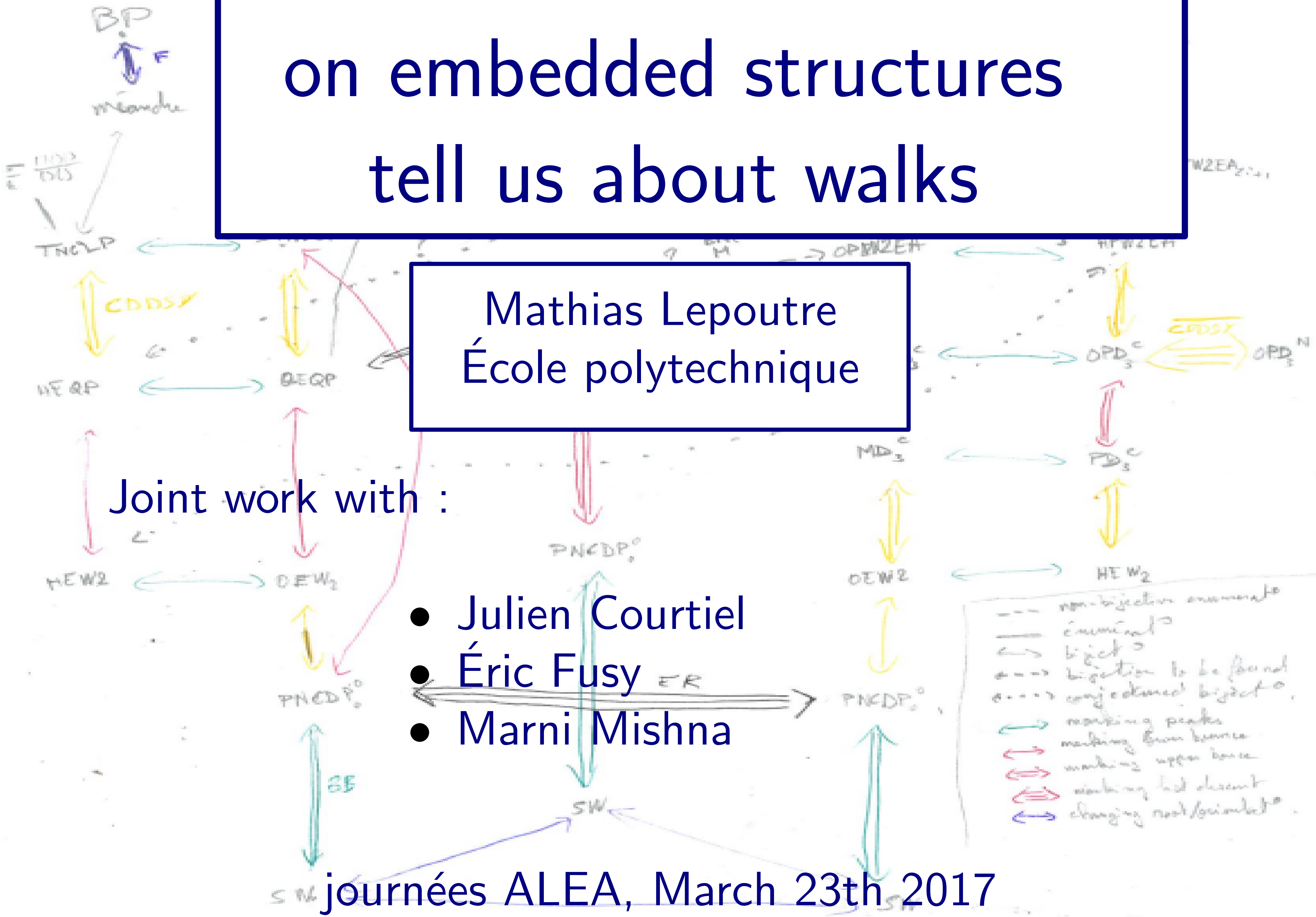
What reflexions on embedded structures tell us about walks

Mathias Lepoutre
École polytechnique



Joint work with :

- Julien Courtiel
- Éric Fusy
- Marni Mishna

journées ALEA, March 23th 2017

Introductory example : Narayana numbers

$N(n,p) = \frac{1}{p}\binom{n}{p-1}\binom{n-1}{p-1}$:

Number of Dyck paths of length $2n$ with p peaks.

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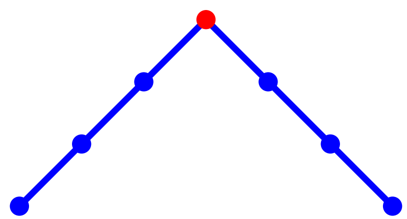
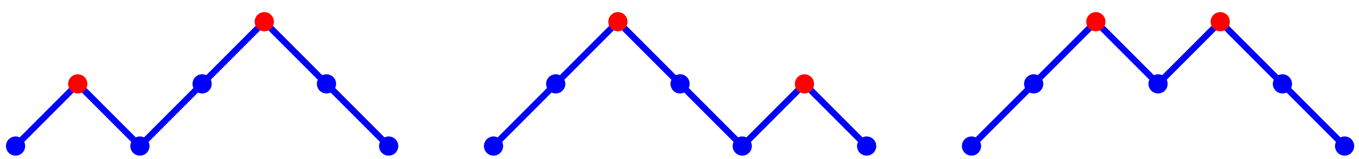
Example for $n = 3$

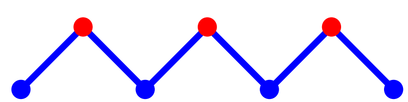
$N(3,1) = 1$



$N(3,2) = 3$



$N(3,3) = 1$



Introductory example : Narayana numbers

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Number of Dyck paths of length $2n$ with p peaks.

Properties :

- $\sum_{p=1}^{n} N(n,p) = C_n$

- $N(n,p) = N(n, n-p+1)$

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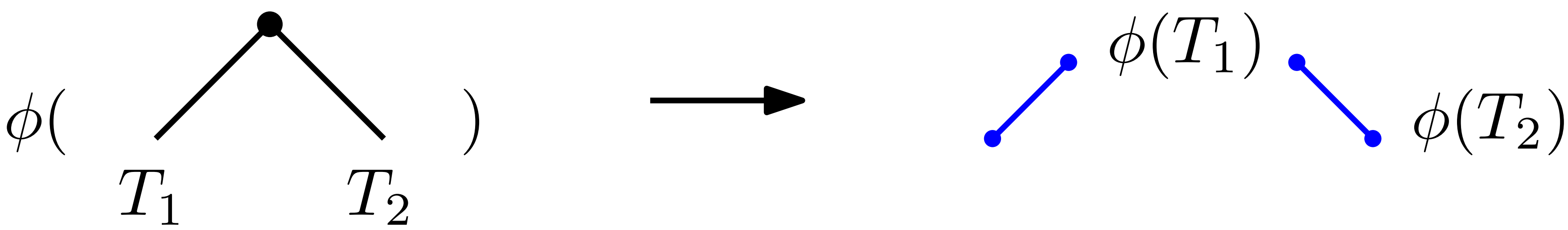
A bijection between plane binary trees with n leaves and Dyck paths of length $2n$:



