

Le lien entre Michael Jordan et Catalan

Jérémie Bettinelli

Éric Fusy

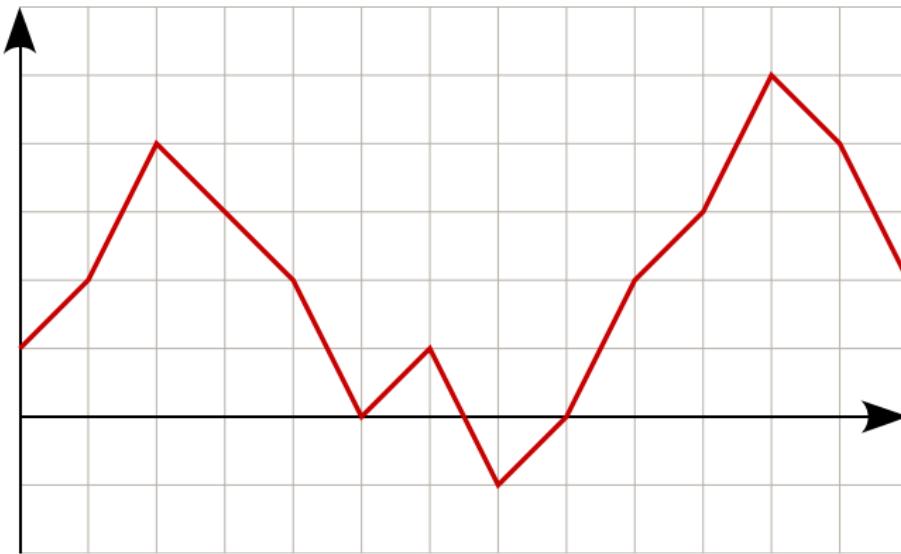
Cécile Mailler

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Basketball walks



Basketball walk: integer-valued walk with step-set $\{-2, -1, +1, +2\}$

Generating functions

Theorem (Banderier & Krattenthaler & Krnik & Kruchinin & Kruchinin & Nguyen & Wallner '16)

The generating function G of basketball walks from 0 to 1 that are positive except at the origin, counted with weight z per step is given by

$$G(z) = -\frac{1}{2} + \frac{1}{2} \sqrt{\frac{2 - 3z - 2\sqrt{1 - 4z}}{z}}.$$

\mathcal{G} := {basketball walks from 0 to 1 that are positive except at the origin}

$|\mathfrak{w}|$: number of steps of \mathfrak{w}



$$\begin{aligned} G(z) &:= \sum_{\mathfrak{w} \in \mathcal{G}} z^{|\mathfrak{w}|} \\ &= \sum_{n=0}^{\infty} |\{\mathfrak{w} \in \mathcal{G} : |\mathfrak{w}| = n\}| z^n \end{aligned}$$

Catalan is everywhere!

The previous authors observed that

$$1 + \textcolor{red}{G}(z) + \textcolor{red}{G}^2(z) = \textcolor{red}{\text{Cat}}(z) \quad (1)$$

where **Cat** is the Catalan generating function.

Cat(z) = $\sum_{n=0}^{\infty} c_n z^n$ where $c_n := \frac{1}{n+1} \binom{2n}{n}$ is the n -th **Catalan number**

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012

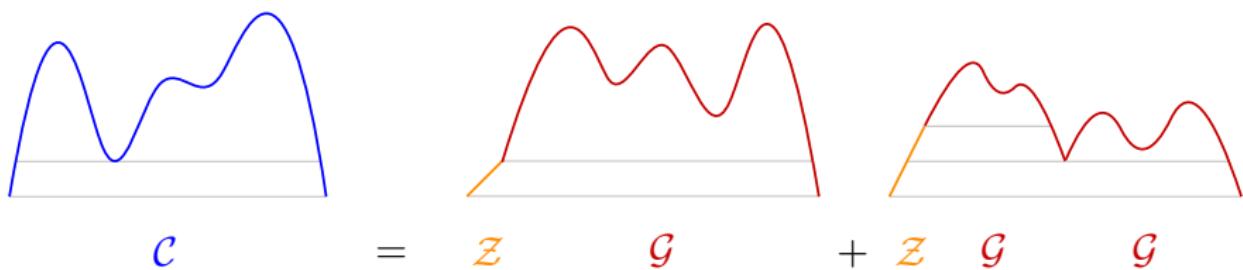
n -edge rooted trees, $n+1$ -leaf binary rooted trees, $2n$ -step Dyck walks, well-parenthesized words with n pairs of parentheses, rooted triangulations of the $n+2$ -gon, noncrossing partitions of the n -set, etc.

C-walks

C-walk: basketball walk from 0 to 0 that visits 1 and is positive except at the extremities

$$\mathcal{C} := \{\text{C-walks}\}$$

$$C(z) := \sum_{w \in \mathcal{C}} z^{|w|}$$

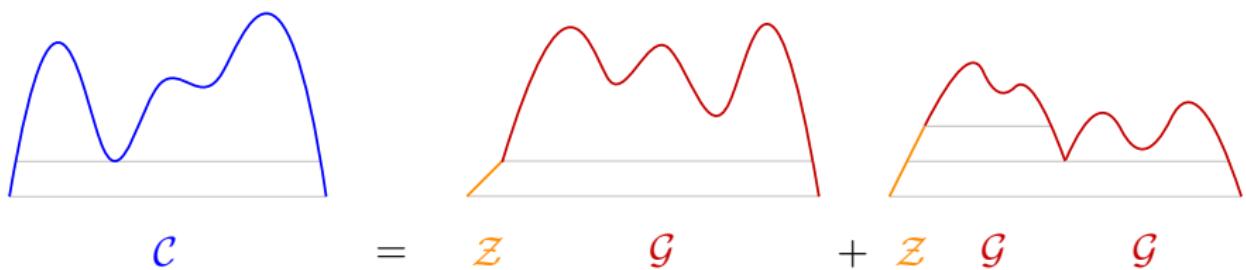


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$$C(z) = \mathcal{Z}G(z) + zG^2(z)$$

Equation (1) becomes $C(z) = z(\text{Cat}(z) - 1)$, which is the generating function of nontrivial binary trees counted with weight z per leaf.

Refined enumeration

even step: step starting at even height

odd step: step starting at odd height

Proposition

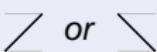
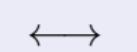
The number of C-walks with $2d \pm 1$ -steps, ℓ odd +2-steps or even -2-steps, and r odd -2-steps or even +2-steps is equal to

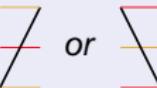
$$\frac{1}{d} \binom{2d-2}{d-1} \binom{\ell+r+2d-2}{\ell+r} \binom{\ell+r}{\ell}.$$

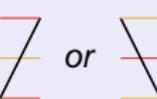
Matched statistics

Proposition

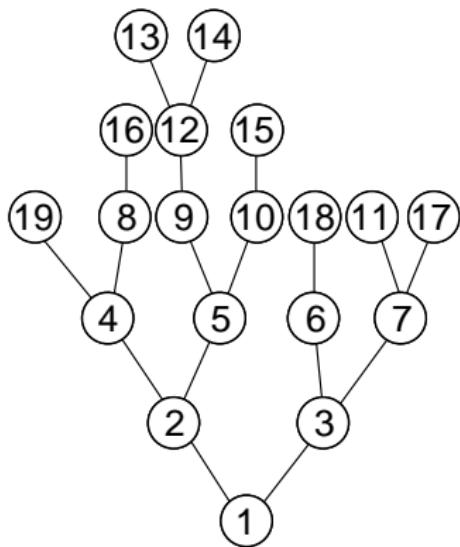
n-step C-walk \longleftrightarrow *n-leaf binary tree*

± 1 -steps  or  \longleftrightarrow  *double leaves*

odd +2 / even -2  or  \longleftrightarrow  *left leaves*

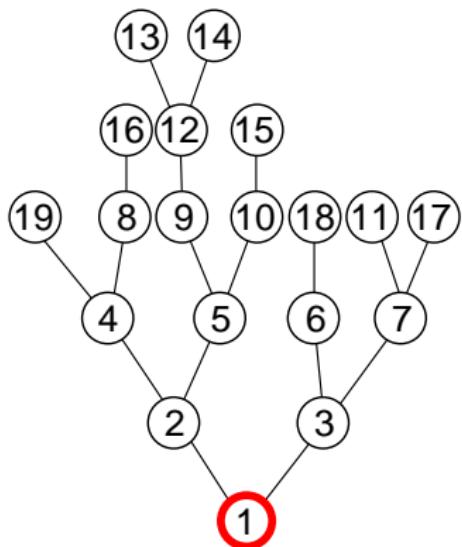
even +2 / odd -2  or  \longleftrightarrow  *right leaves*

Increasing unary-binary tree



increasing unary-binary tree of size n :
plane tree with n vertices labeled 1, 2, ..., n such that each vertex has at most 2 children, and all have larger labels

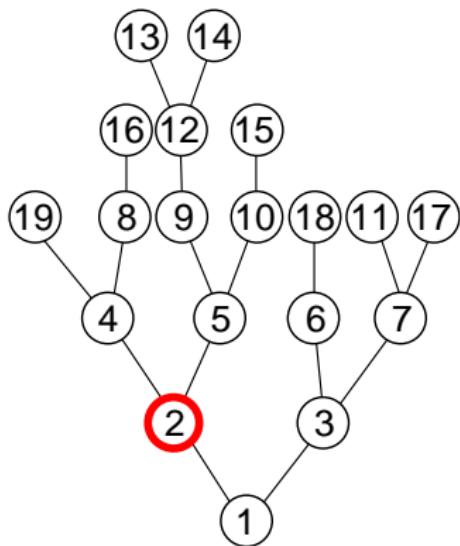
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We associate with it the permutation obtained by reading the labels of the tree in breadth-first search order

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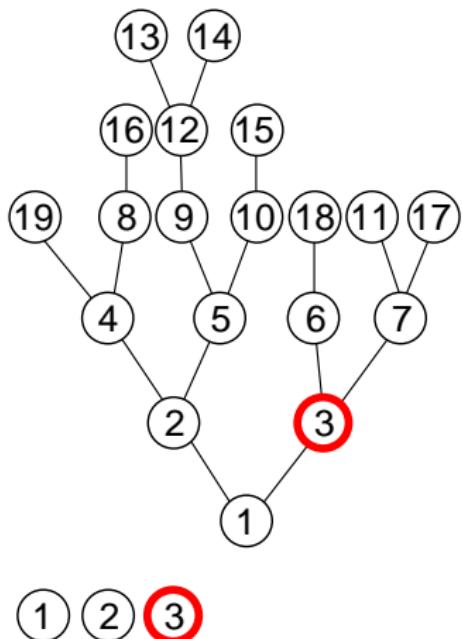


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① ②

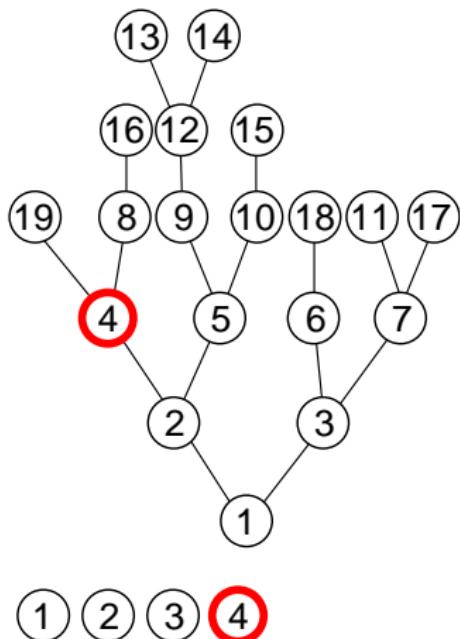
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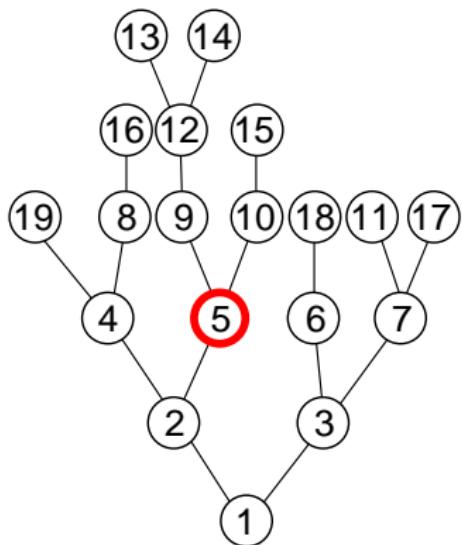
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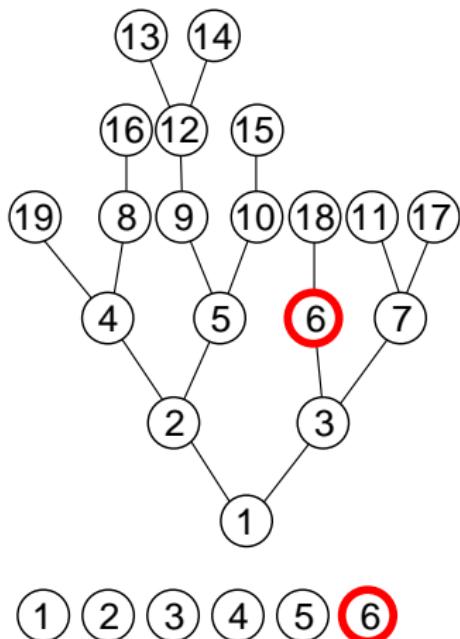


