On the mixing time of the flip walk on triangulations of the sphere

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ENS Paris

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Planar maps

Definitions

- A *planar map* is a finite, connected graph embedded in the sphere in such a way that no two edges cross (except at a common endpoint), considered up to orientation-preserving homeomorphism.
- A planar map is a *rooted type-I triangulation* if all its faces have degree 3 and it has a distinguished oriented edge. It may contain multiple edges and loops.

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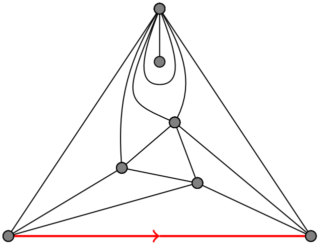

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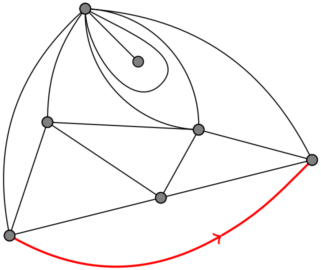

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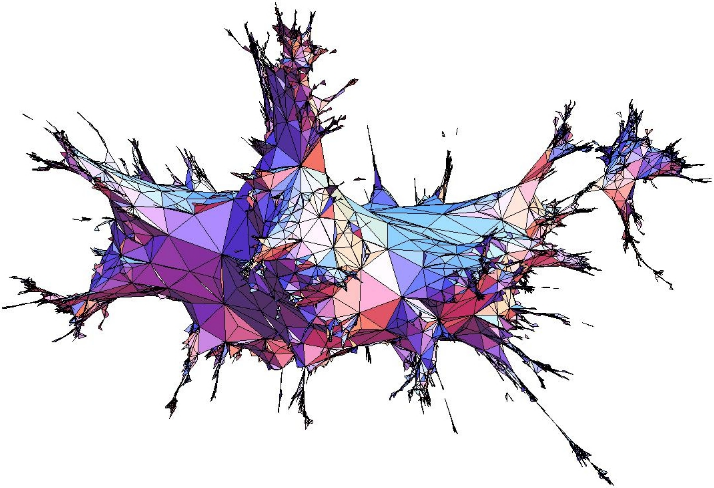
- the Brownian map is homeomorphic to the sphere [Le Gall–Paulin].

A uniform triangulation of the sphere with 10 000 vertices



How to sample a large uniform triangulation ?

- "Modern" tools : bijections with trees, peeling process.
- Back in the 80's : Monte Carlo methods : we look for a Markov chain on $\mathscr{T}_n$ for which the uniform measure is stationary.
- A simple local operation on triangulations : flips.

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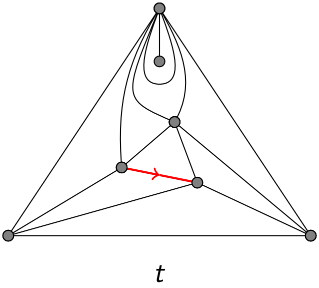

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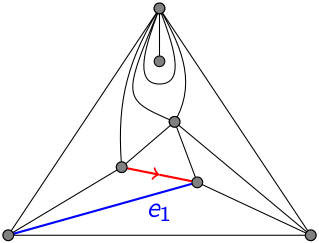

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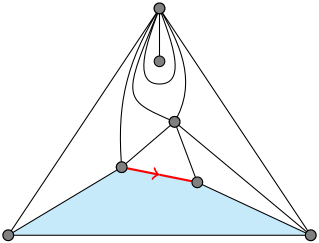

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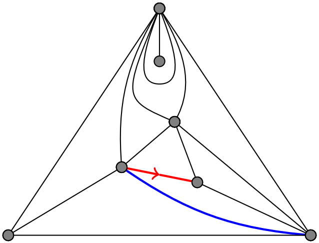

$\mathfrak{flip}(t, e_1)$

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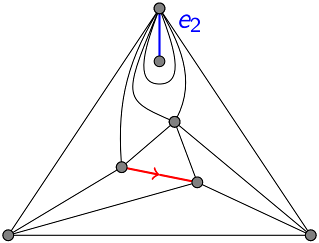

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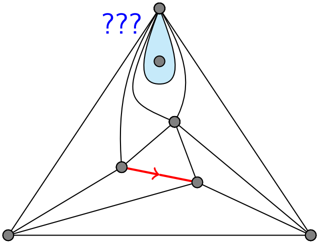

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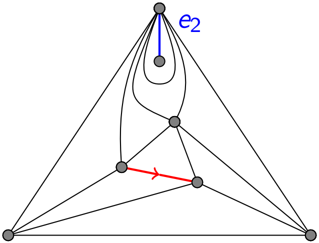

$$\mathfrak{flip}(t, e_2) = t$$

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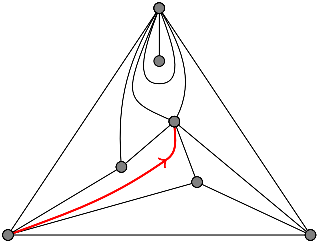

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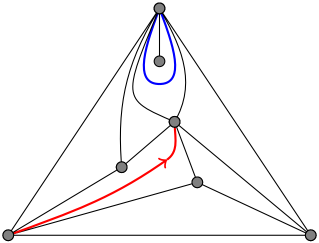

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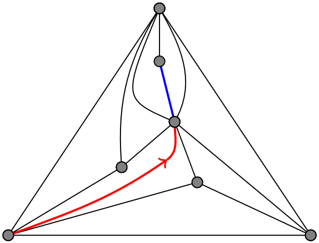

A Markov chain on $\mathscr{T}_n$

- We fix $t_0 \in \mathscr{T}_n$ and take $T_n(0) = t_0$.
- Conditionally on $(T_n(k))_{0 \leq i \leq k}$, let e_k be a uniform edge of $T_n(k)$ and $T_n(k+1) = \mathfrak{flip}\left(T_n(k), e_k\right)$.

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- The chain T_n is irreducible (the flip graph is connected [Wagner 36]) and aperiodic (non flippable edges), so it converges to the uniform measure.
- Question : how quick is the convergence ?

- For $n \geq 3$ and $0 < \varepsilon < 1$ we define the mixing time $t_{mix}(\varepsilon, n)$ as the smallest k such that

$$\max_{t_0 \in \mathscr{T}_n} \max_{A \subset \mathscr{T}_n} |\mathbb{P}(T_n(k) \in A) - \mathbb{P}(T_n(\infty) \in A)| \leq \varepsilon,$$

where we recall that $T_n(\infty)$ is uniform on $\mathscr{T}_n$.

Mixing time of T_n

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Theorem (B., 2016)

For all $0 < \varepsilon < 1$, *there is a constant* $c > 0$ *such that*

$$t_{mix}(\varepsilon, n) \geq cn^{5/4}.$$

Sketch of proof

We will be interested in the existence of small separating cycles.

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Theorem ($\approx$ Le Gall–Paulin, 2008)

Let $\ell_n = o(n^{1/4})$. Then, with probability going to 1 as $n \to +\infty$, there is no cycle in $T_n(\infty)$ of length at most ℓ_n that separates $T_n(\infty)$ in two parts, each of which contains at least $\frac{n}{4}$ vertices.