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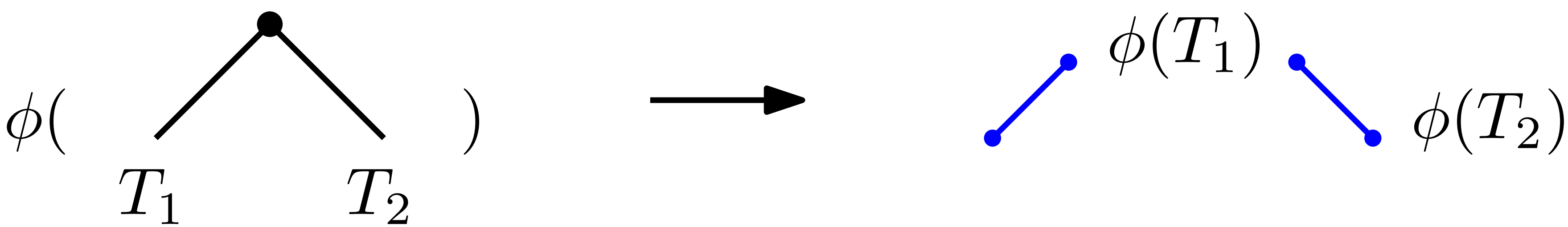
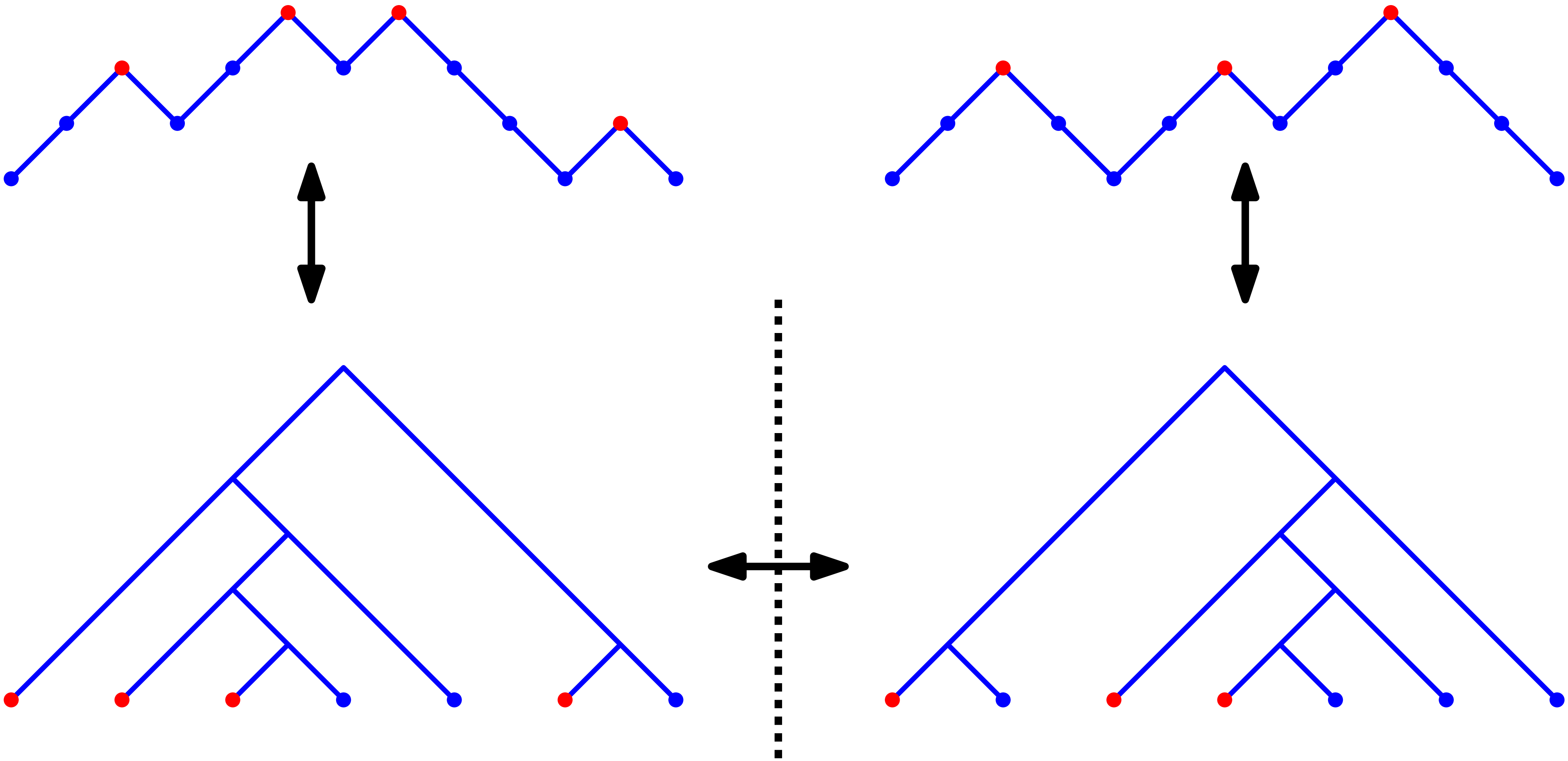


Tracking an interesting parameter :

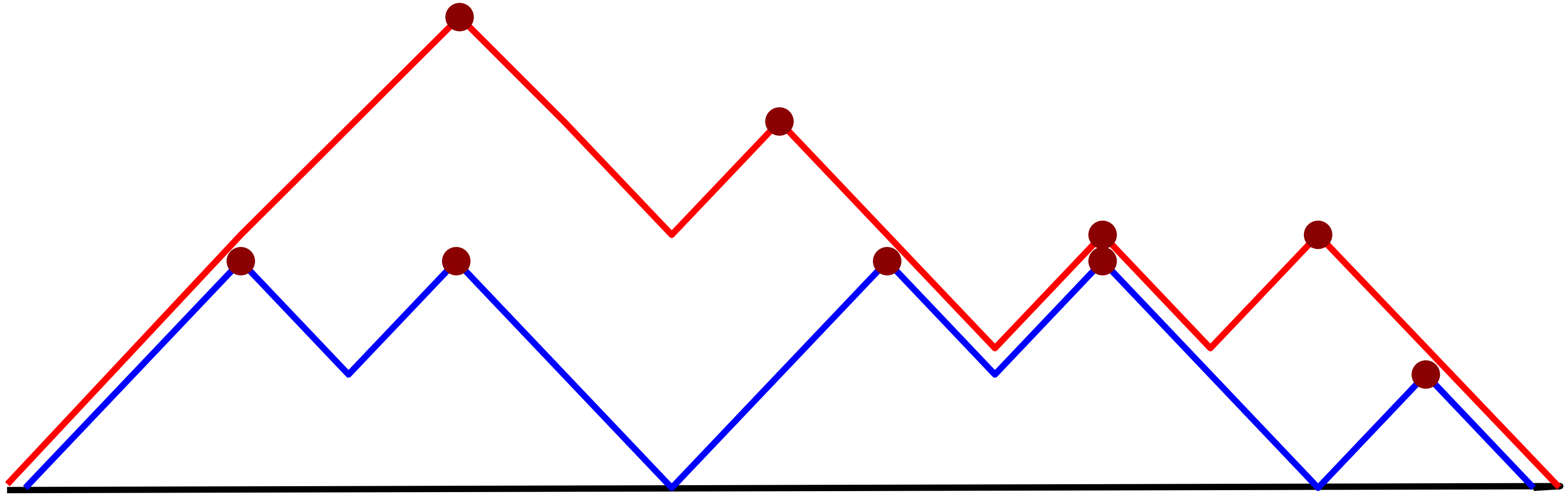
Number of left leaves $\longrightarrow$ Number of peaks