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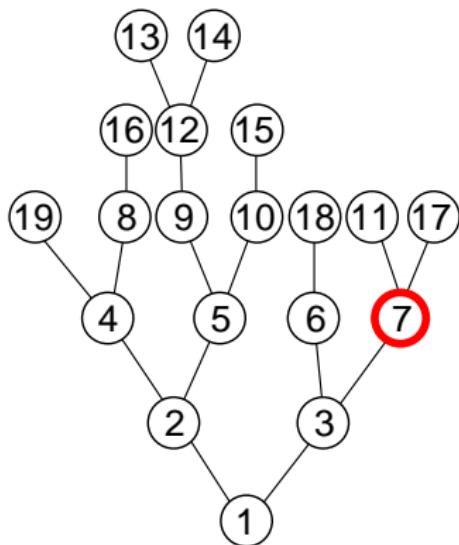
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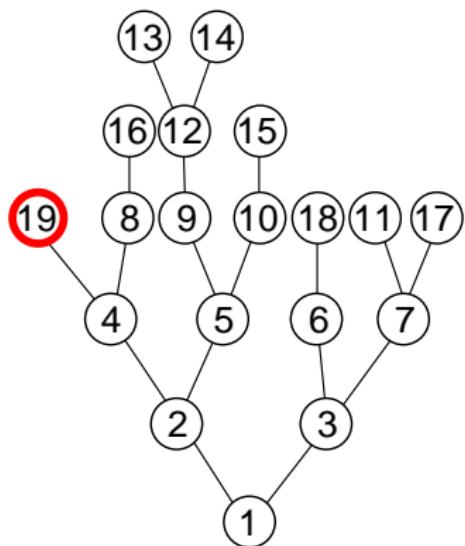


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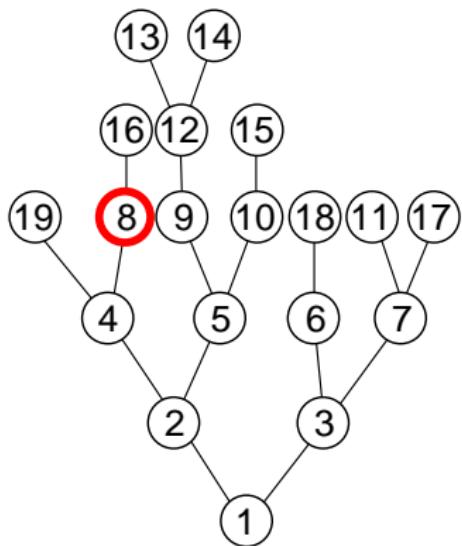


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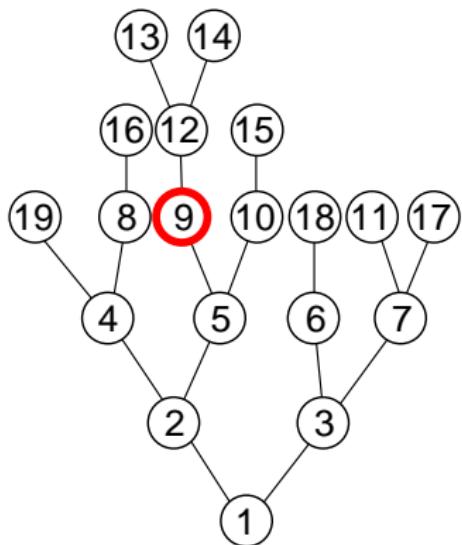


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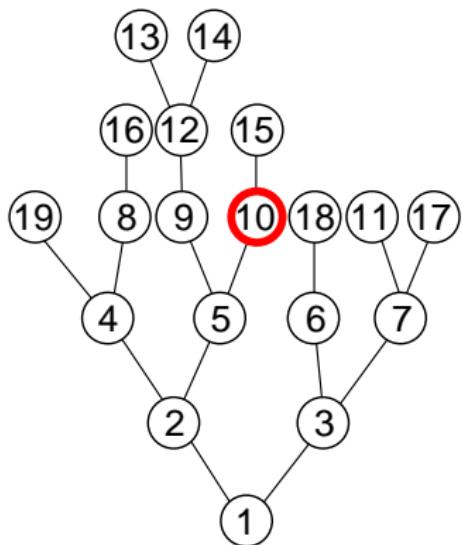


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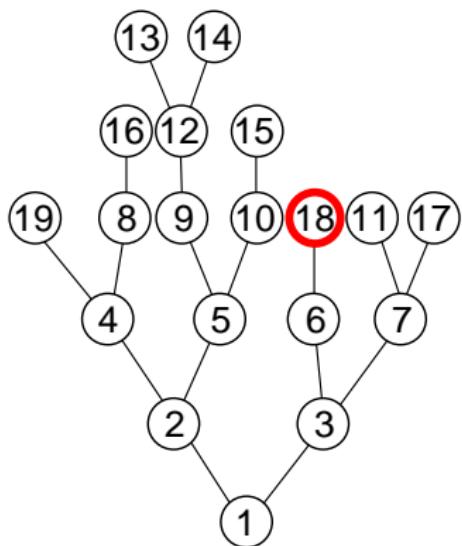


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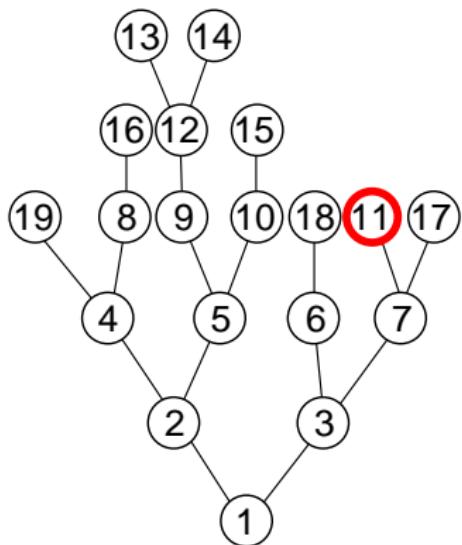


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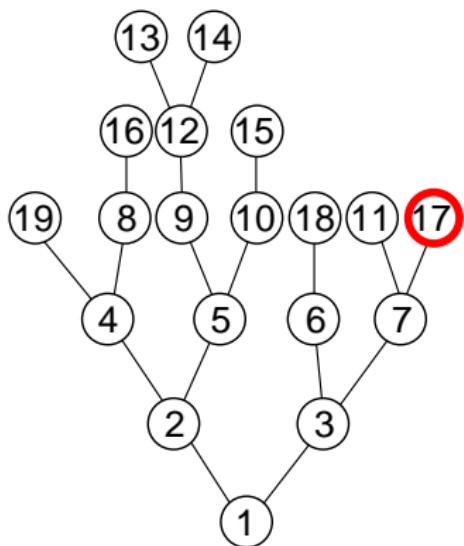


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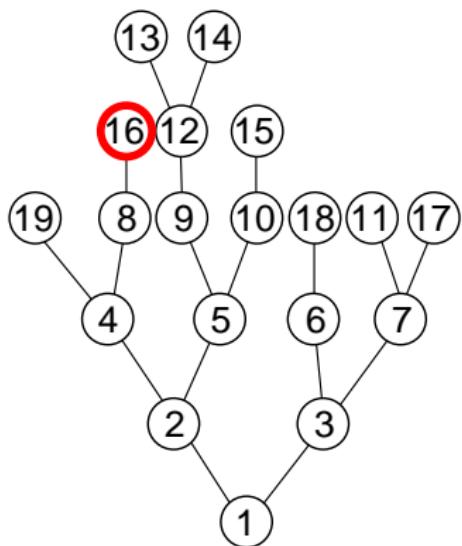


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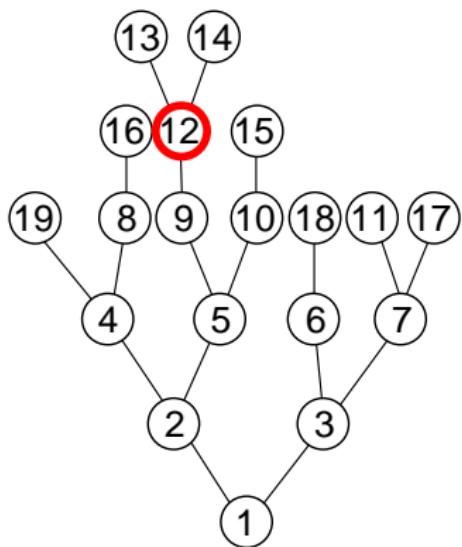


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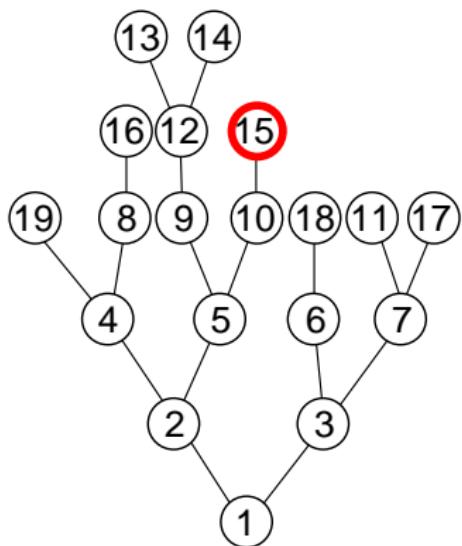


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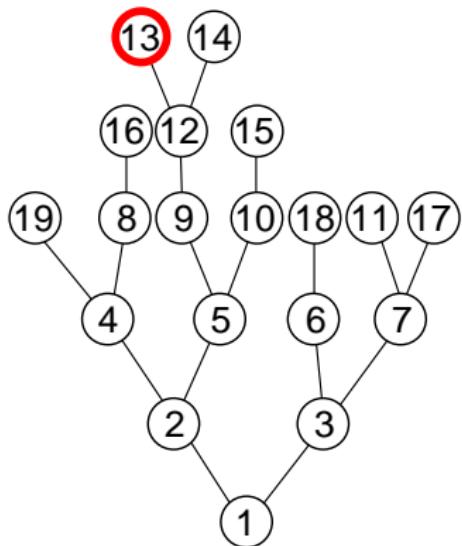


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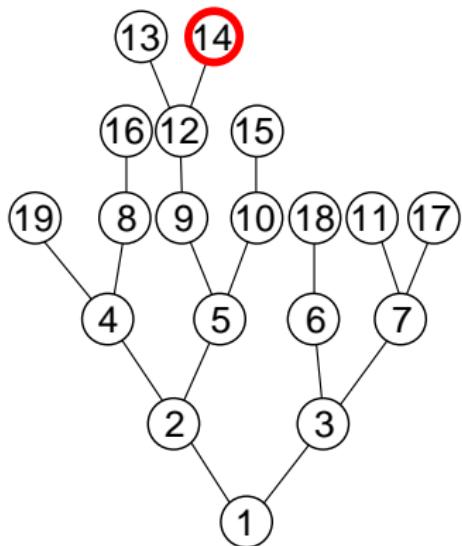


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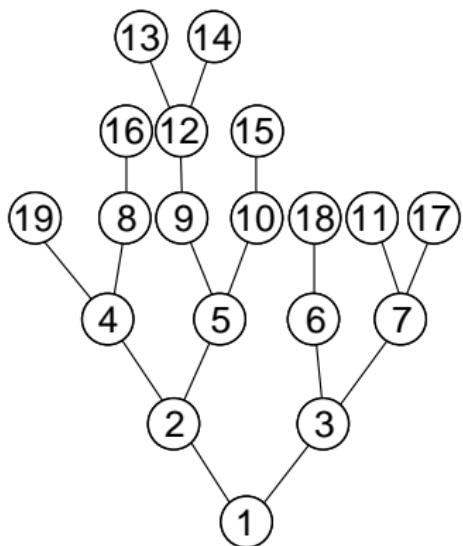


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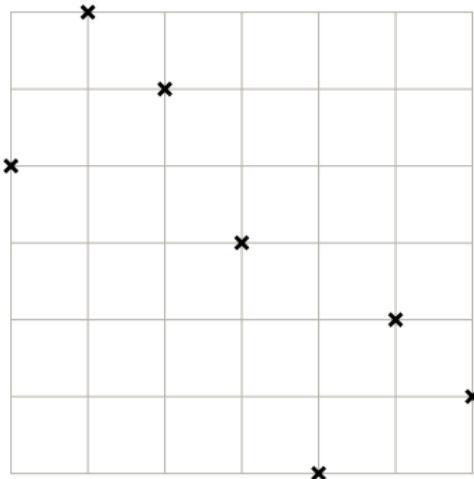
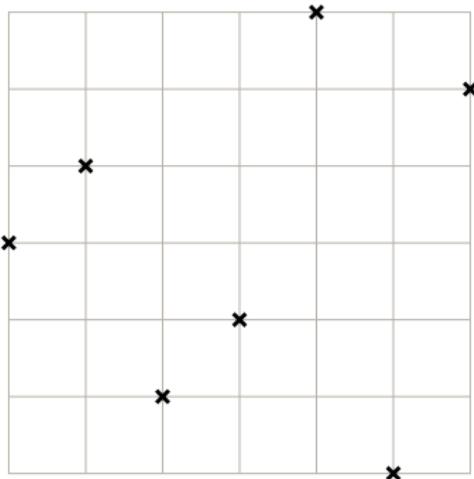


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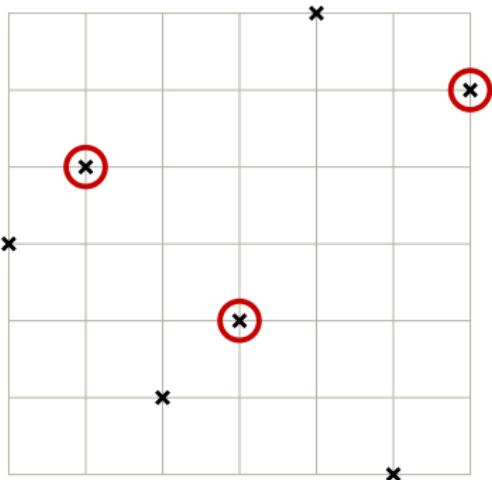
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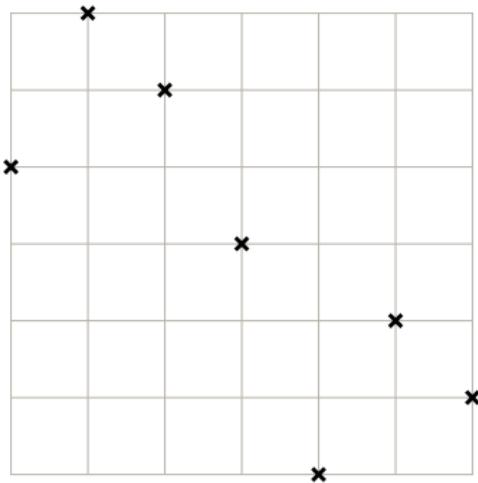
Permutation avoiding 213



Permutation avoiding 213



contains 213



avoids 213

Counting IUBTs

IUBT: increasing unary-binary tree with associated permutation avoiding 213

Theorem

IUBTs are counted by G-walks (basketball walks from 0 to 1 that are positive except at the origin).