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Let $T_n^1(0)$ and $T_n^2(0)$ be two independent uniform triangulations of a 1-gon with $\frac{n}{2}$ inner vertices each, and $T_n(0)$ the gluing of $T_n^1(0)$ and $T_n^2(0)$ along their boundary.

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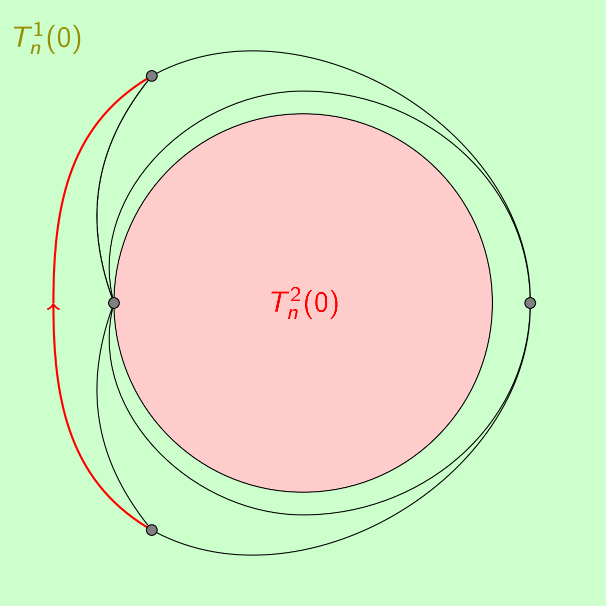
Proposition

Let $k_n = o(n^{5/4})$. There is a cycle γ in $T_n(k_n)$ of length $o(n^{1/4})$ in probability that separates $T_n(k_n)$ in two parts, each of which contains at least $\frac{n}{4}$ vertices.

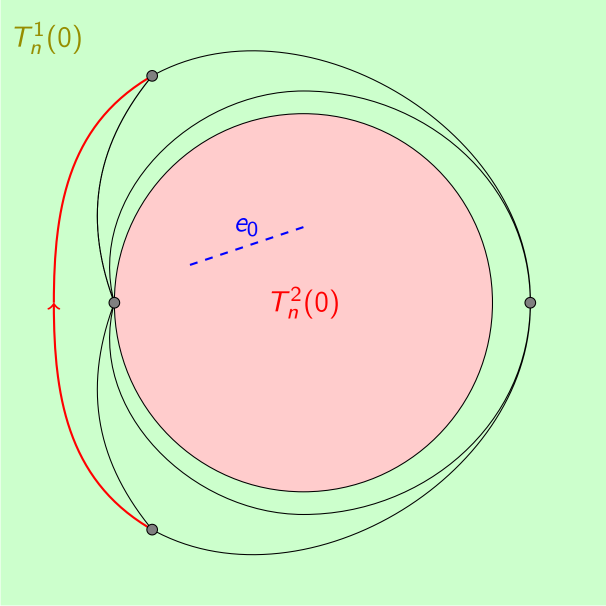
Exploration of $T_n(k)$



Perimeter :
$\widetilde{P}_n(0) = 1$
Explored volume :
$\widetilde{V}_n(0) = 1$

exploration steps :

Exploration of $T_n(k)$



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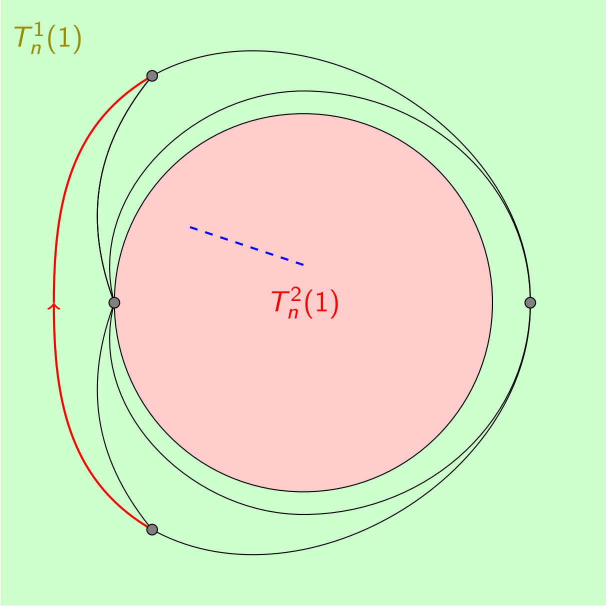
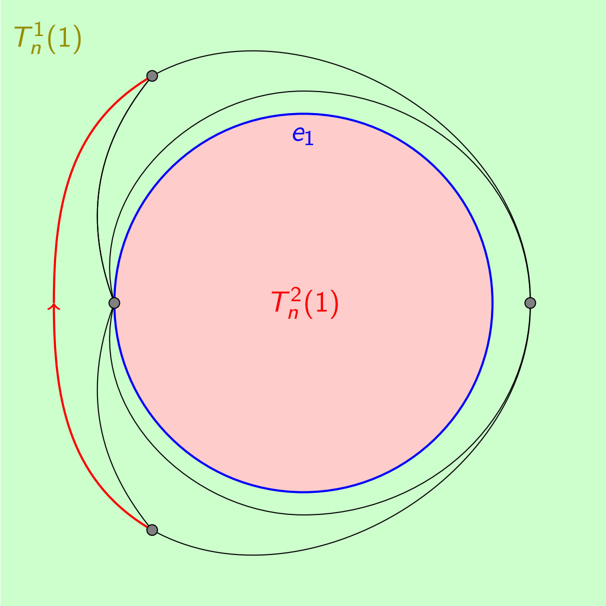


Perimeter :
$\widetilde{P}_n(1) = 1$
Explored volume :
$\widetilde{V}_n(1) = 1$

exploration steps :

Exploration of $T_n(k)$



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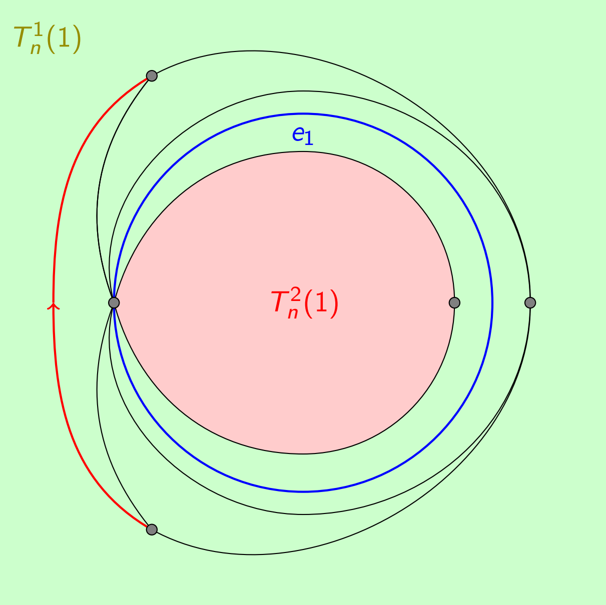
Exploration of $T_n(k)$



Perimeter :

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Explored volume :