Introductory example : Narayana numbers

A bijective proof of Narayana numbers symmetry:



Generalisation :
Peaks of the pairs of non-crossing Dyck paths



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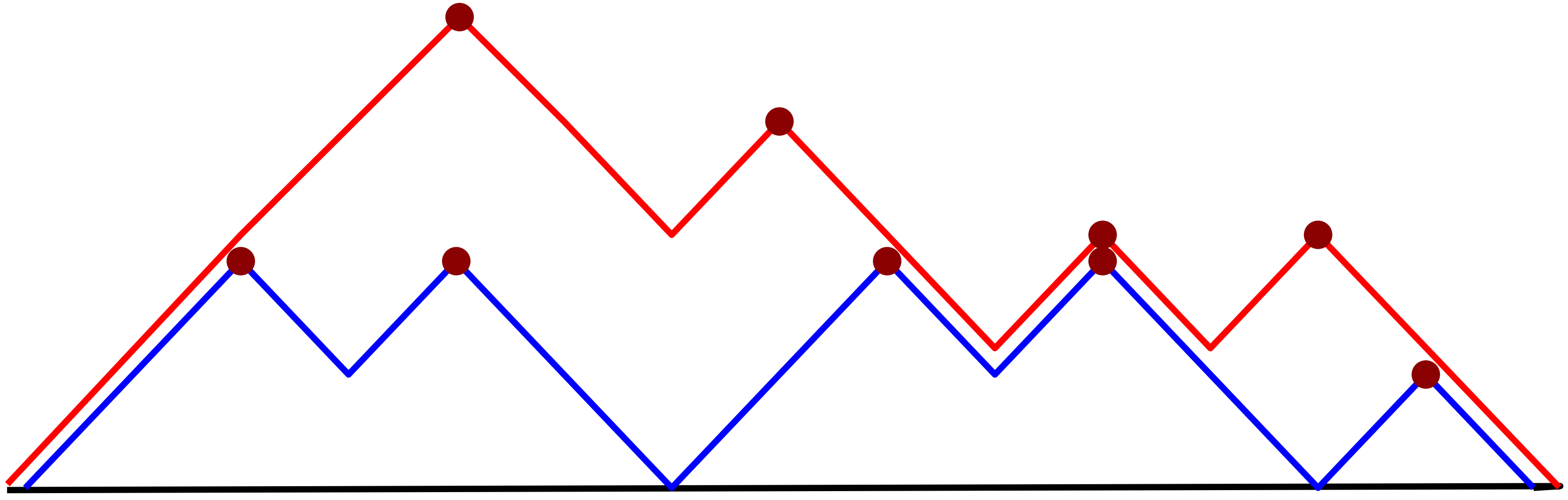
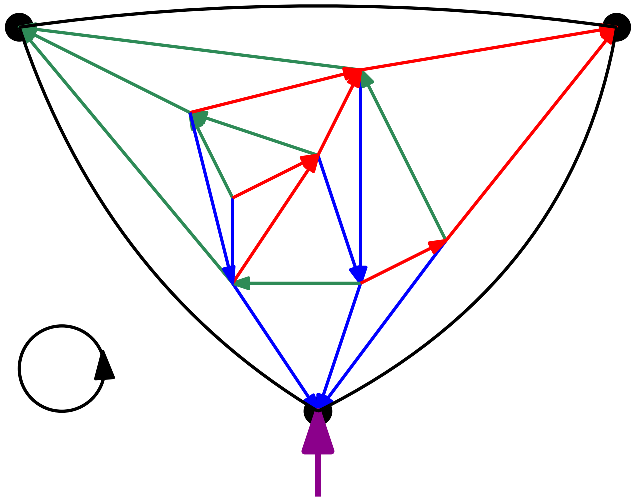
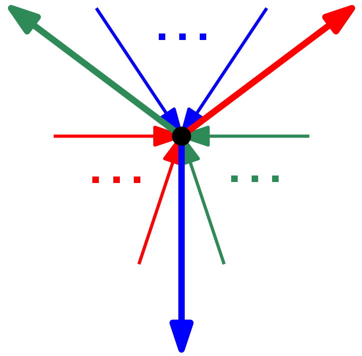


Let $N(n, p, q)$ be the number of pairs of non-crossing Dyck paths of length $2n$ with p upper peaks and q lower peaks.
Then: $N(n, p, q) = N(n, n - q + 1, n - p + 1)$

Generalisation :
Peaks of the pairs of non-crossing Dyck paths

Schnyder woods of triangulations





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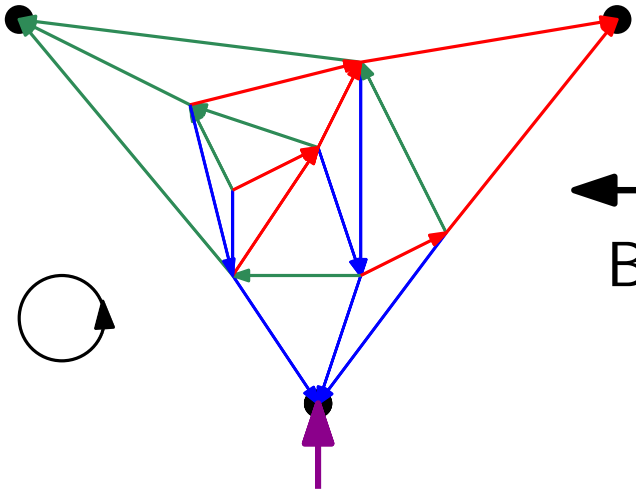

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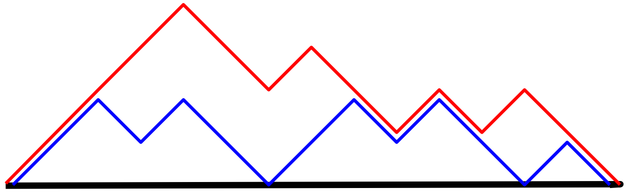

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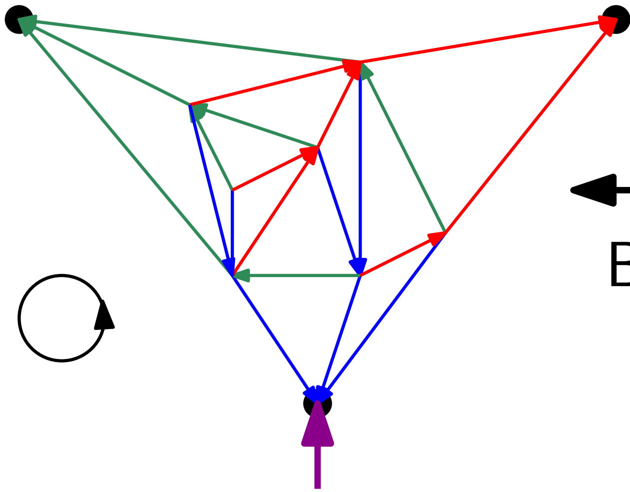