Proposition

For $n \geq 1$ and $0 \leq k \leq \lfloor(n-1)/2\rfloor$, the number of n -vertex IUBTs with exactly $n-1-2k$ unary nodes is

$$\frac{1}{n} \binom{2n}{k} \binom{n-k}{k+1}.$$

Matched statistics

IUBT: increasing unary-binary tree with associated permutation avoiding 213

Proposition

n-step G-walk

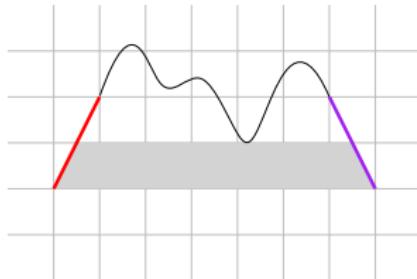


n-vertex IUBT

staggered ± 2 -steps



unary nodes



The red and purple steps are **paired**.

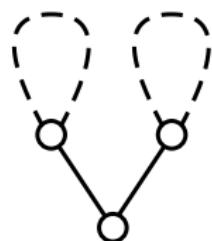
A **staggered ± 2 -step** is a ± 2 -step that is not paired with any other ± 2 -step.

Decomposition of binary trees

\mathcal{N} : class of nontrivial binary trees counted by number of leaves

Goal

Understand bijectively that $\mathcal{C} = \mathcal{N}$



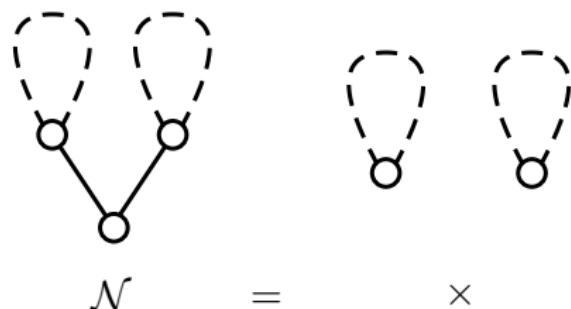
\mathcal{N}

Decomposition of binary trees

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Decomposition of binary trees

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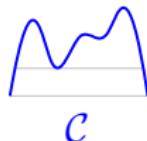
Goal

Understand bijectively that $\mathcal{C} = \mathcal{N}$

$$\mathcal{N} = \dots \times \text{ (Diagram)} \times \text{ (Diagram)} = (\mathcal{Z} + \mathcal{N}) \times (\mathcal{Z} + \mathcal{N})$$

The diagram shows a nontrivial binary tree with two children. It is decomposed into two parts: a single node and a tree with one child. This decomposition is multiplied by itself, resulting in four terms: a single node, a tree with one child, a tree with two children, and a tree with three children.

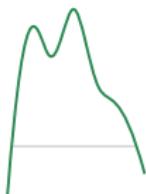
Elementary decomposition of basketball walks



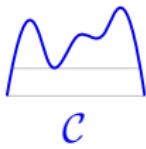
Elementary decomposition of basketball walks



A



B

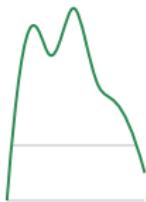


C

Elementary decomposition of basketball walks



\mathcal{A}

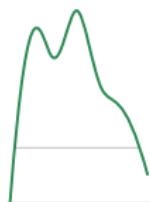


\mathcal{B}



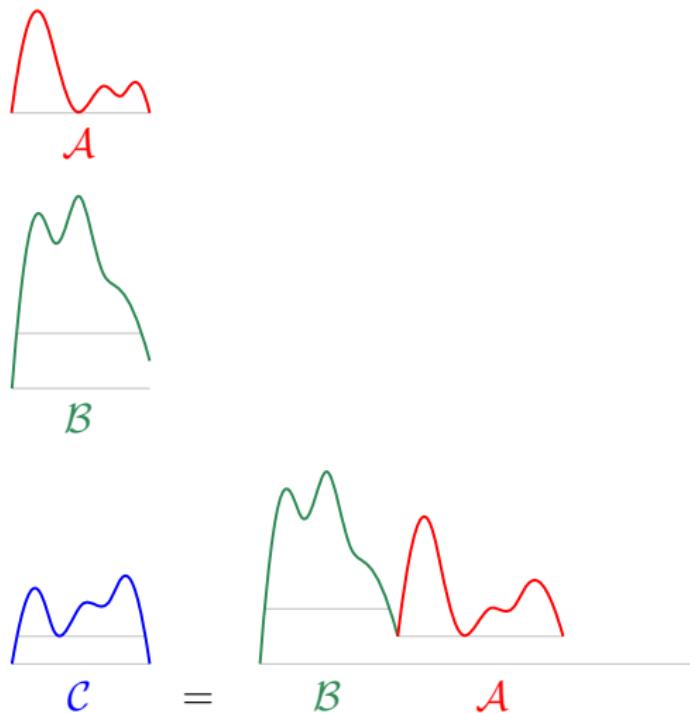
\mathcal{C}

=



\mathcal{B}

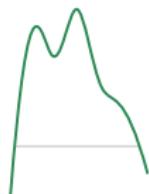
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Elementary decomposition of basketball walks



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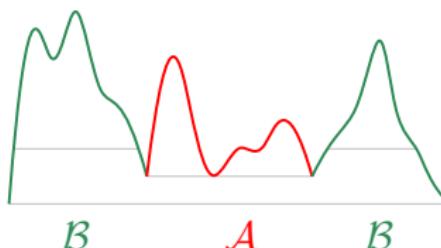


B

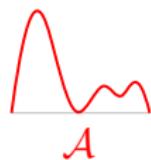


C

=



Elementary decomposition of basketball walks

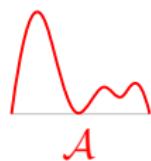


$$= \begin{smallmatrix} Z \\ Z \end{smallmatrix} + \begin{smallmatrix} / \\ Z \end{smallmatrix}$$



$$= \begin{smallmatrix} B \\ B \end{smallmatrix} \quad \begin{smallmatrix} A \\ A \end{smallmatrix} \quad \begin{smallmatrix} B \\ B \end{smallmatrix}$$

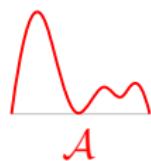
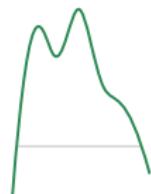
Elementary decomposition of basketball walks



$$B = \begin{matrix} Z \\ + \end{matrix} \begin{matrix} Z \\ A \end{matrix}$$

$$C = \begin{matrix} B \\ + \end{matrix} \begin{matrix} A \\ B \end{matrix}$$

Elementary decomposition of basketball walks

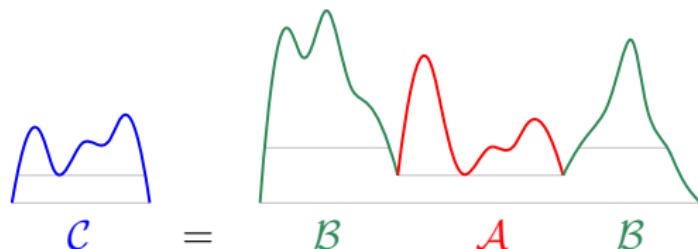
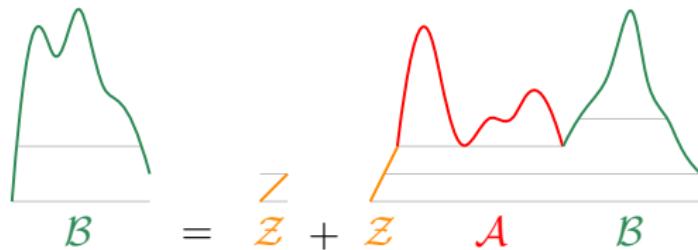
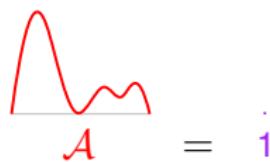
*A**B*

$$= \begin{matrix} \textcolor{brown}{Z} \\ + \end{matrix} \begin{matrix} \textcolor{orange}{Z} \\ \textcolor{red}{A} \end{matrix} \begin{matrix} \textcolor{green}{B} \end{matrix}$$

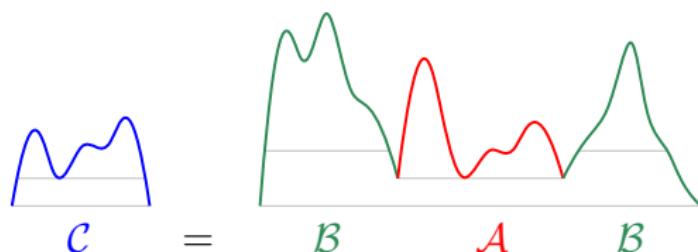
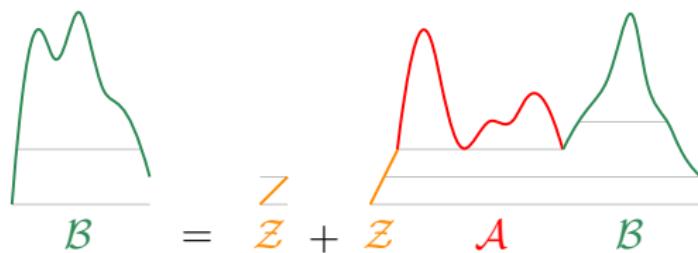
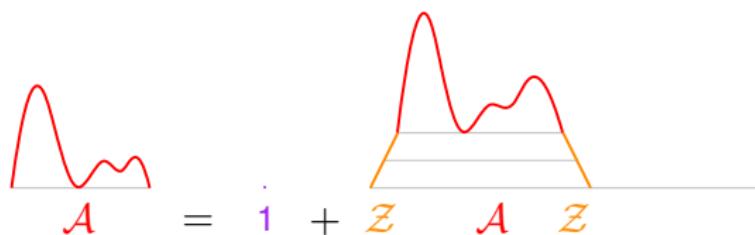
*C*

$$= \begin{matrix} \textcolor{green}{B} \\ \textcolor{red}{A} \end{matrix} \begin{matrix} \textcolor{green}{B} \end{matrix}$$

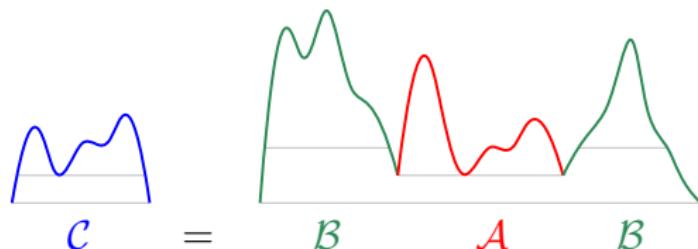
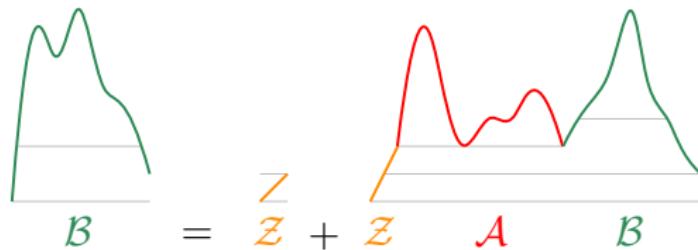
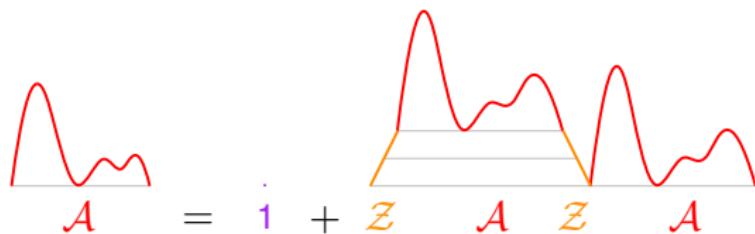
Elementary decomposition of basketball walks