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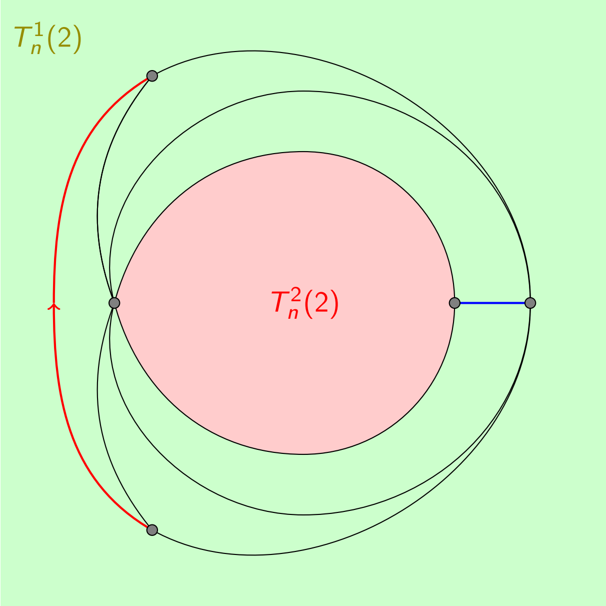
exploration steps :

1

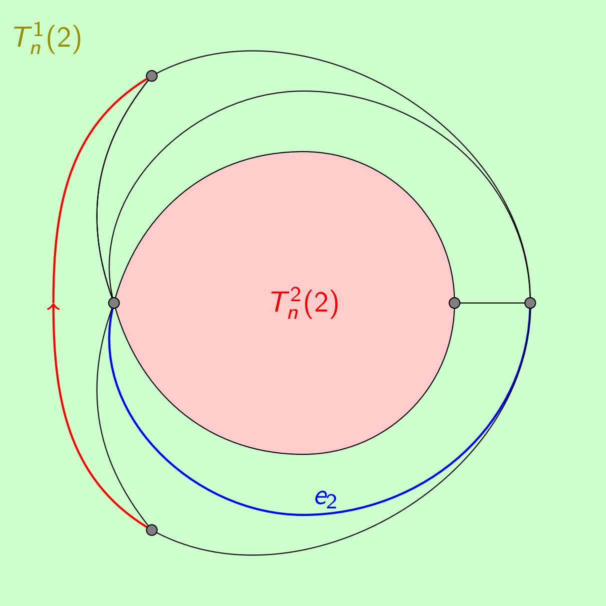
Exploration of $T_n(k)$



Perimeter :
$\widetilde{P}_n(2) = 2$
Explored volume :
$\widetilde{V}_n(2) = 2$

exploration steps :
1

Exploration of $T_n(k)$



Perimeter :

$\widetilde{P}_n(2) = 2$

Explored volume :

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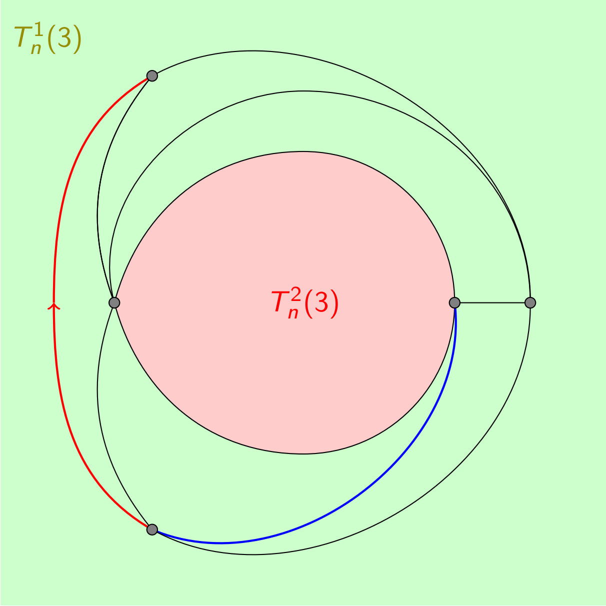
1

Exploration of $T_n(k)$



Perimeter :

$\widetilde{P}_n(3) = 2$

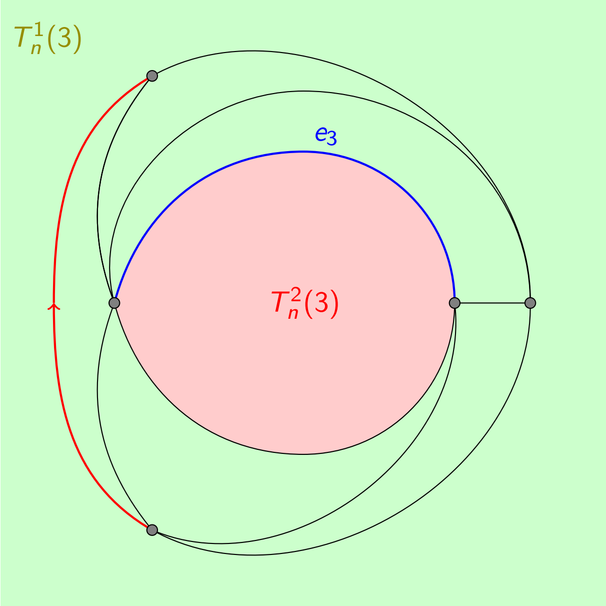
Explored volume :

$\widetilde{V}_n(3) = 2$

exploration steps :

1

Exploration of $T_n(k)$



Perimeter :
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Explored volume :
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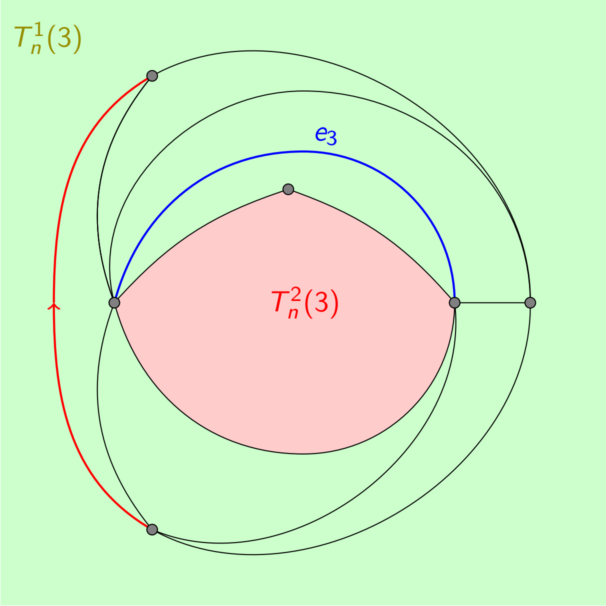
1

Exploration of $T_n(k)$



Perimeter :

$\widetilde{P}_n(3) = 2$

Explored volume :