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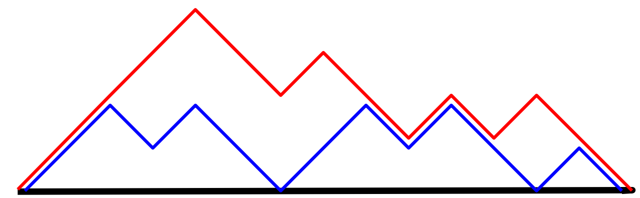
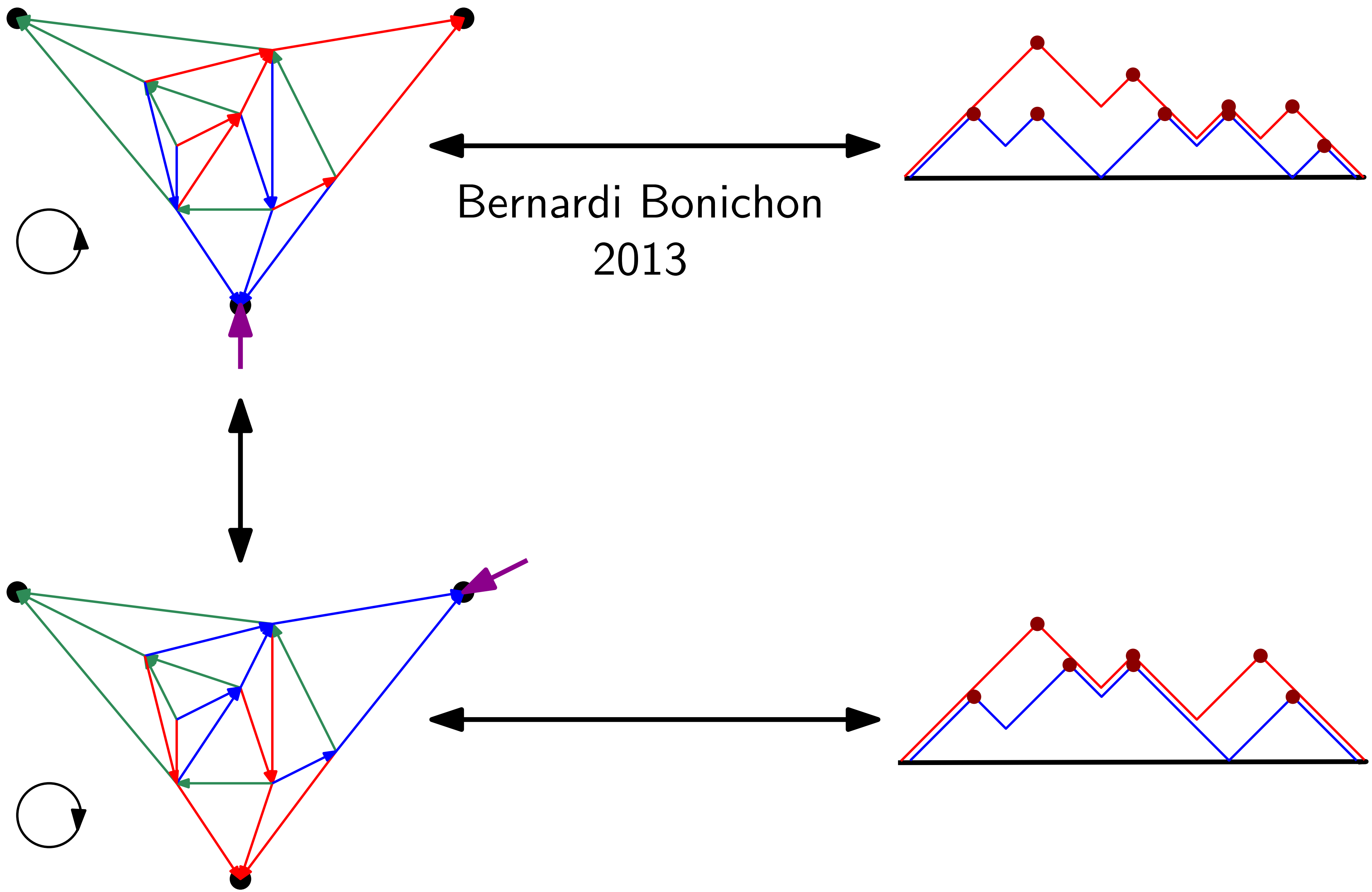


- Number of blue leaves
- Number of red internal vertices

↔

- Number of blue peaks
- Number of red peaks

Generalisation : Peaks of the pairs of non-crossing Dyck paths



Can this be further generalized?

Let $N(n, p_1 ... p_k)$ be the number of k-tuples of non-crossing Dyck paths of length $2n$ with p_i peaks on the i-th paths from the top.
Do we have: $N(n, p_1 ... p_k) = N(n, n - p_k + 1 ... n - p_1 + 1)$?

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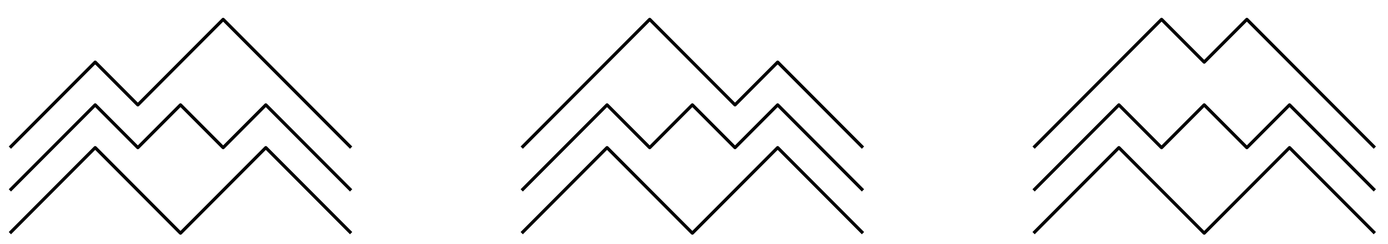
Do we have: $N(n, p_1 ... p_k) = N(n, n - p_k + 1 ... n - p_1 + 1)$?

No!

$N(4, 3, 2, 3) = 2$

$N(4, 2, 3, 2) = 3$



A result on walks in the plane

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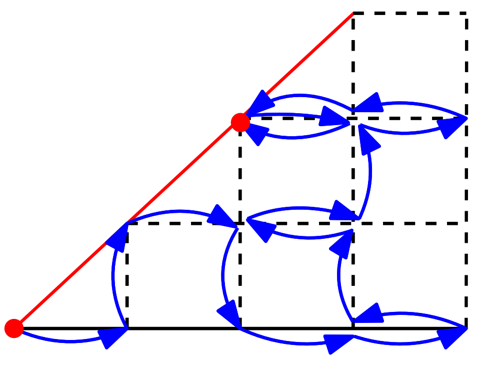
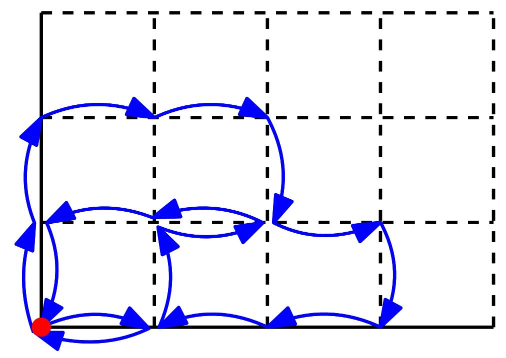
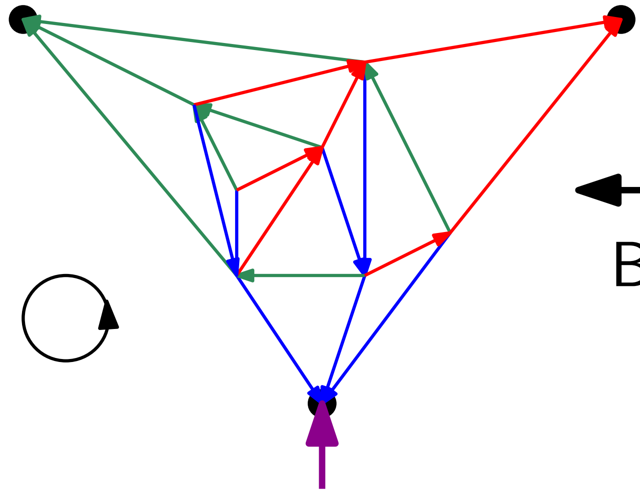




At given size, there are as many walks in the first octant that end on the x-axis than excursions in the quarter plane.