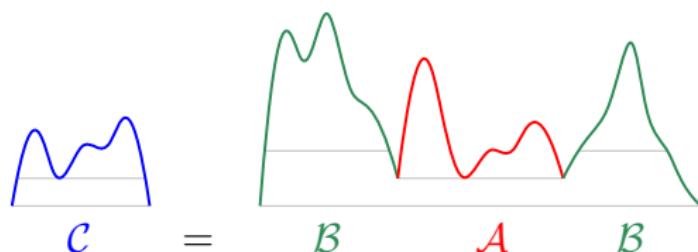
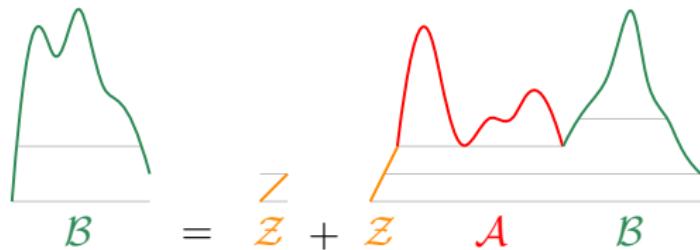
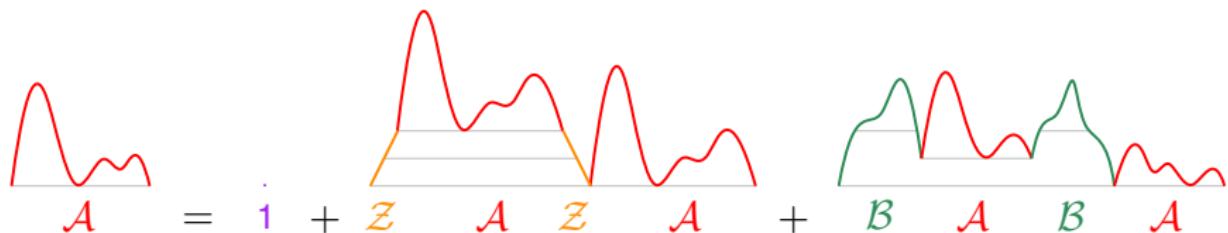
Elementary decomposition of basketball walks



Elementary decomposition of basketball walks



Elementary decomposition of basketball walks



The bijection

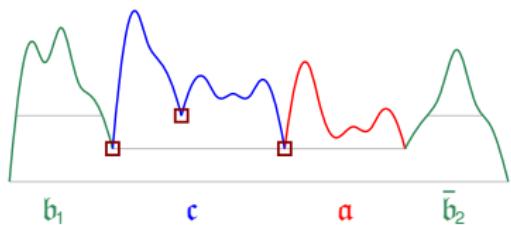


The bijection

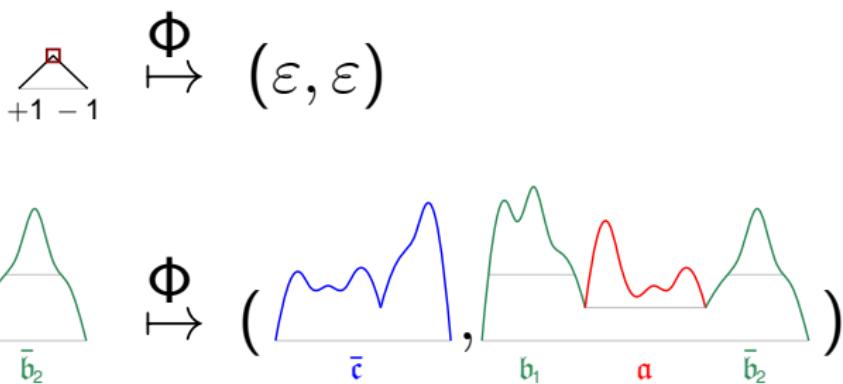
$$\begin{array}{c} \text{Diagram of a binary tree node with a red square at the top, labeled } +1 \text{ and } -1 \text{ below it.} \\ \xrightarrow{\Phi} (\varepsilon, \varepsilon) \end{array}$$

The bijection

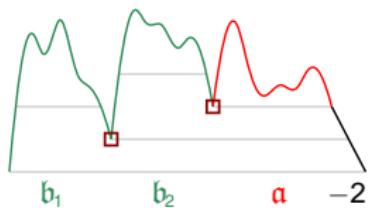
$$\begin{array}{c} \text{Φ} \\ \mapsto \\ \begin{array}{c} \text{---} \\ \text{---} \end{array} \end{array} (\varepsilon, \varepsilon)$$



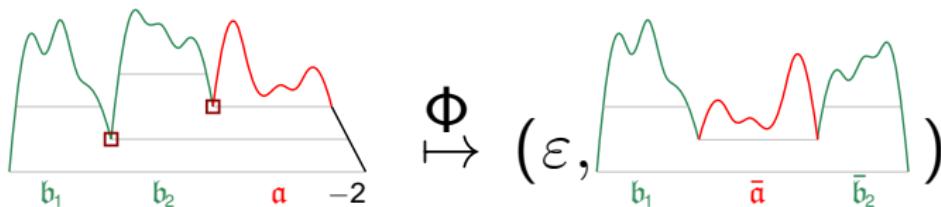
The bijection



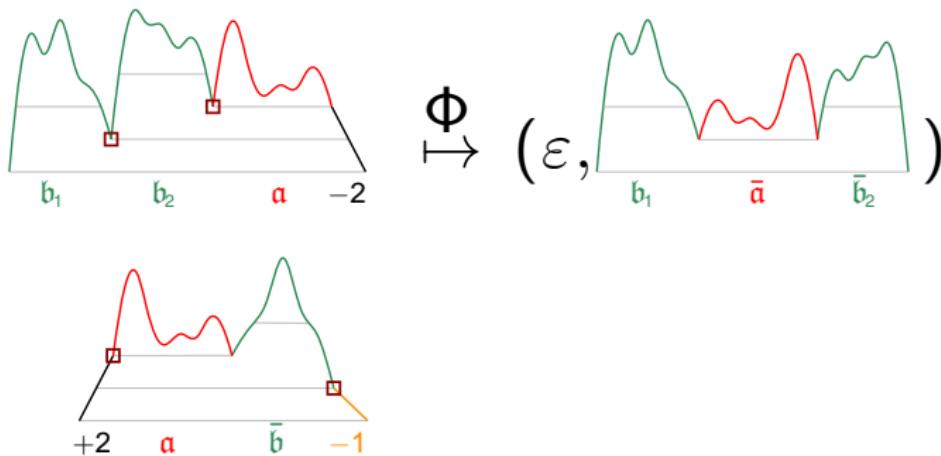
The bijection



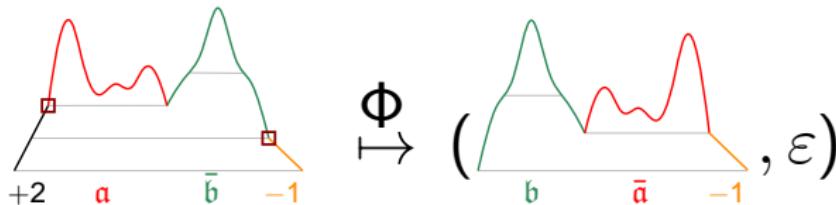
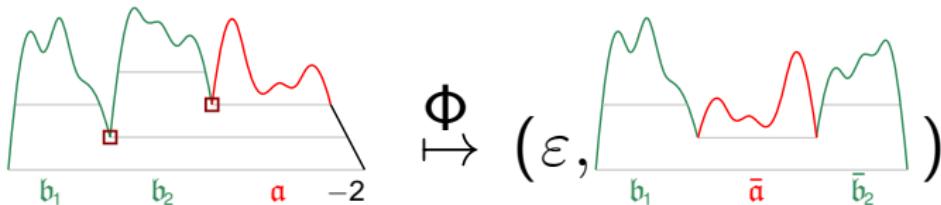
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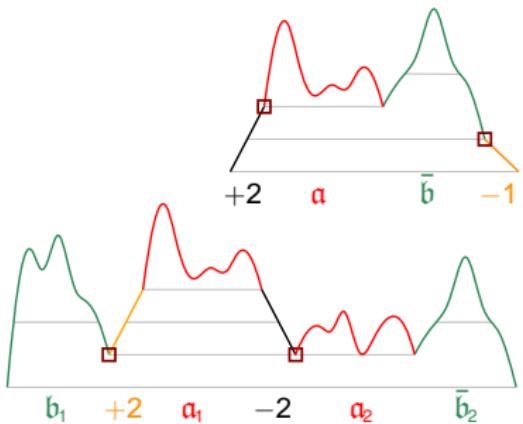
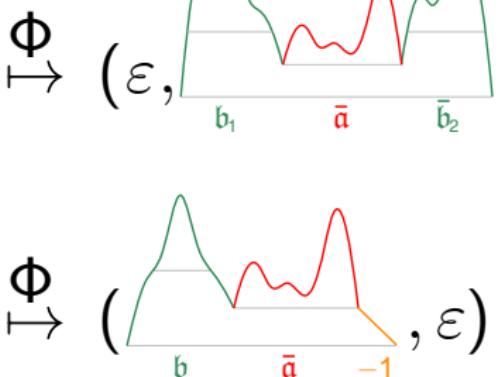
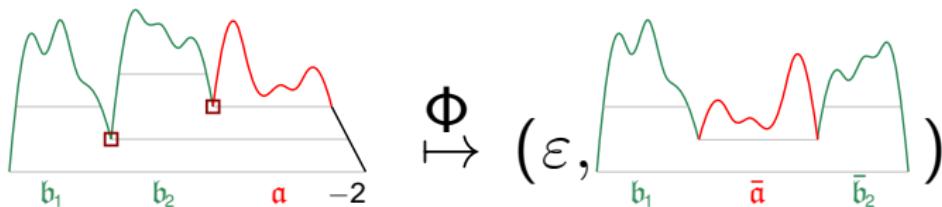
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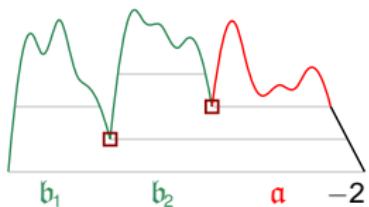
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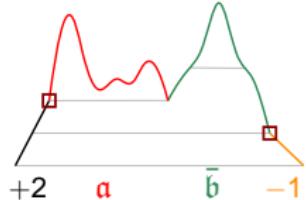
The bijection



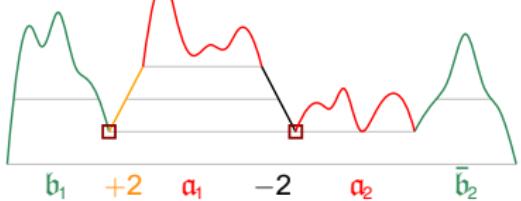
The bijection



$$\Phi \mapsto (\varepsilon, (\text{green path}, \text{red path}))$$

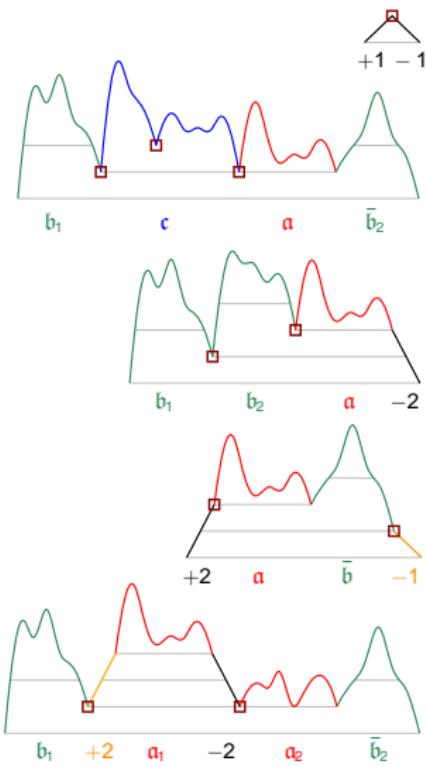
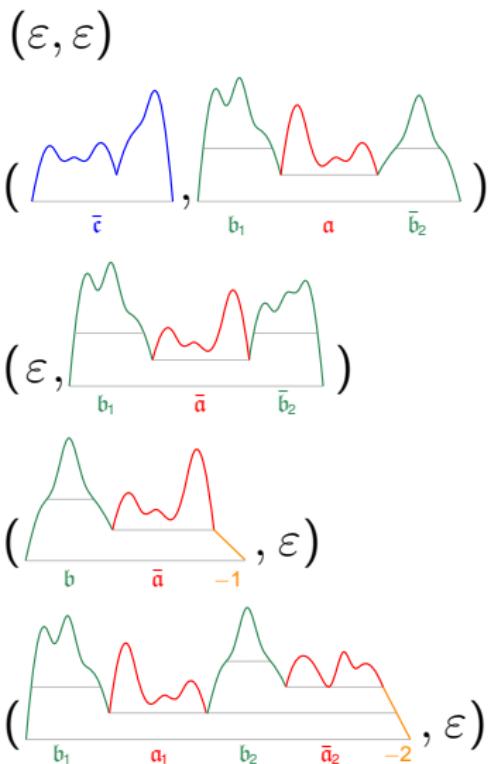


$$\Phi \mapsto ((\text{green path}, \text{red path}), \varepsilon)$$



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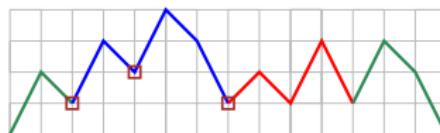