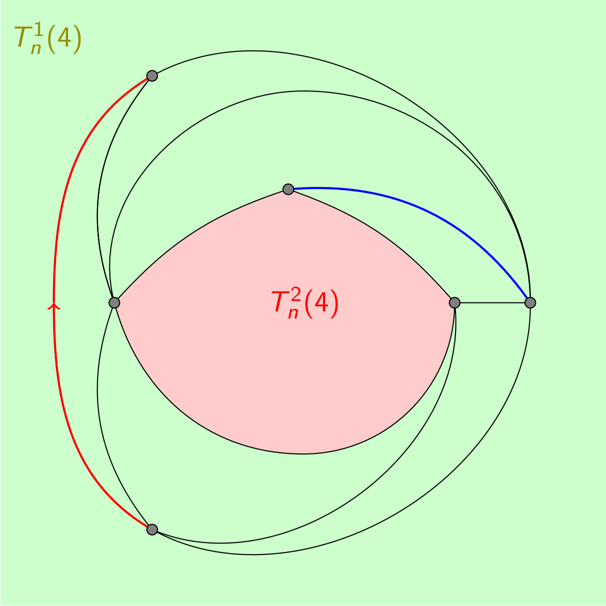
$\widetilde{V}_n(3) = 2$

exploration steps :

1

3

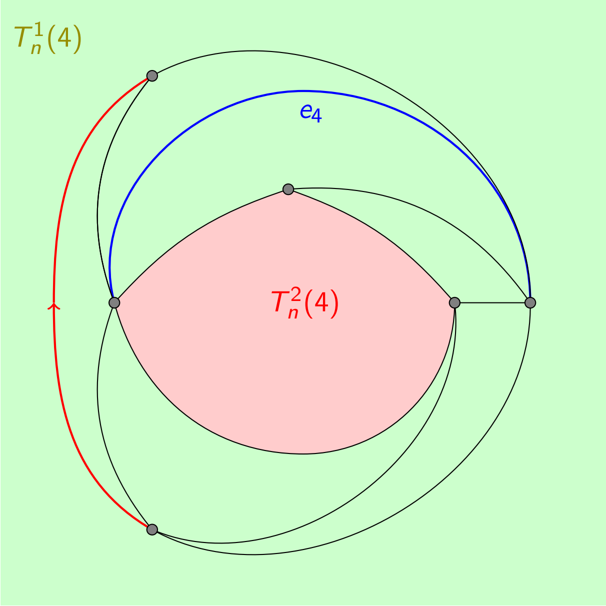
Exploration of $T_n(k)$



Perimeter :
$\widetilde{P}_n(4) = 3$
Explored volume :
$\widetilde{V}_n(4) = 3$

exploration steps :
1
3

Exploration of $T_n(k)$



Perimeter :

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Explored volume :

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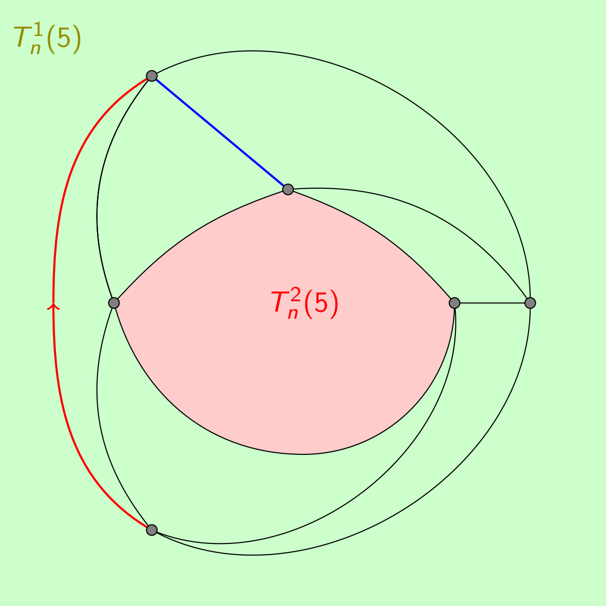
exploration steps :

1
3

Exploration of $T_n(k)$



Perimeter :
$\widetilde{P}_n(5) = 3$
Explored volume :
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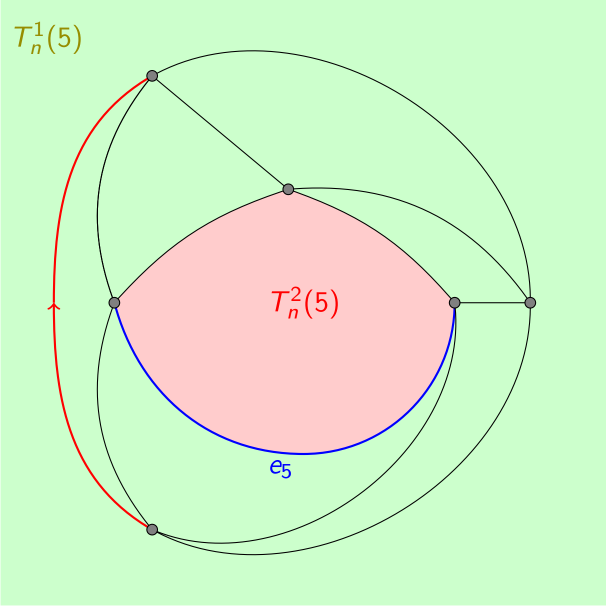
exploration steps :
1
3

Exploration of $T_n(k)$



Perimeter :
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Explored volume :
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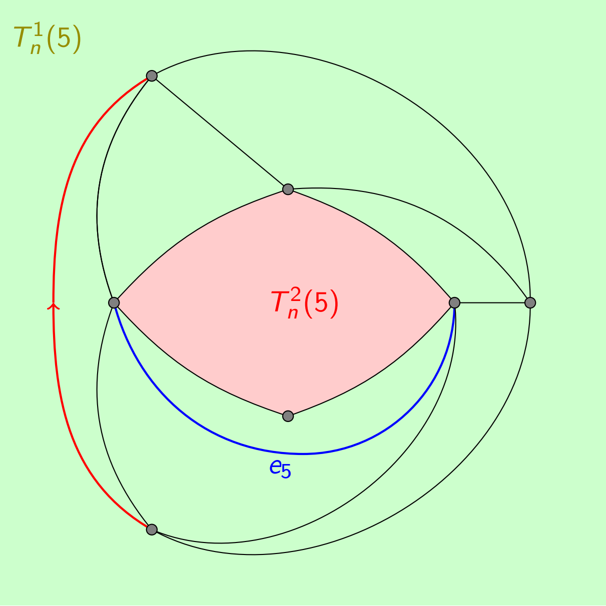
exploration steps :
1
3

Exploration of $T_n(k)$



Perimeter :

$\widetilde{P}_n(5) = 3$

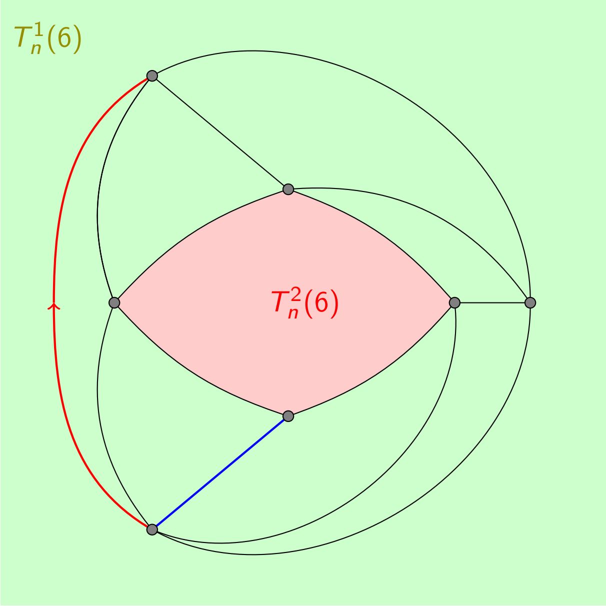
Explored volume :