A result on walks in the plane



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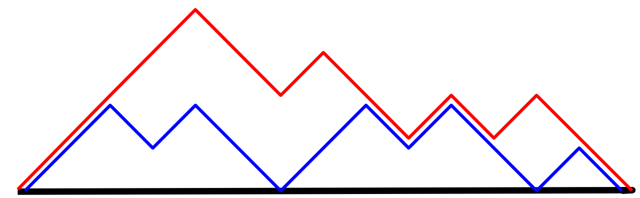


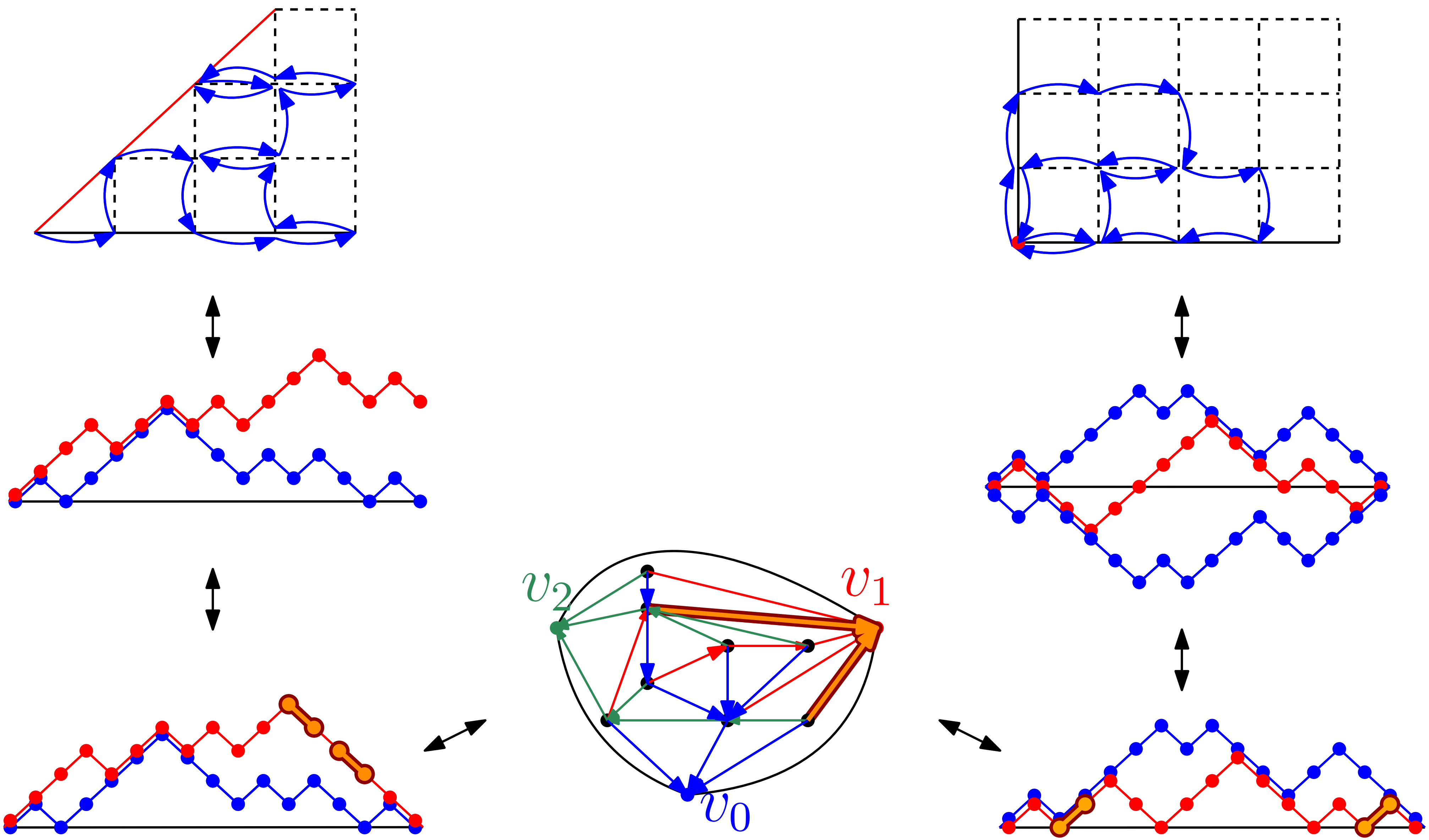
- Number of blue leaves
- Number of red internal vertices