 $\Phi \mapsto$




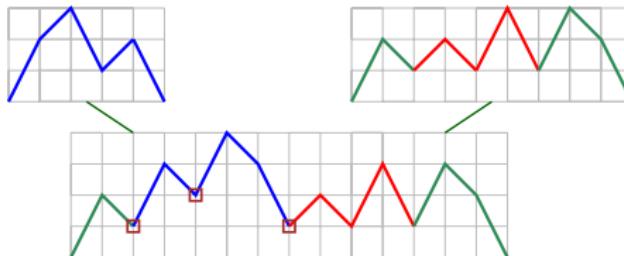
The bijection



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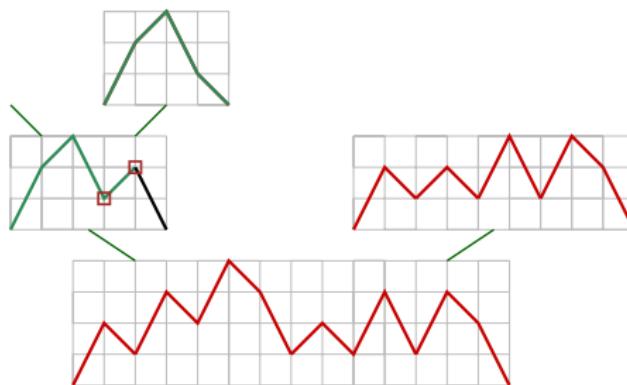
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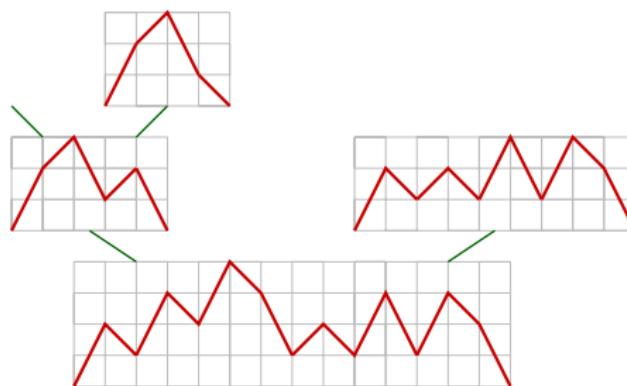
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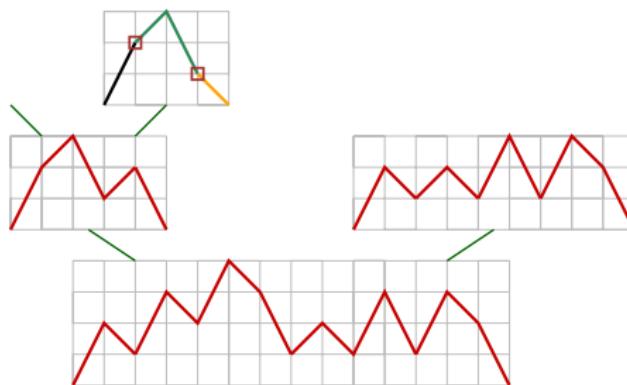
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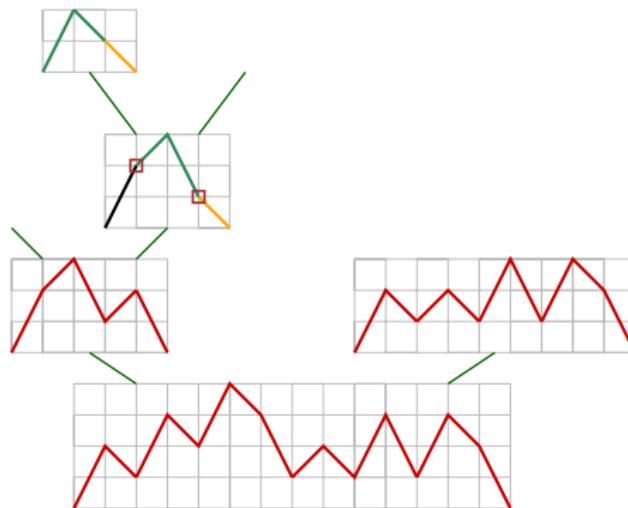
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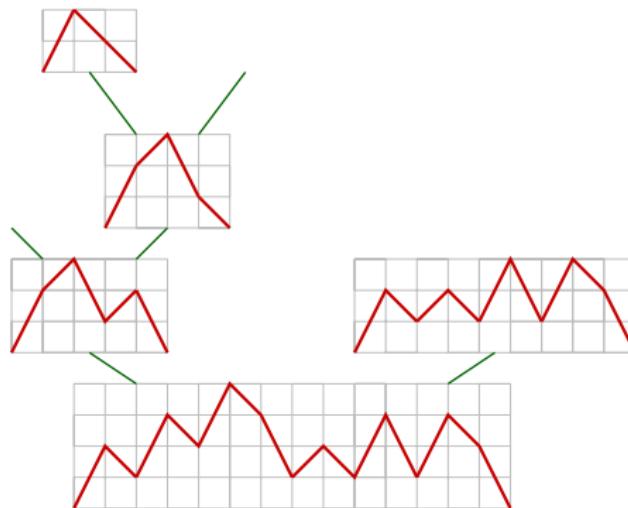
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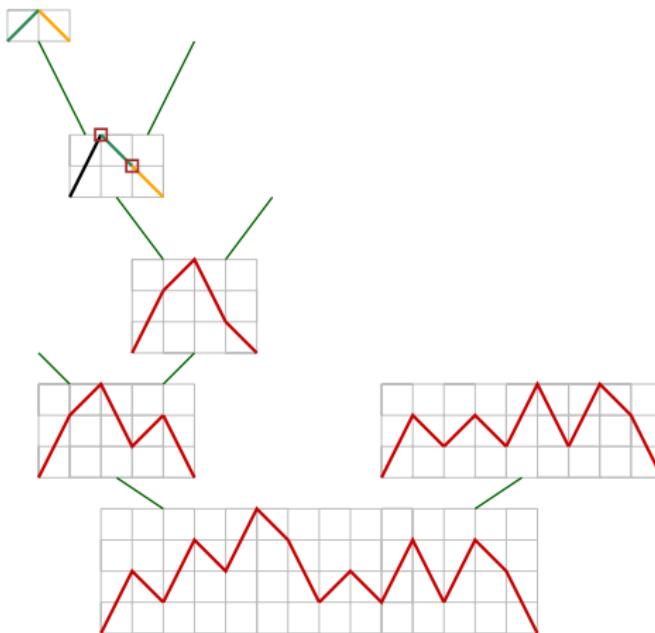
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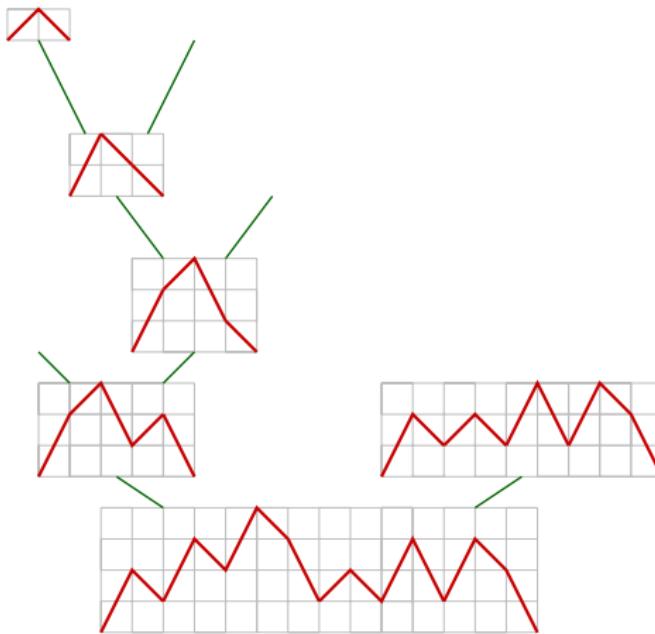
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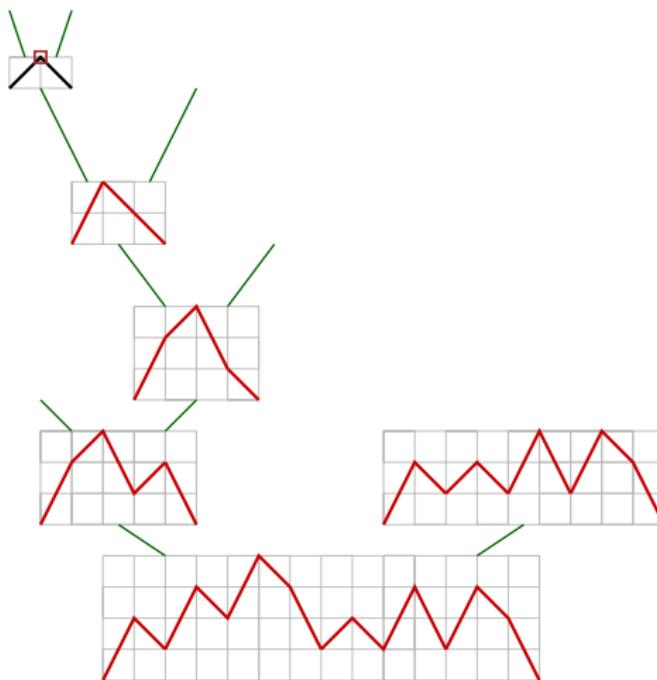
The bijection



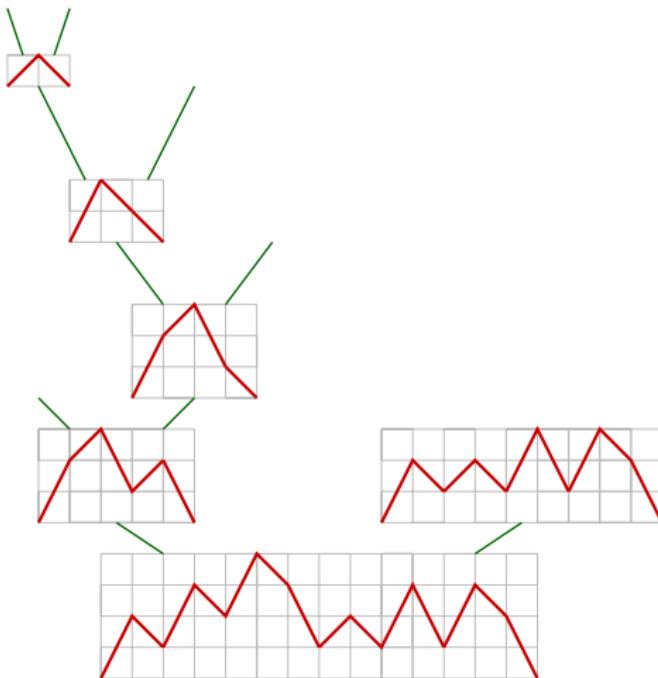
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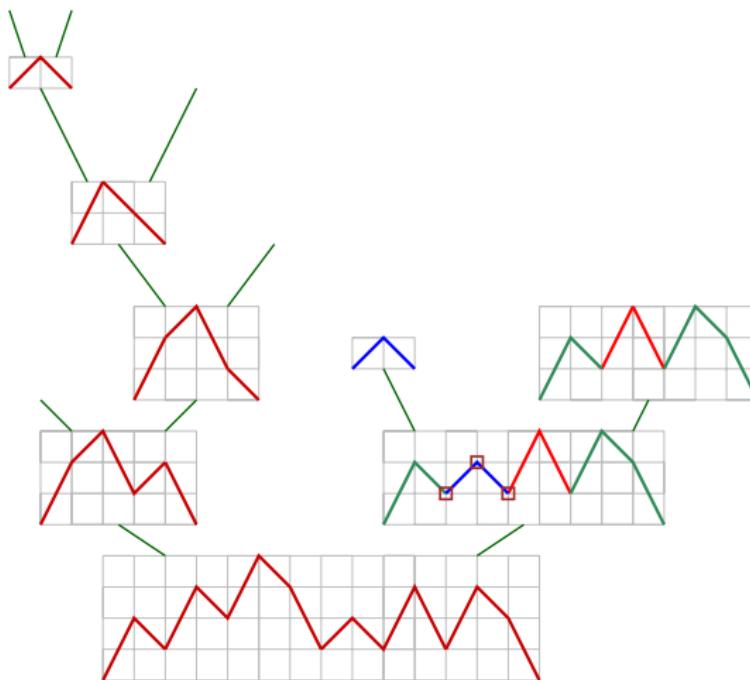
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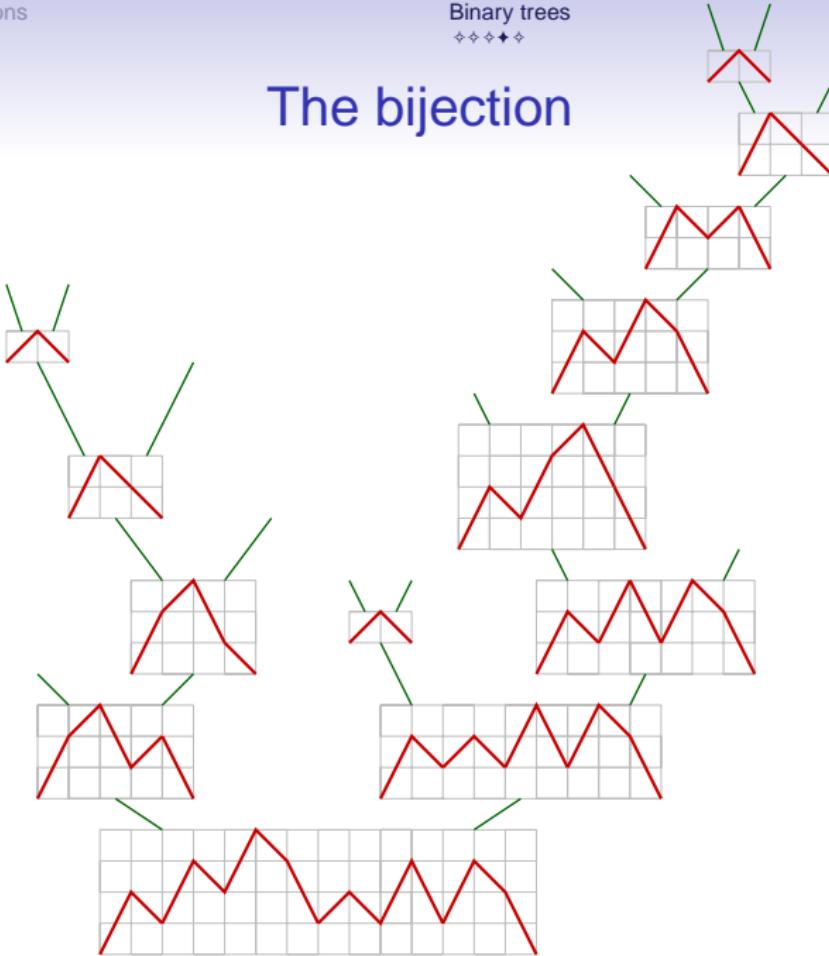
The bijection



