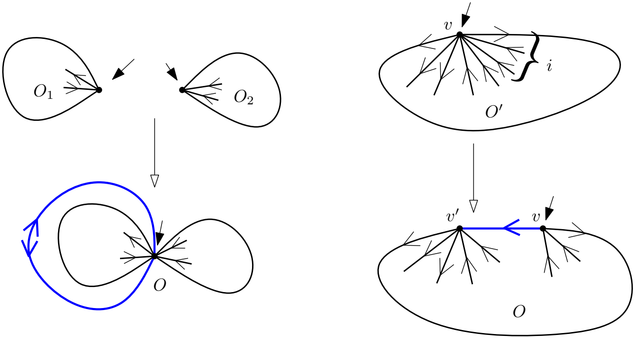

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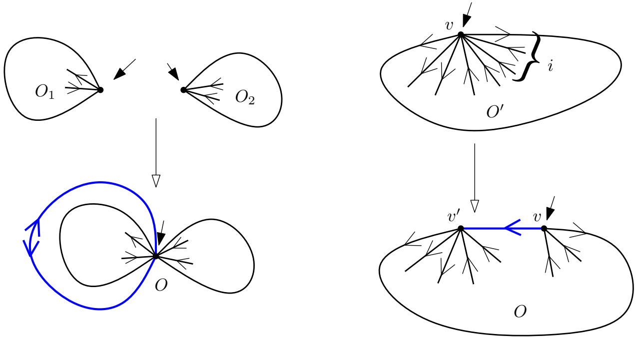

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Remember the **full orientation** around the root: no recurrence relation with a finite number of parameters

Computing the first terms

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n	$o(n)$	n	$o(n)$	n	$o(n)$
0	1	6	37 548	12	37 003 723 200
1	2	7	350 090	13	393 856 445 664
2	10	8	3 380 520	14	4 240 313 009 272
3	66	9	33 558 024	15	46 109 094 112 170
4	504	10	340 670 720		
5	4 216	11	3 522 993 656		

Not already in the OEIS!

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Variant of Fekete's Lemma (1923): $\mu = \sup_{n\geq 1} o(n)^{1/n} \in \mathbb{R}_+^*$
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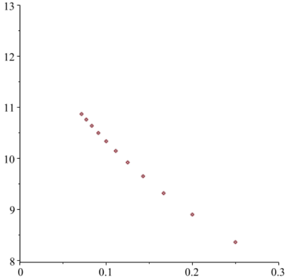
$\frac{o(n+1)}{o(n)}$ as a function of $1/n \to$



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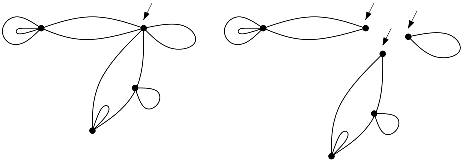

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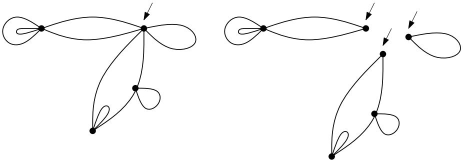

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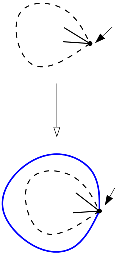

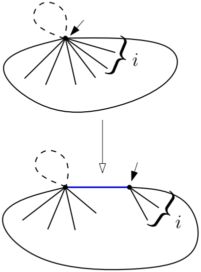

Operations to create a prime map:

- Add a loop around any map
- Split at index $i \leq \Delta(P_\ell)$ in the last prime P_ℓ of any map

Subsets (and supersets) of $\mathcal{O}$

Two families of sets of orientations $\mathcal{O}_k^-$ and $\mathcal{O}_k^+$ s.t.
$\mathcal{O}_k^- \subset \mathcal{O}_{k+1}^- \subset \mathcal{O} \subset \mathcal{O}_{k+1}^+ \subset \mathcal{O}_k^+$

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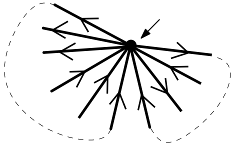
The atomic map (one vertex, no edges) is in $\mathcal{O}_k^-$.
Fewer splits allowed → the number of orientations necessary to look at form now a word of finite length , which we can use as a parameter

ALGEBRAIC SYSTEM FOR $\mathcal{O}^{(k)-} \equiv \mathcal{O}^{-}$

The **root-word** $w(O)$ of a map O is the binary word formed as follows in counterclockwise order around the root-vertex:

- 1 if there is an out-edge,
- 0 if there is an in-edge.



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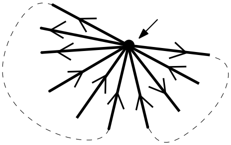

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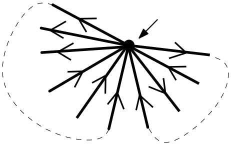

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$F_w(t)$: g.f. of the set $\{O \in \mathcal{O}^- | \mathbf{w}(O) = w\}$
$L_w(t)$: g.f. of the set $\{O \in \mathcal{O}^- | \mathbf{w}(O) = uw$ for some $u\}$
$F'_w(t)$, $L'_w(t)$: their counterparts for **prime** maps of $\mathcal{O}^-$.