- Number of blue peaks
- Number of red peaks

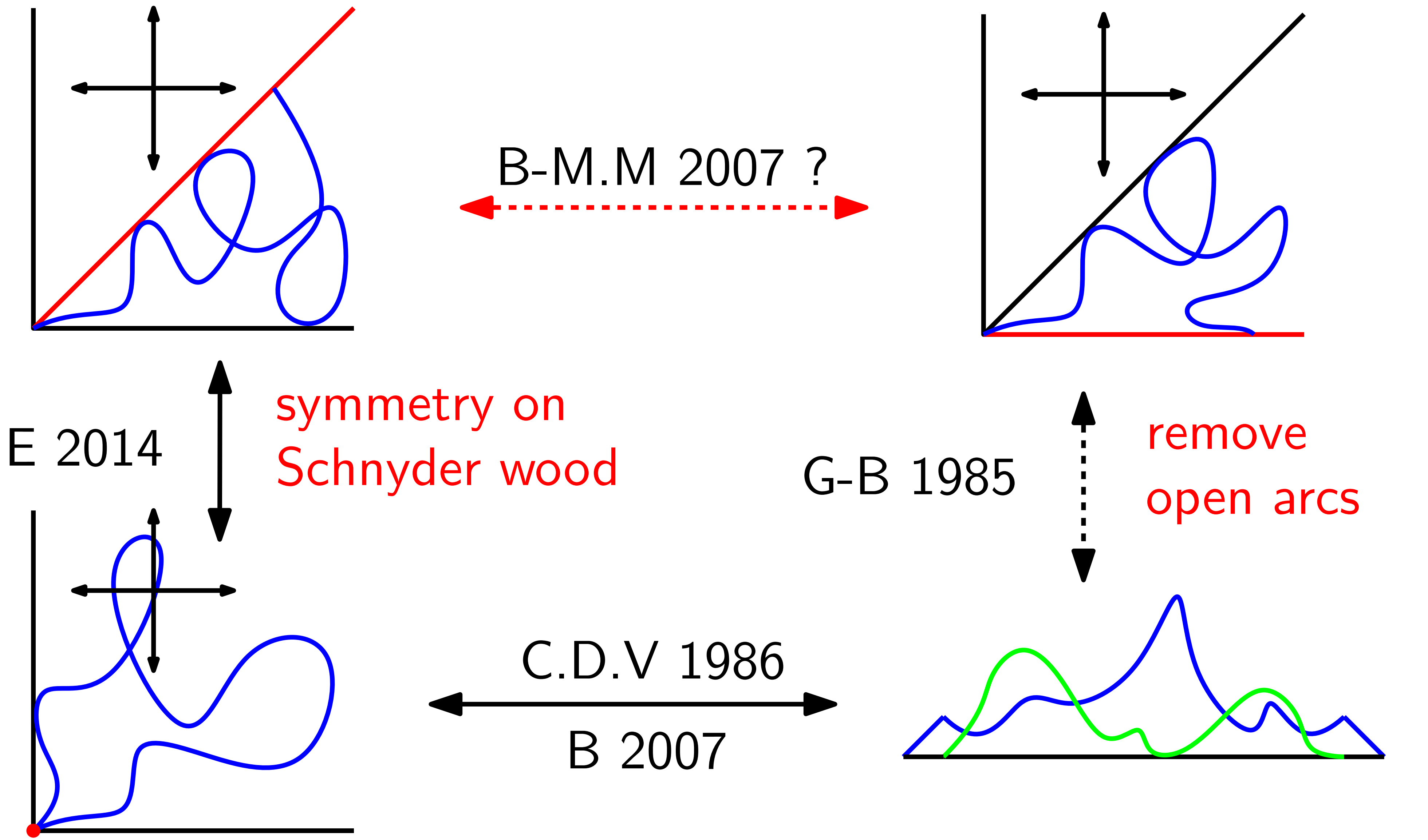
- Blue root-degree
- Red root-degree

- Number of blue steps leaving the axis
- Length of the red last descent

A result on walks in the plane



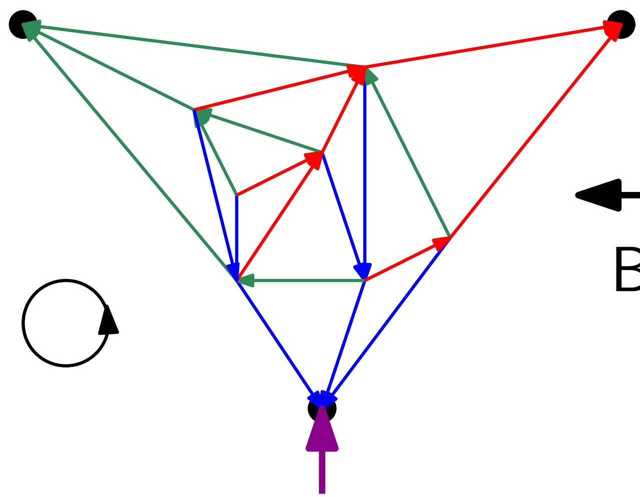
A look at another problem



Another result to prove a conjecture

There exists an explicit involution on pairs of non-crossing Dyck paths that preserves the size and the number of upper peaks, while exchanging the number of lower steps leaving the axis and the number of common up-steps.

Another result to prove a conjecture



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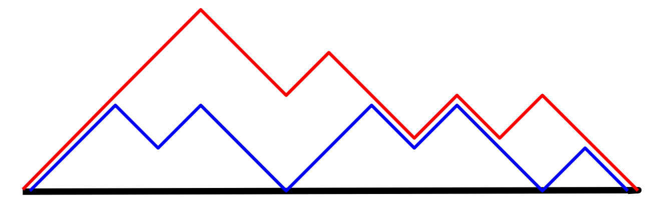
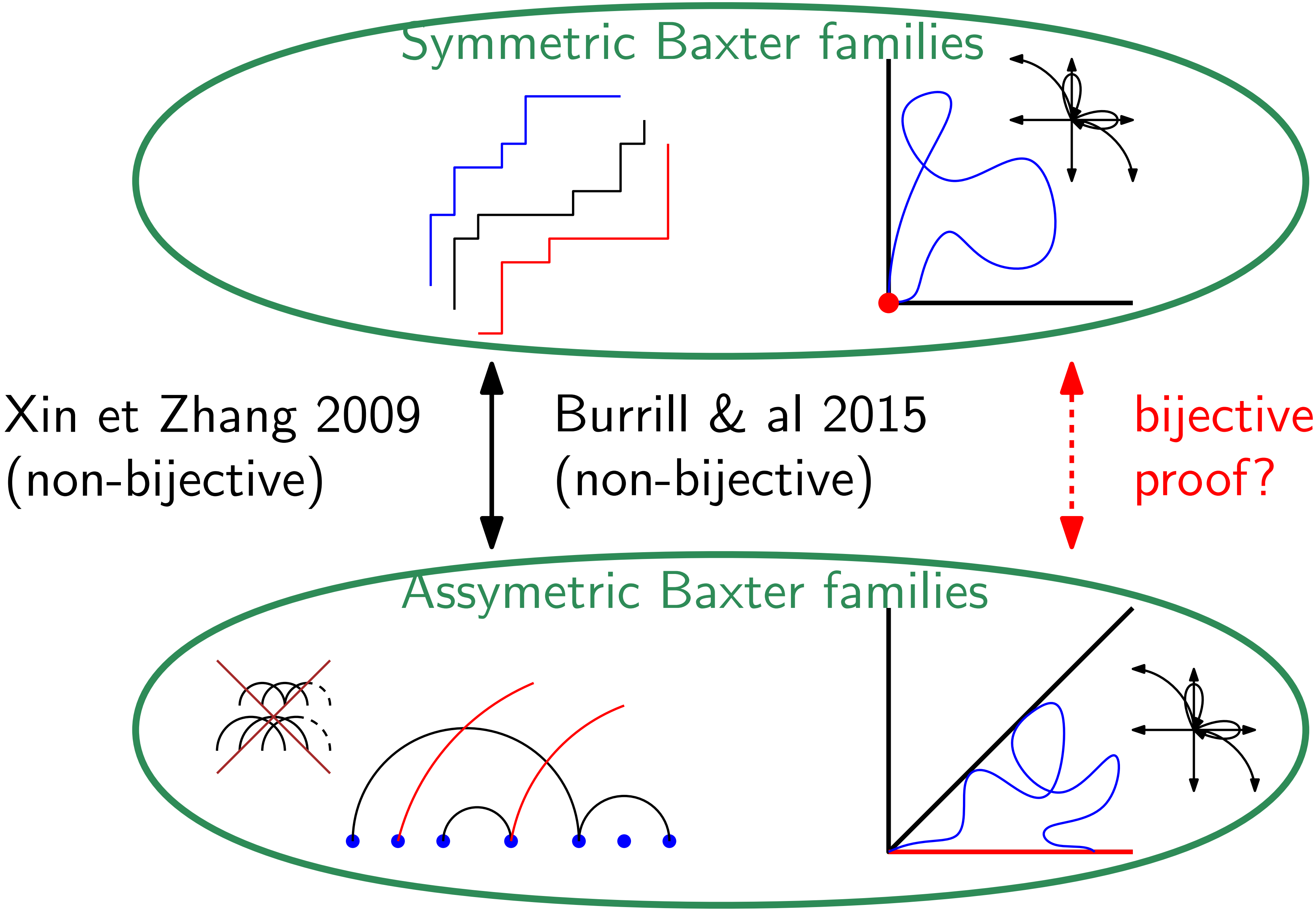
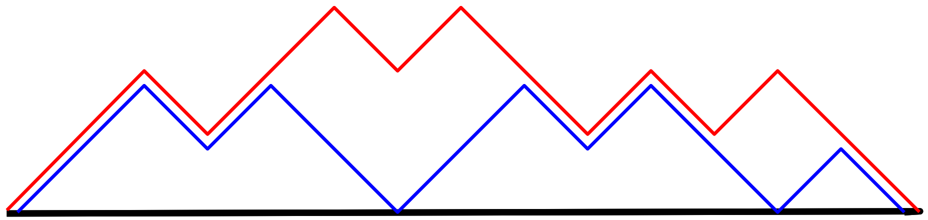