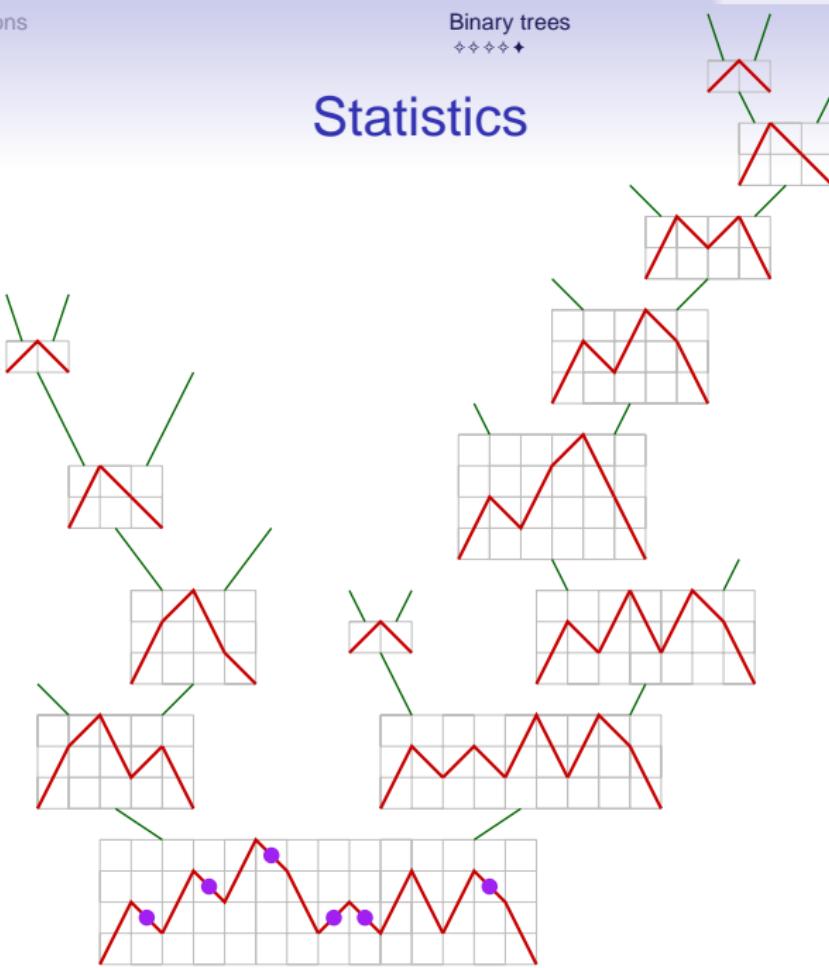
The bijection



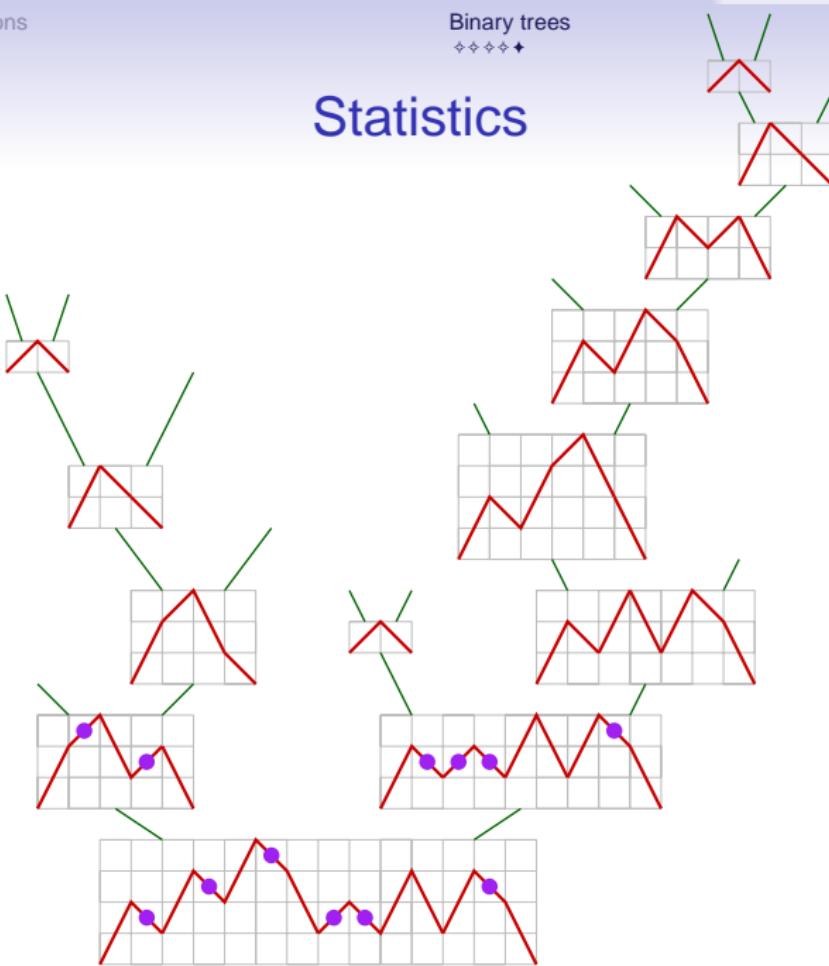
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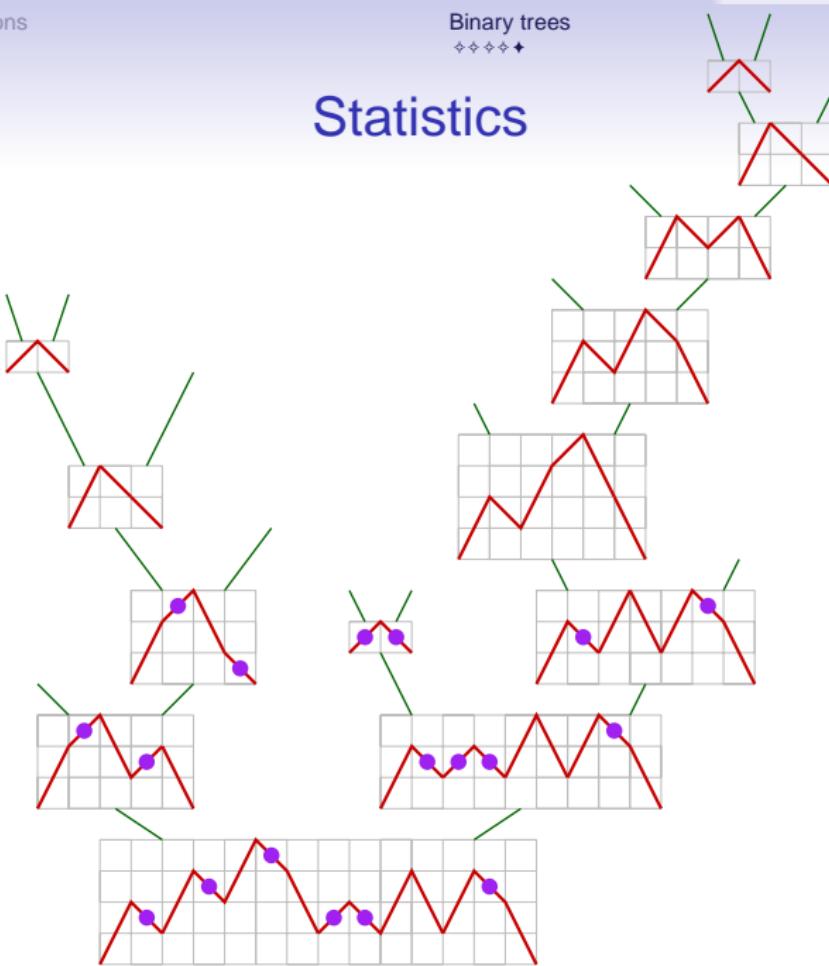
Statistics