An example: equation for $F'_w(t)$

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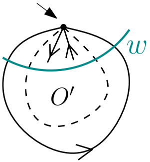
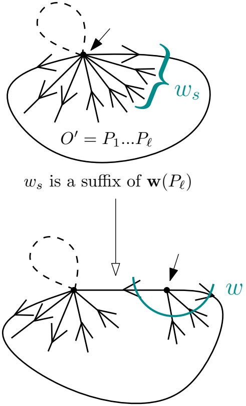
$$F'_w = tF_{w_c} + tL_\varepsilon L'_{w_s}$$



$\mathbf{w}(O') = w_c$



Algebraic system for $\mathcal{O}^{(k)-}$

$$\begin{cases} F_w = \sum_{w=uv} F_u F'_v & |w| \leq 2k-2 \\ L_w = \begin{cases} L_\varepsilon L'_w + \sum_{w=uv, u\neq\varepsilon} L_u F'_v & |w| \leq 2k-2 \\ 1 + L_\varepsilon L'_\varepsilon & w = \varepsilon \end{cases} \\ F'_w = tF_{w_c} + tL_\varepsilon L'_{w_s} & |w| \leq 2k \\ L'_w = \begin{cases} tL_{w_p} + tF_{w_c} + tL_\varepsilon L'_w + \\ \quad tL_\varepsilon \sum_{\substack{u=vw \\ u \text{ balanced} \\ 0<|u|\leq 2k}} (L'_{u_s} - F_u) + tL_\varepsilon (L'_{w'} - F'_w) & |w| \leq 2k-2 \\ 2tL_\varepsilon + tL_\varepsilon L'_\varepsilon & w = \varepsilon \end{cases} \end{cases}$$

$w = \varepsilon \Rightarrow F_w = 1, F'_w = 0$
w non-balanced $\Rightarrow F_w = F'_w = 0.$

Small example: subsets, $k = 1$

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Eliminating all series but L_ε: cubic equation for L_ε:

$$t^2 L_\varepsilon^3 + t(t-4)L_\varepsilon^2 + (2t+1)L_\varepsilon - 1 = 0$$

Small example: subsets, $k = 2$

$$
\begin{cases}
F_{01} = F_{10} & = F'_{01}, \\
F'_{10} = F'_{01} & = t + tL_\varepsilon L'_1, \\
F'_{1100} & = tF_{10} + tL_\varepsilon L'_{100}, \\
F'_{1010} & = tF_{01} + tL_\varepsilon L'_{010}, \\
F'_{0110} & = tL_\varepsilon L'_{110}, \\
L_\varepsilon & = 1 + L_\varepsilon L'_\varepsilon, \\
L_0 = L_1 & = L_\varepsilon L'_0, \\
L_{00} = L_{11} & = L_\varepsilon L'_{00}, \\
L_{01} = L_{10} & = L_\varepsilon L'_{01}, \\
L'_\varepsilon & = 2tL_\varepsilon + tL_\varepsilon (L'_\varepsilon + 2(L'_0 - F'_{10} + L'_{100} - F'_{1100} + L'_{010} - F'_{1010} + L'_{110} - F'_{0110})), \\
L'_0 = L'_1 & = tL_\varepsilon + tL_\varepsilon (L'_0 + L'_0 - F'_{10} + L'_{100} - F'_{1100} + L'_{010} - F'_{1010} + L'_{110} - F'_{0110}), \\
L'_{00} & = tL_0 + tL_\varepsilon (L'_{00} + L'_{100} - F'_{1100}), \\
L'_{10} = L'_{01} & = tL_1 + t + tL_\varepsilon (L'_{10} + L'_1 - F'_{01} + L'_{010} - F'_{1010} + L'_{110} - F'_{0110}), \\
L'_{100} & = tL_{10} + tL_\varepsilon (L'_{100} + L'_{100} - F'_{1100}), \\
L'_{010} & = tL_{01} + tL_\varepsilon (L'_{010} + L'_{010} - F'_{1010}), \\
L'_{110} & = tL_{11} + tL_\varepsilon (L'_{110} + L'_{110} - F'_{0110}).
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Finding L_ε

Generate the systems automatically then **eliminate** the variables with Maple (keeping L_ε)

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nature	k	degree	growth rate
inf	1	3	10.60
inf	2	6	10.97
inf	3	20	11.22
inf	4	258	$11.44^{(*)}$
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inf	6	–	$11.68^{(*)}$
PEO	–	–	?

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Let μ_k^- be the growth rate of the set $\mathcal{O}_k^-$. Then $\mu_k^- \to_{k\to\infty} \mu$.

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General idea: allowing splits at indices $i \geq \Delta(P_\ell)$, creating non Eulerian orientations

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For $k = 1$:

$$\begin{cases} L_\varepsilon(t,x) &= 1 + L_\varepsilon(t,x) L'_\varepsilon(t,x), \\ L'_\varepsilon(t,x) &= 2txL_\varepsilon(t,x) + tL_\varepsilon(t,1)\left(2xL'_0(t,1) + \dfrac{x}{x-1}\left(L'_\varepsilon(t,x) - xL'_\varepsilon(t,1)\right)\right), \\ L'_0(t,x) &= txL_\varepsilon(t,x) + tL_\varepsilon(t,1)\left(xL'_0(t,1) + \dfrac{x}{x-1}\left(L'_0(t,x) - xL'_0(t,1)\right)\right). \end{cases}$$

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nature	k	degree	growth rate
PEO	–	–	?
sup	5	–	$13.005^{(*)}$
sup	4	–	$13.017^{(*)}$
sup	3	–	$13.031^{(*)}$
sup	2	28	13.047
sup	1	3	13.065

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