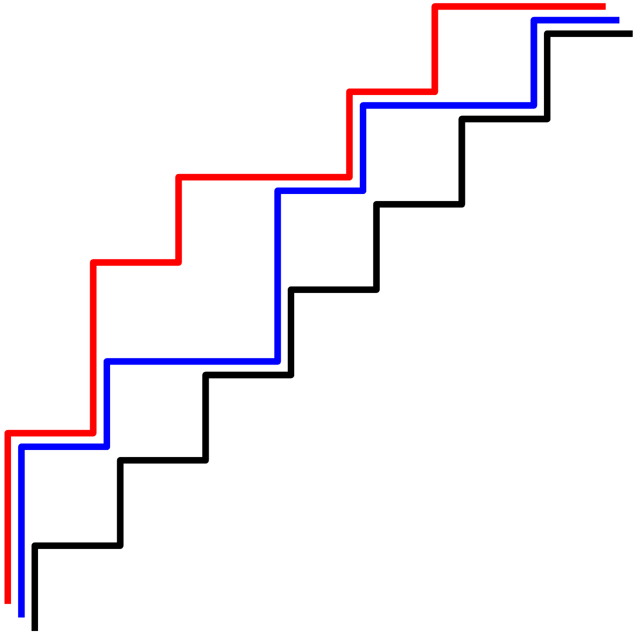
• Number of blue leaves • Number of red internal vertices	↔	• Number of blue peaks • Number of red peaks
• Blue root-degree • Red root-degree	↔	• Number of blue steps leaving the axis • Length of the red last descent
• Green root-degree	↔	• Number of common up-steps

Another result to prove a conjecture



Extending this last result to triples of paths making use of plane bipolare orientations





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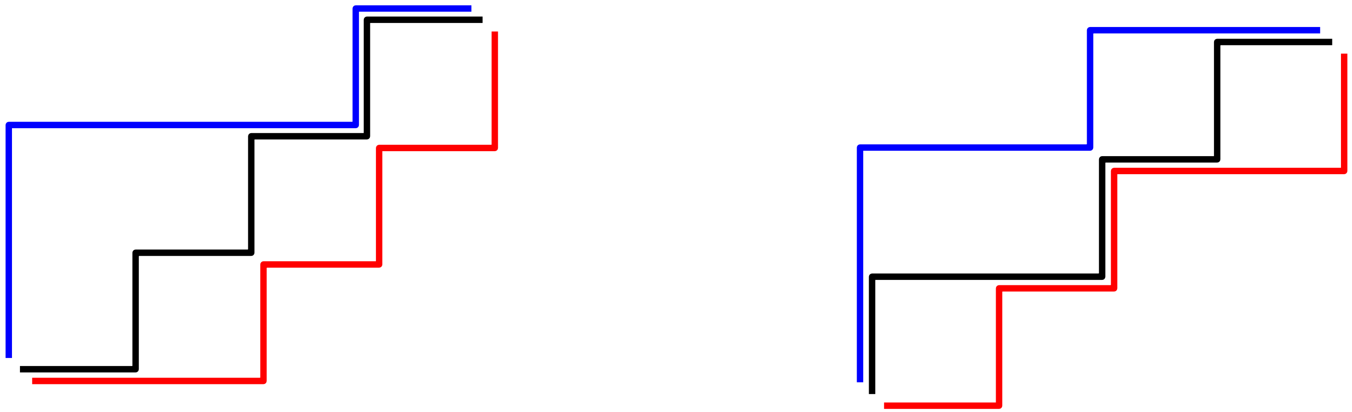
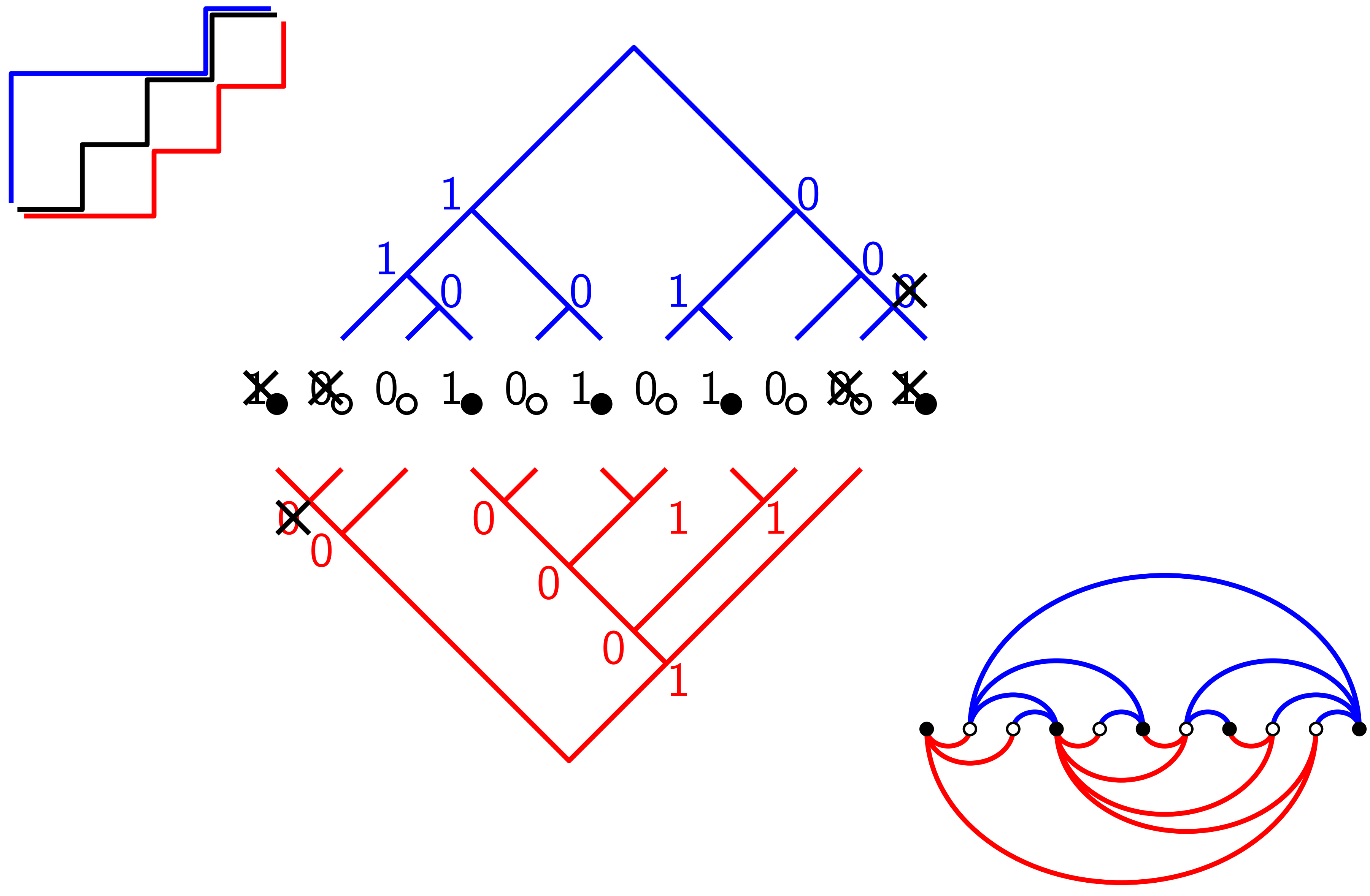


There exists an explicit involution on triples of non-crossing lattice paths that preserves the size, the number of upper peaks, and the number of lower valleys, while exchanging the number higher horizontal contacts and the number of lower horizontal contacts.