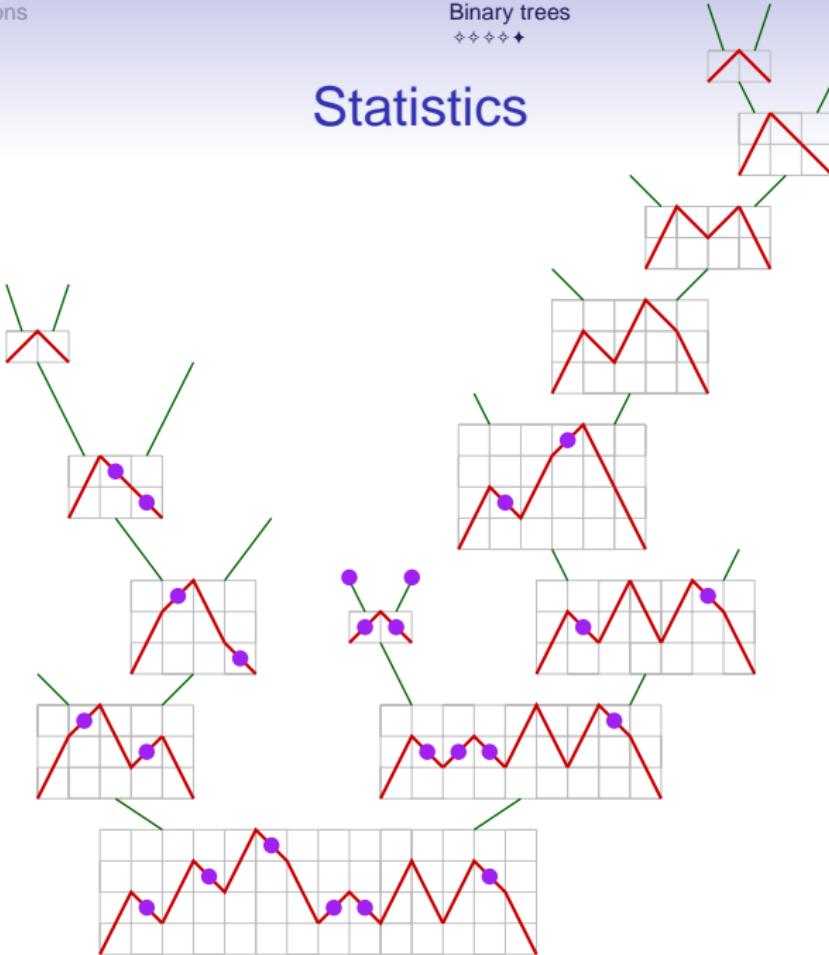
Statistics



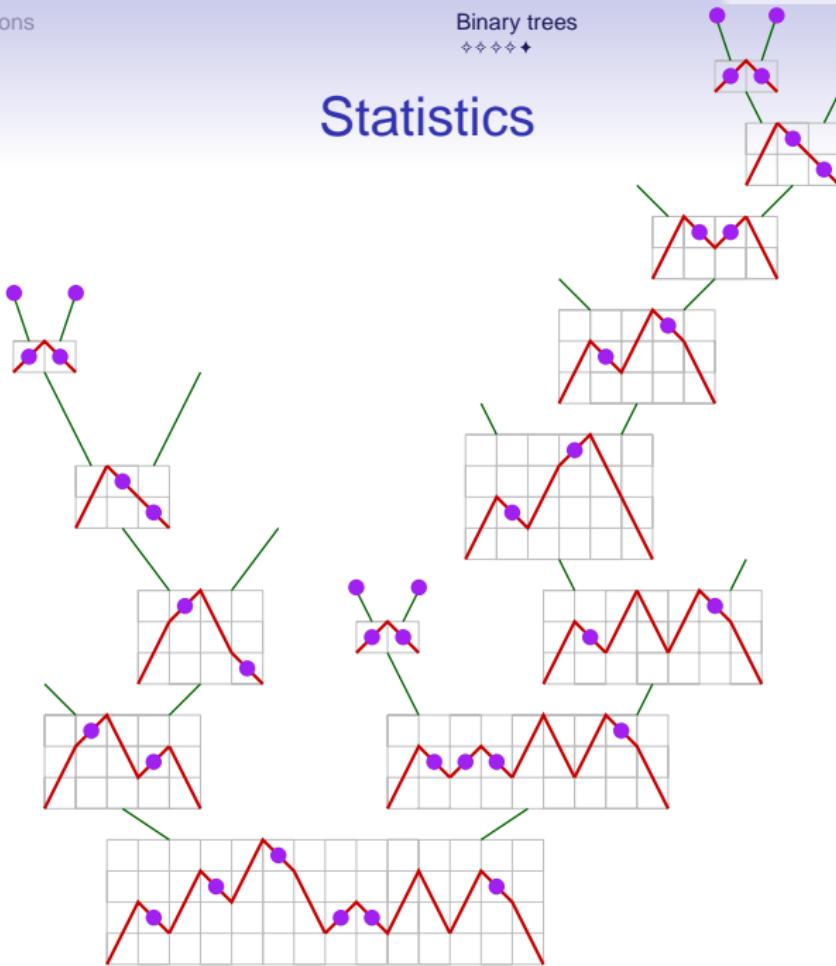
Statistics



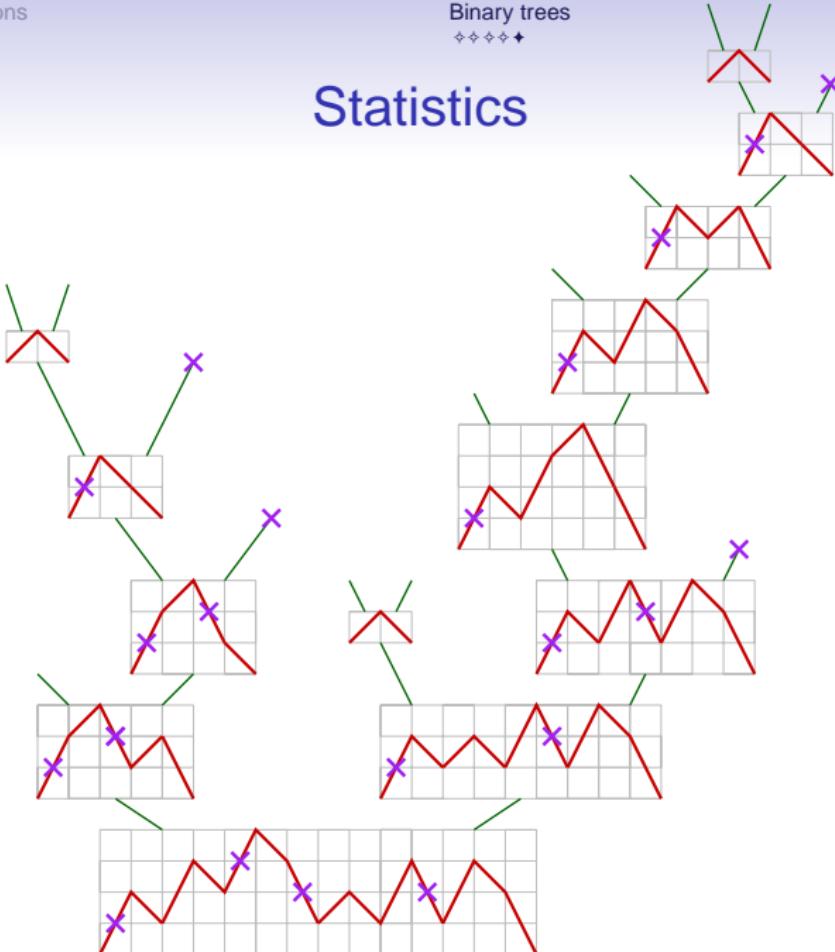
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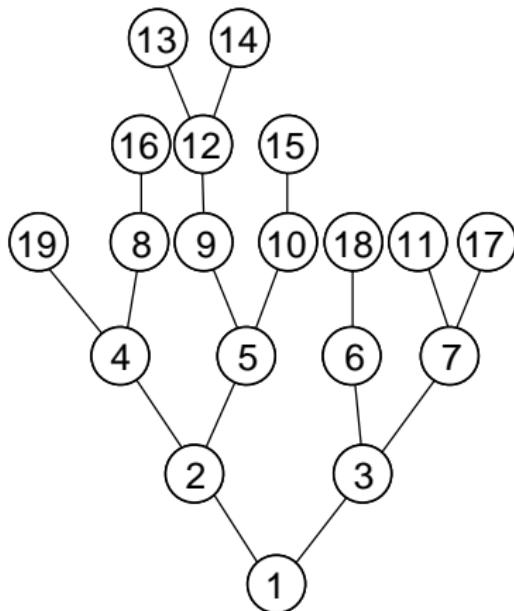
Statistics



Statistics



Valid permutations on a unary-binary tree

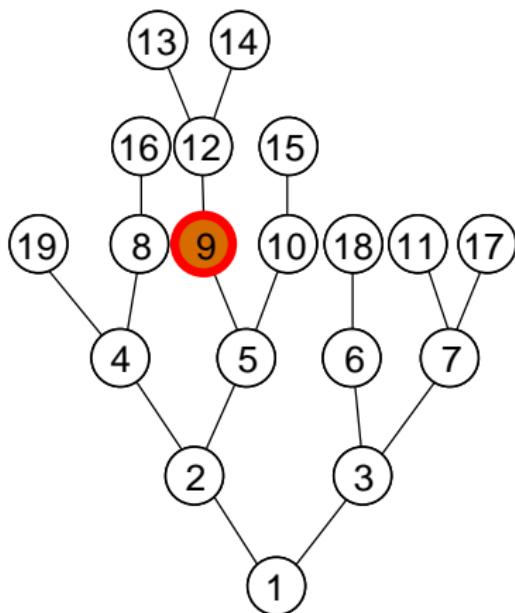


Lemma

A permutation is valid for a tree if and only if it avoids 213 and, the value taken at each node is a right-to-left minimum.



Valid permutations on a unary-binary tree

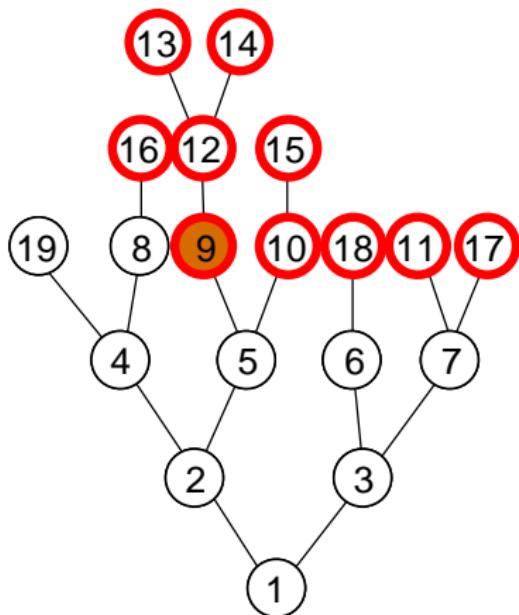


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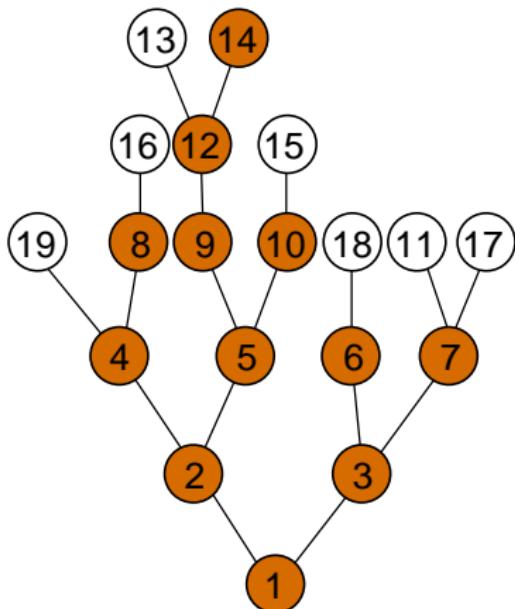


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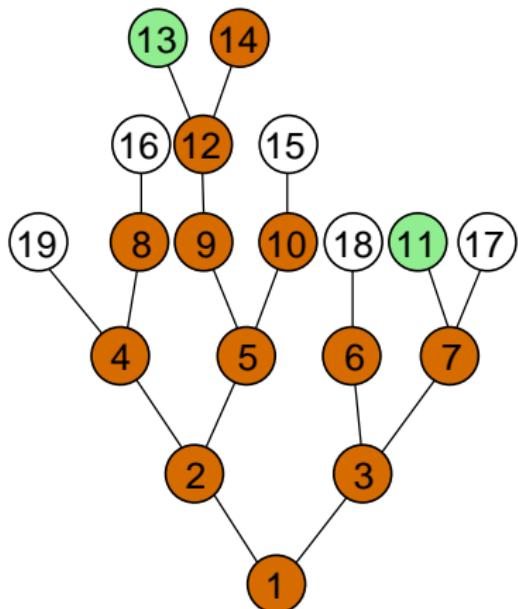


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Valid permutations on a unary-binary tree

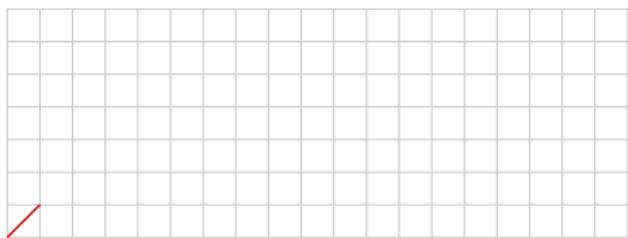
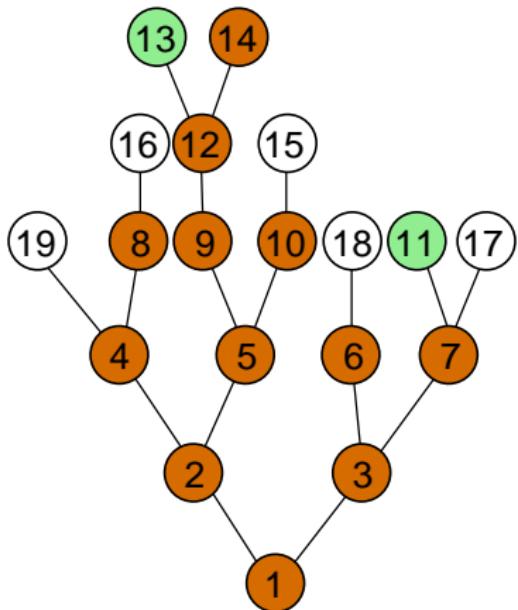


Lemma

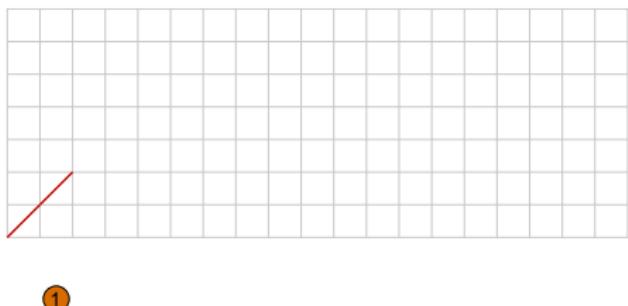
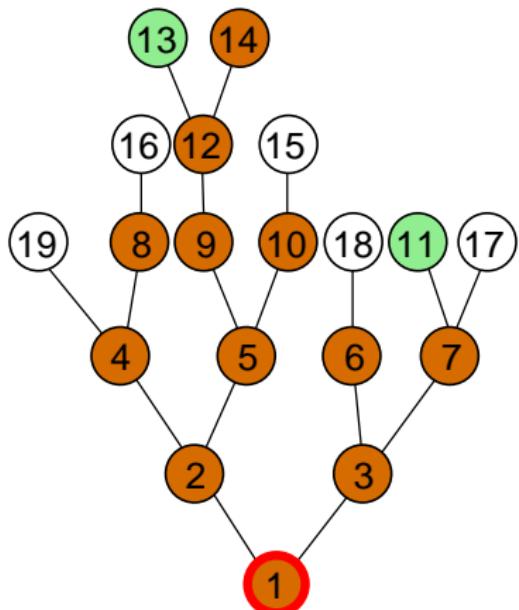
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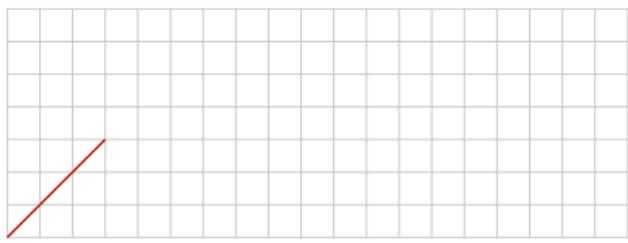
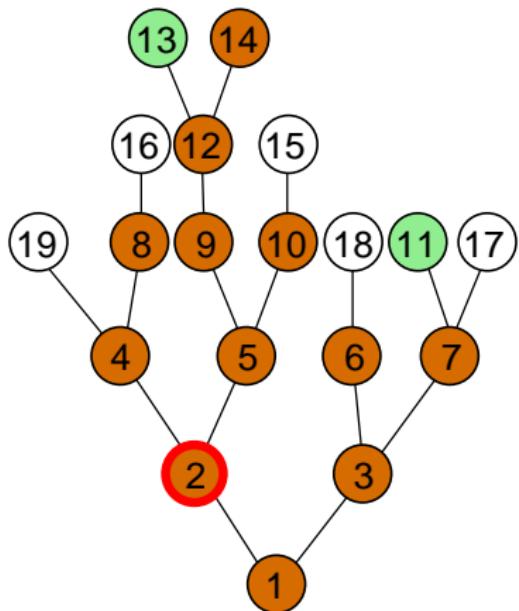
Encoding by decorated Motzkin paths