$\widetilde{V}_n(5) = 3$

exploration steps :

1
3
5

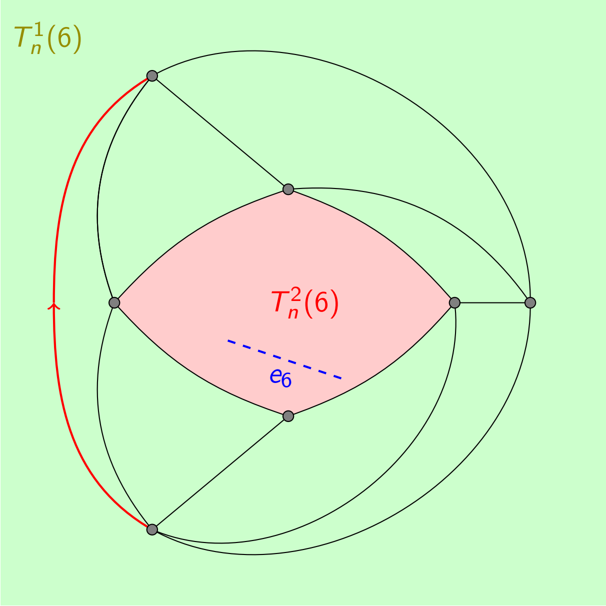
Exploration of $T_n(k)$



Perimeter :
$\widetilde{P}_n(6) = 4$
Explored volume :
$\widetilde{V}_n(6) = 4$

exploration steps :
1
3
5

Exploration of $T_n(k)$



Perimeter :

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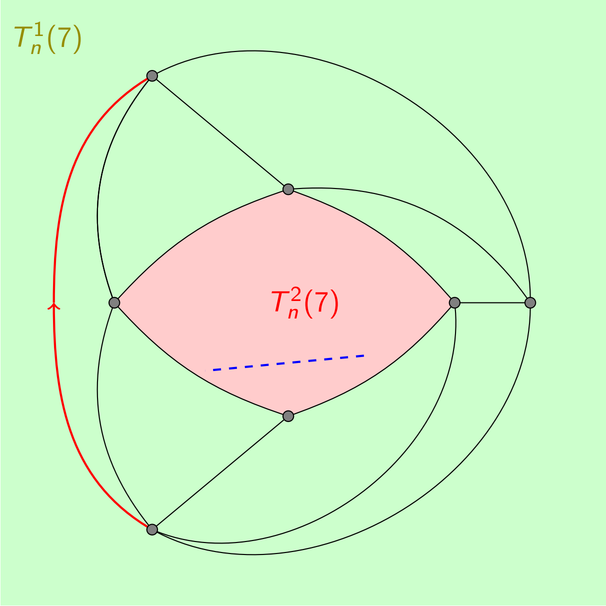
Explored volume :

$\widetilde{V}_n(6) = 4$

exploration steps :

1
3
5

Exploration of $T_n(k)$

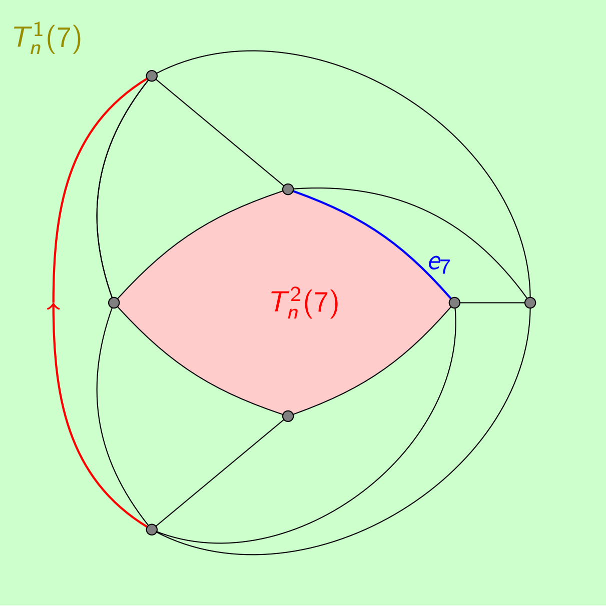


Perimeter :
$\widetilde{P}_n(7) = 4$
Explored volume :
$\widetilde{V}_n(7) = 4$

exploration steps :

1
3
5

Exploration of $T_n(k)$



Perimeter :
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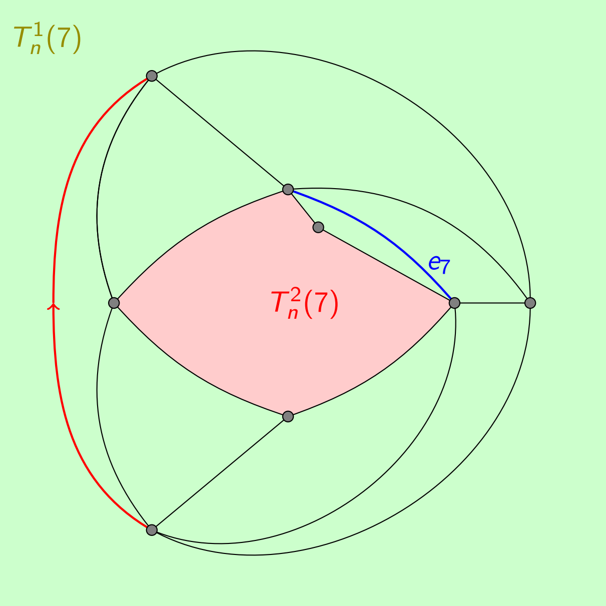
Explored volume :
$\widetilde{V}_n(7) = 4$

exploration steps :

1
3
5

Exploration of $T_n(k)$



Perimeter :

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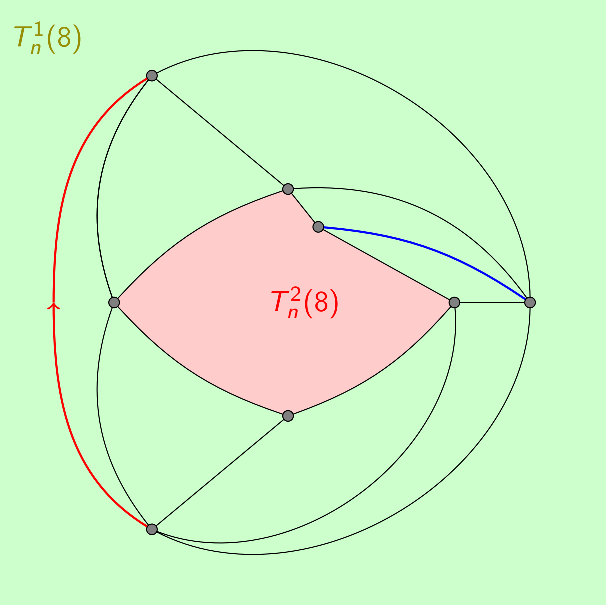
Explored volume :

$\widetilde{V}_n(7) = 4$

exploration steps :

1
3
5
7

Exploration of $T_n(k)$



Perimeter :