Plane bipolar orientations



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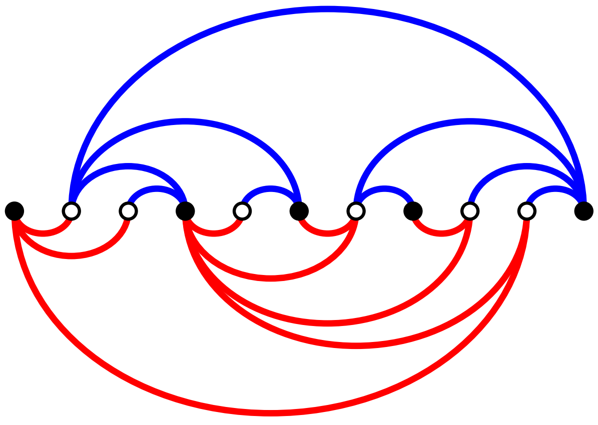

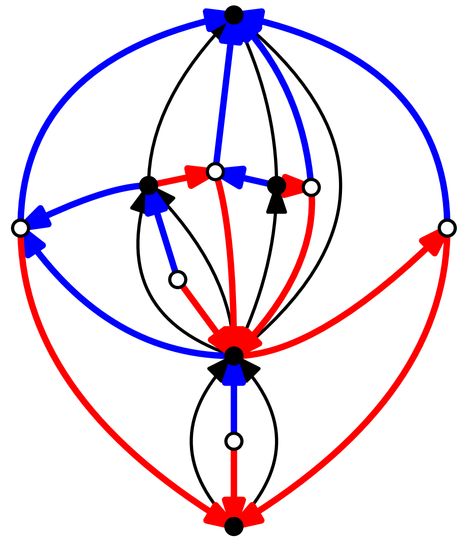

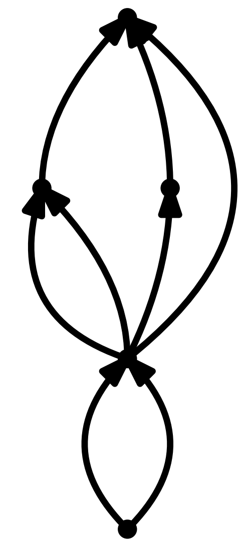

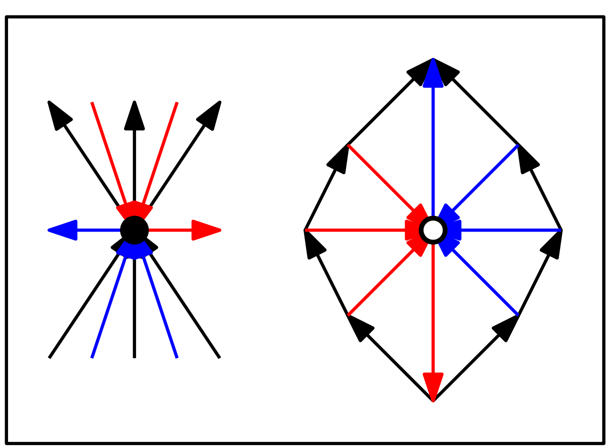

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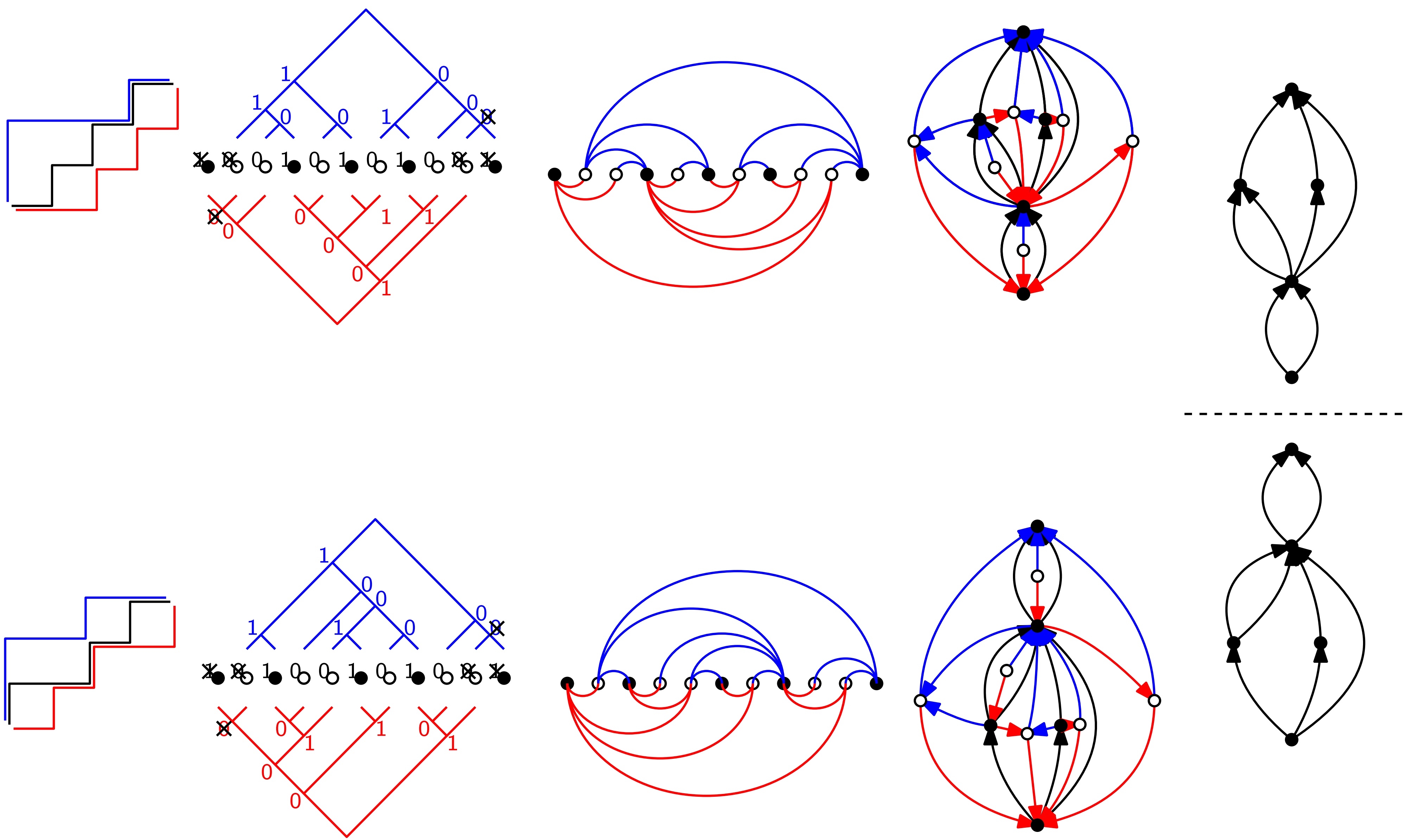

Merci pour votre attention