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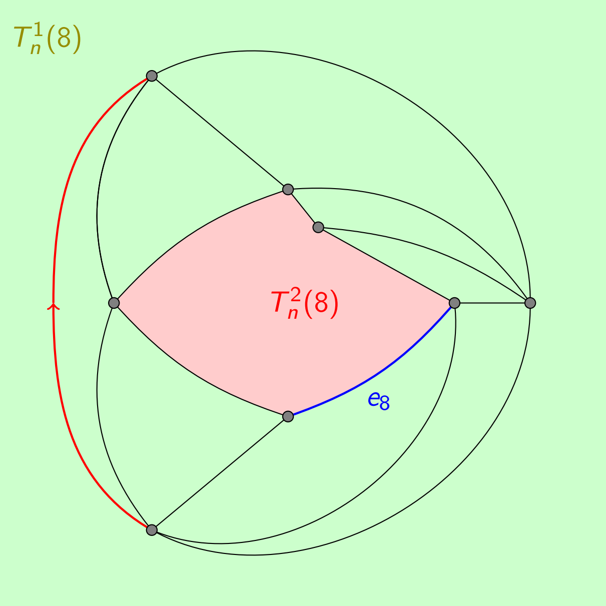
Explored volume :

$\widetilde{V}_n(8) = 5$

exploration steps :

1
3
5
7



Perimeter :
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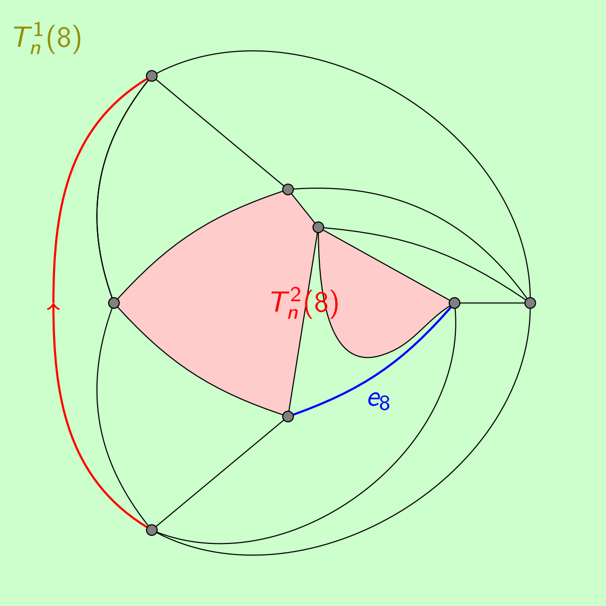
exploration steps :

1
3
5
7

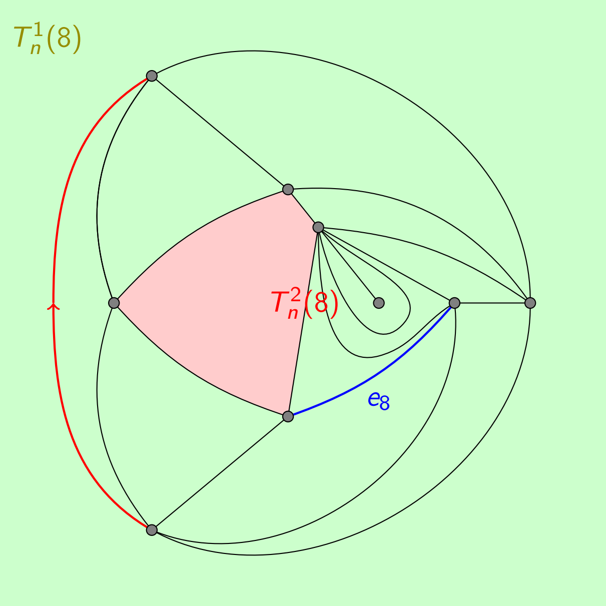


Perimeter :
$\widetilde{P}_n(8) = 5$
Explored volume :
$\widetilde{V}_n(8) = 5$

exploration steps :

1
3
5
7
8

Exploration of $T_n(k)$



Perimeter :

$\widetilde{P}_n(8) = 5$

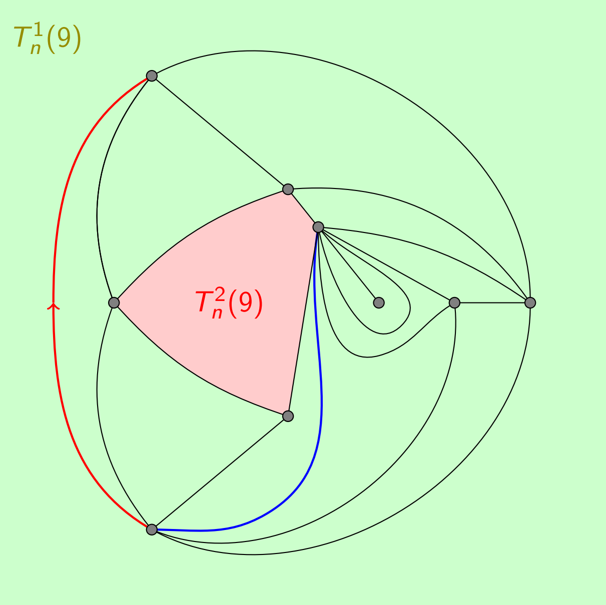
Explored volume :

$\widetilde{V}_n(8) = 5$

exploration steps :

1
3
5
7
8

Exploration of $T_n(k)$

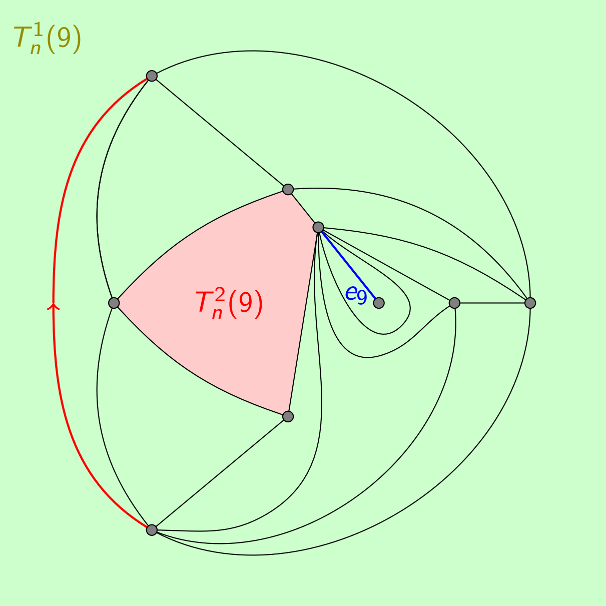


Perimeter :
$\widetilde{P}_n(9) = 4$
Explored volume :
$\widetilde{V}_n(9) = 6$

exploration steps :

1
3
5
7
8

Exploration of $T_n(k)$



Perimeter :

$\widetilde{P}_n(9) = 4$

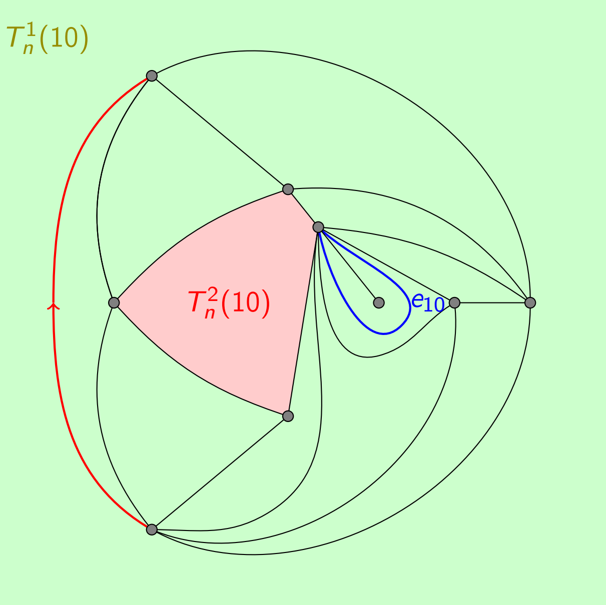
Explored volume :

$\widetilde{V}_n(9) = 6$

exploration steps :

1
3
5
7
8

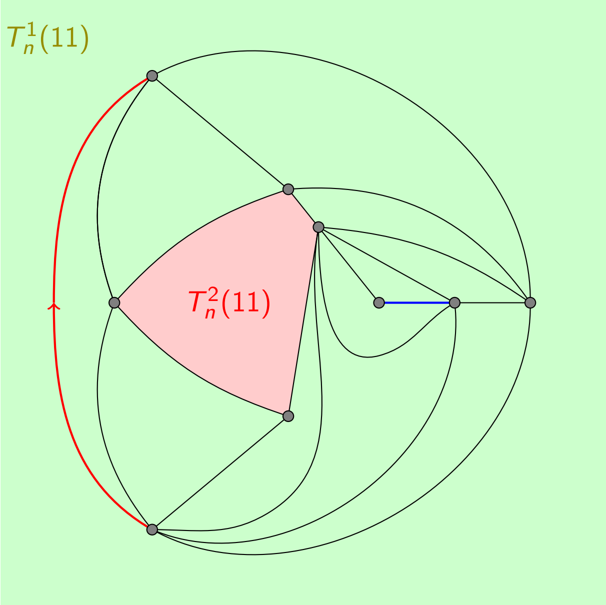
Exploration of $T_n(k)$



Perimeter :
$\widetilde{P}_n(10) = 4$
Explored volume :
$\widetilde{V}_n(10) = 6$

exploration steps :
1
3
5
7
8

Exploration of $T_n(k)$

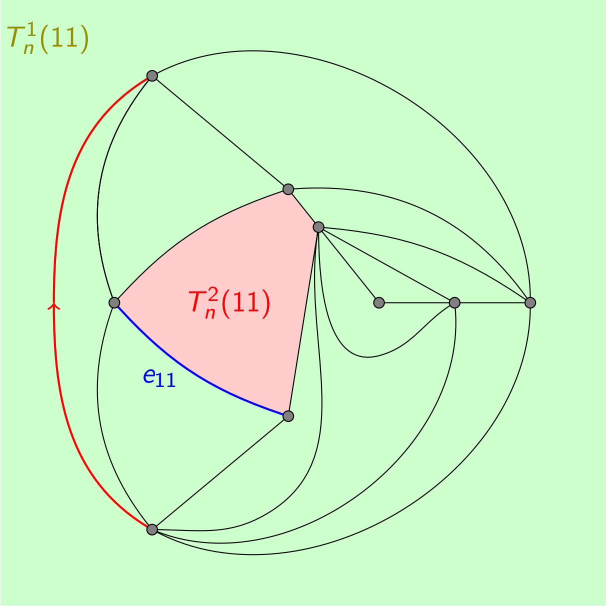


Perimeter :
$\widetilde{P}_n(11) = 4$
Explored volume :
$\widetilde{V}_n(11) = 6$

exploration steps :

1
3
5
7
8

Exploration of $T_n(k)$

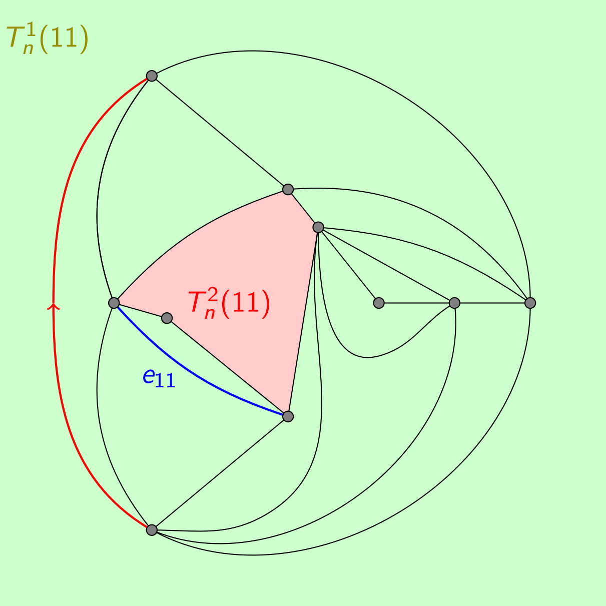


Perimeter :
$\widetilde{P}_n(11) = 4$
Explored volume :
$\widetilde{V}_n(11) = 6$

exploration steps :

1
3
5
7
8

Exploration of $T_n(k)$

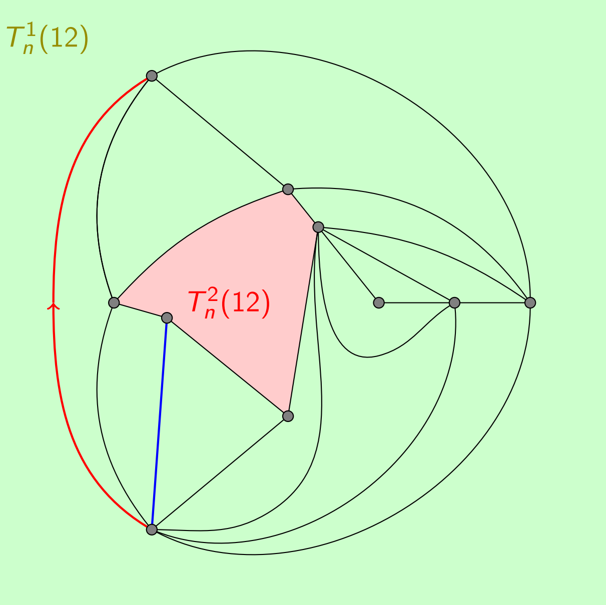


Perimeter :
$\widetilde{P}_n(11) = 4$
Explored volume :
$\widetilde{V}_n(11) = 6$

exploration steps :

1
3
5
7
8
11

Exploration of $T_n(k)$

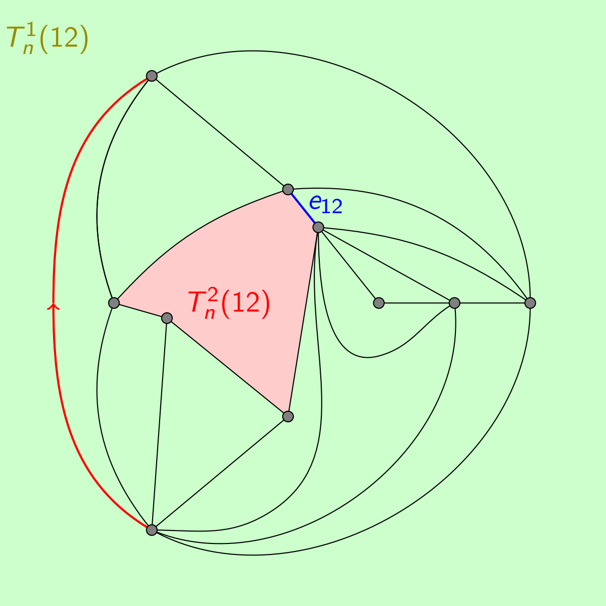


Perimeter :
$\widetilde{P}_n(12) = 5$
Explored volume :
$\widetilde{V}_n(12) = 7$

exploration steps :

1
3
5
7
8
11

Exploration of $T_n(k)$

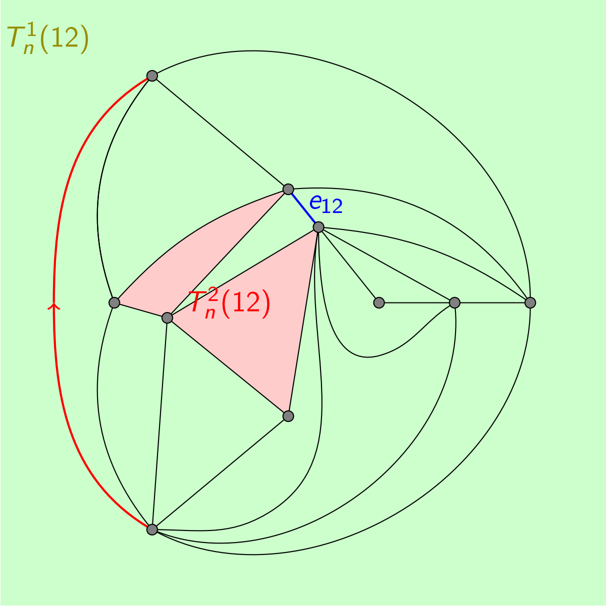


Perimeter :
$\widetilde{P}_n(12) = 5$
Explored volume :
$\widetilde{V}_n(12) = 7$

exploration steps :

1
3
5
7
8
11

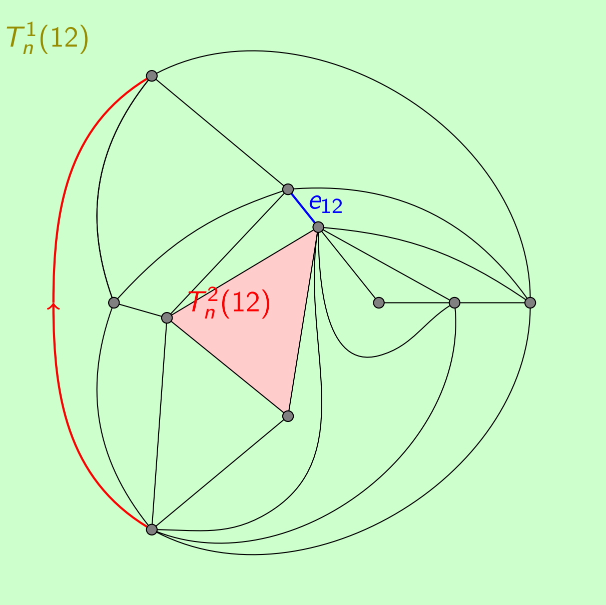
Exploration of $T_n(k)$



Perimeter :
$\widetilde{P}_n(12) = 5$
Explored volume :
$\widetilde{V}_n(12) = 7$

exploration steps :

1
3
5
7
8
11
12



Perimeter :

$\widetilde{P}_n(12) = 5$

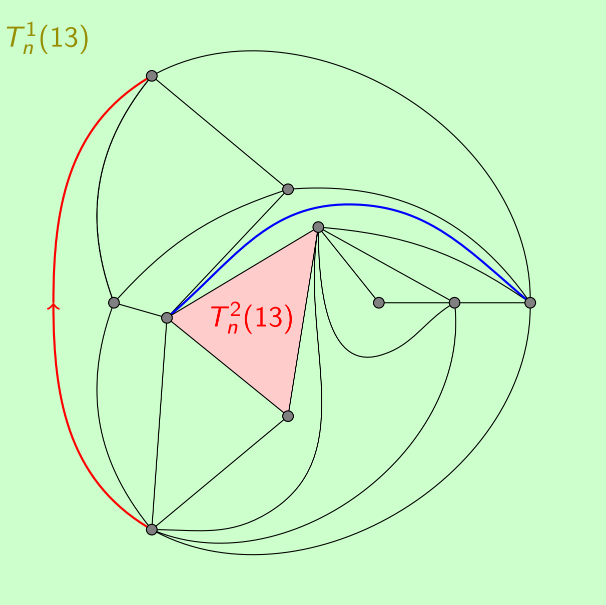
Explored volume :

$\widetilde{V}_n(12) = 7$

exploration steps :

1
3
5
7
8
11
12

Exploration of $T_n(k)$



Perimeter :
$\widetilde{P}_n(13) = 3$
Explored volume :
$\widetilde{V}_n(13) = 7$

exploration steps :

1
3
5
7
8
11
12

Peeling estimates

Claim

For all $k \geq 0$, conditionally on $(T_n^1(i))_{0 \leq i \leq k}$, the triangulation $T_n^2(k)$ is a uniform triangulation with a boundary of length $|\partial T_n^1(k)|$ and $n - |T_n^1(k)|$ inner vertices.

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- We have $P_n(j) \approx j^{2/3}$ and $V_n(j) \approx j^{4/3}$ as long as $j \ll n^{3/4}$ [Curien–Le Gall].

Time change estimates

Conditionally on (P_n, V_n), the $\tau_{i+1} - \tau_i$ are independent and geometric with parameters $\frac{P_n(i)}{3n-6}$, so for $\varepsilon > 0$ small,

$$\mathbb{E}\left[\tau_{\varepsilon n^{3/4}} | P_n\right] = \sum_{i=1}^{\varepsilon n^{3/4}} \frac{3n-6}{P_n(i)} > \frac{n \times \varepsilon n^{3/4}}{\sqrt{n}} = \varepsilon n^{5/4},$$

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$$\widetilde{P}_n(k_n) = P_n(o(n^{3/4})) = o(\sqrt{n}),$$

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We can find a separating cycle of length $\sqrt{\widetilde{P}_n(k_n)} = o(n^{1/4})$ in $T_n^2(k_n)$ [Krikun].

Is the lower bound sharp ?

- Back-of-the-enveloppe computation :
  - in a typical triangulation, the distance between two typical vertices x and y is $\approx n^{1/4}$.
  - The probability that a flip hits a geodesic is $\approx n^{-3/4}$.
  - The distance between x and y changes $\approx kn^{-3/4}$ times before time k.
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- For triangulations of a convex polygon (no inner vertices), the lower bound $n^{3/2}$ is believed to be sharp but the best known upper bound is n^5 [McShine–Tetali].
- Prove that the mixing time is polynomial ?

MERCI !