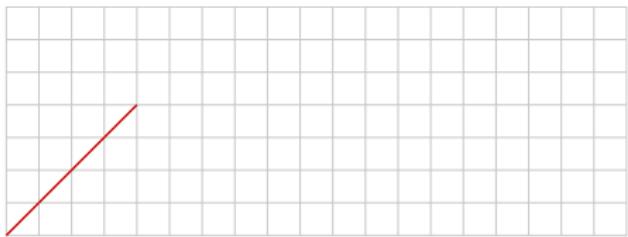
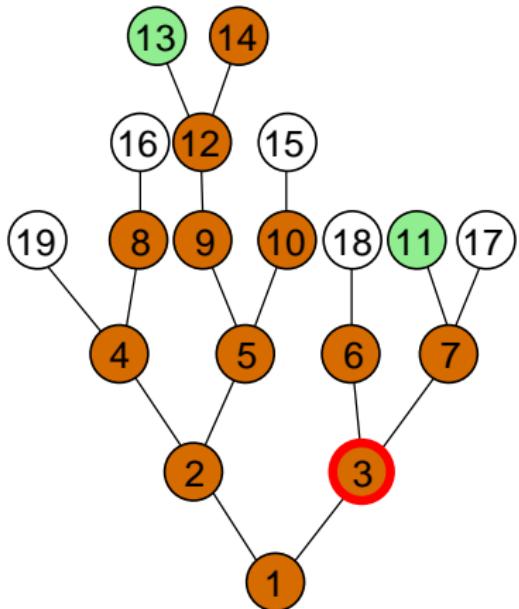
Encoding by decorated Motzkin paths



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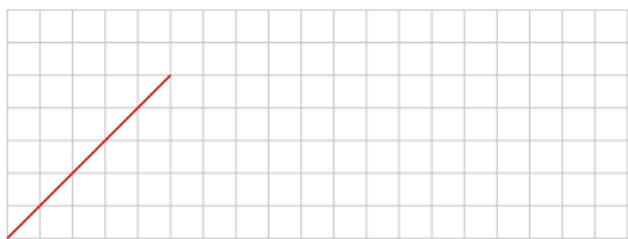
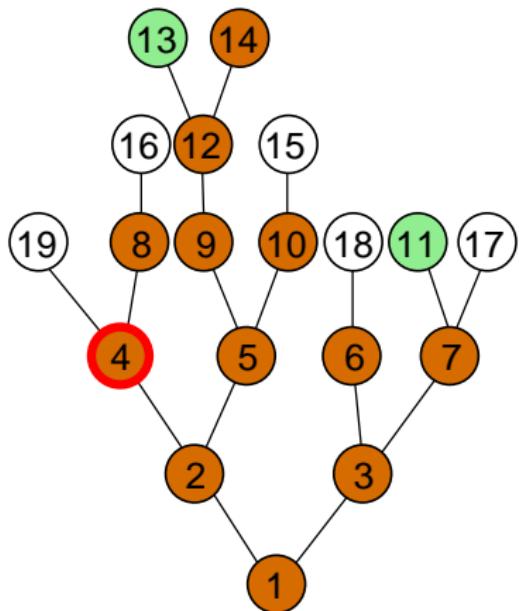


Encoding by decorated Motzkin paths

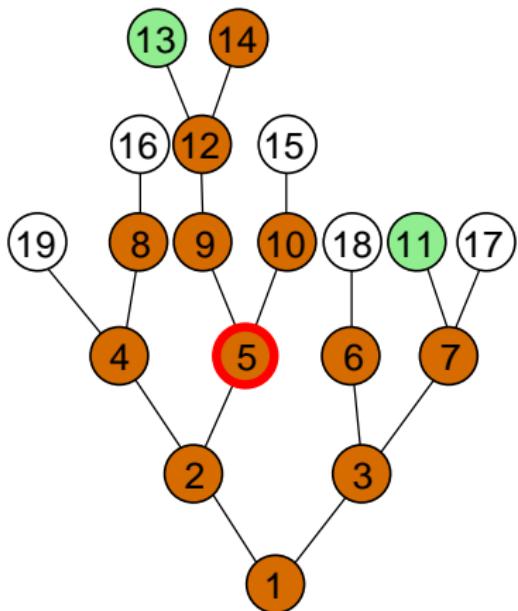


1 2 3

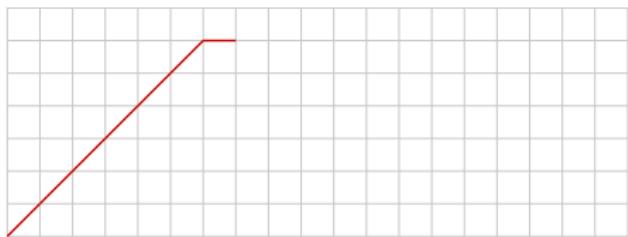
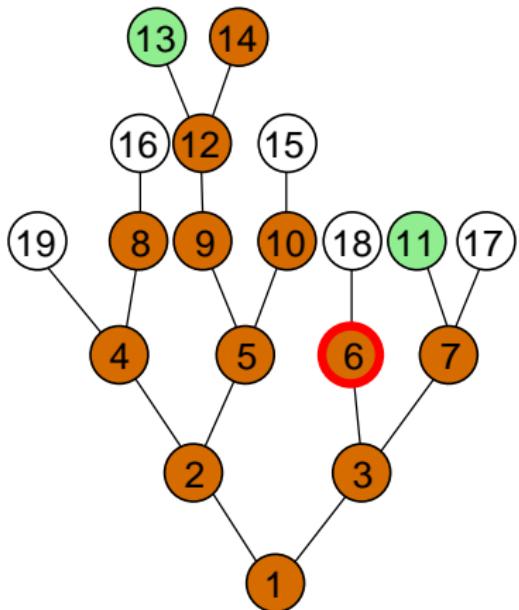
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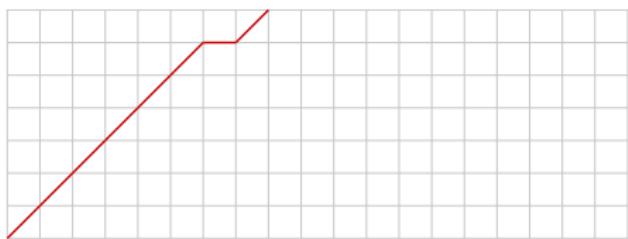
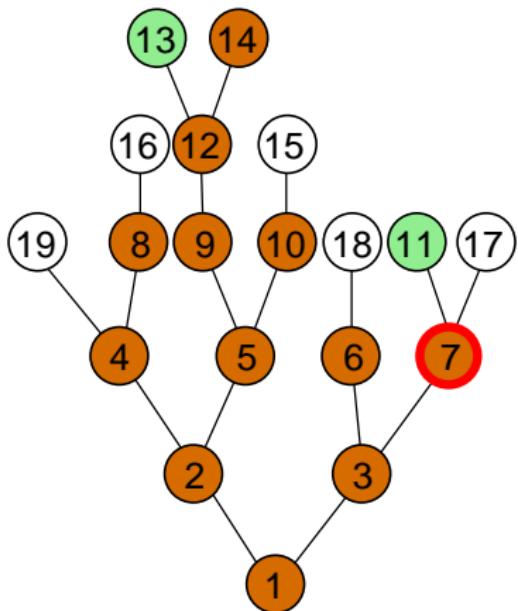
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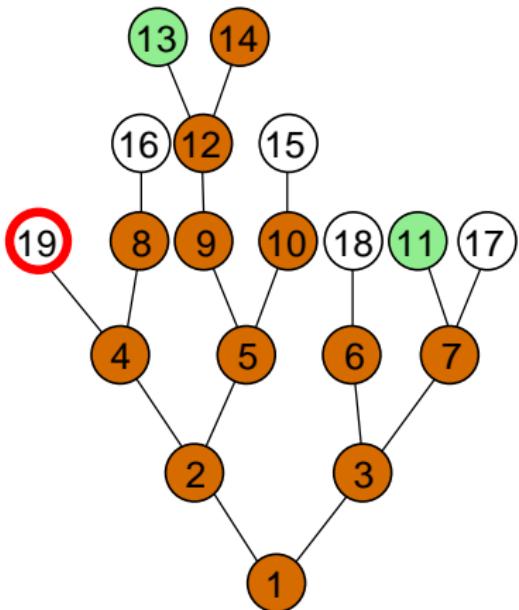


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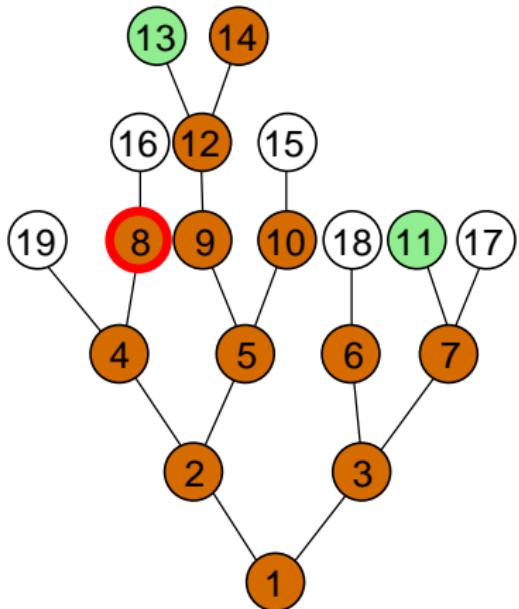
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Encoding by decorated Motzkin paths



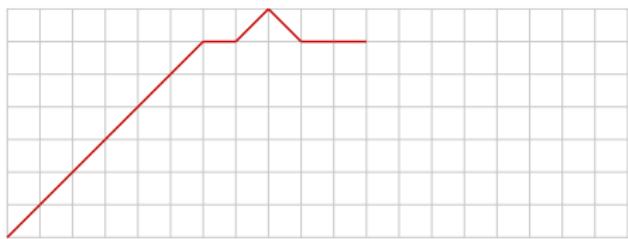
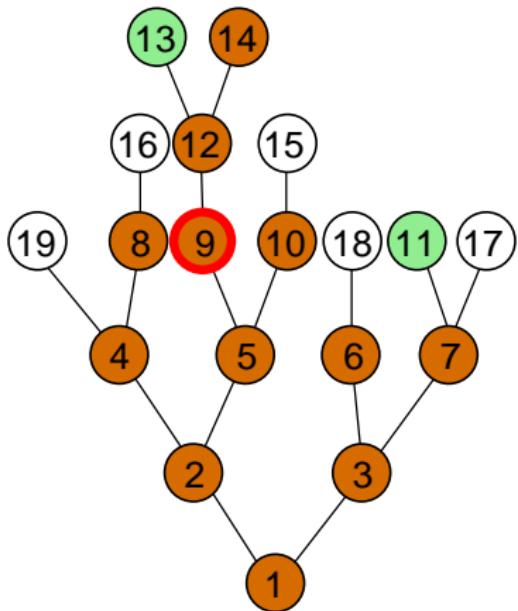
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Encoding by decorated Motzkin paths



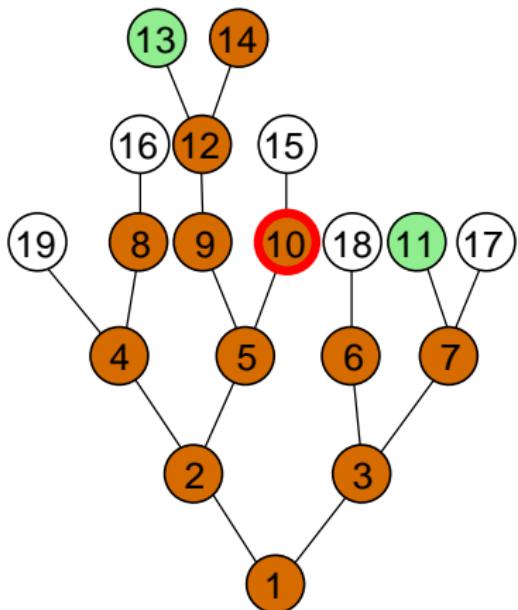
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Encoding by decorated Motzkin paths

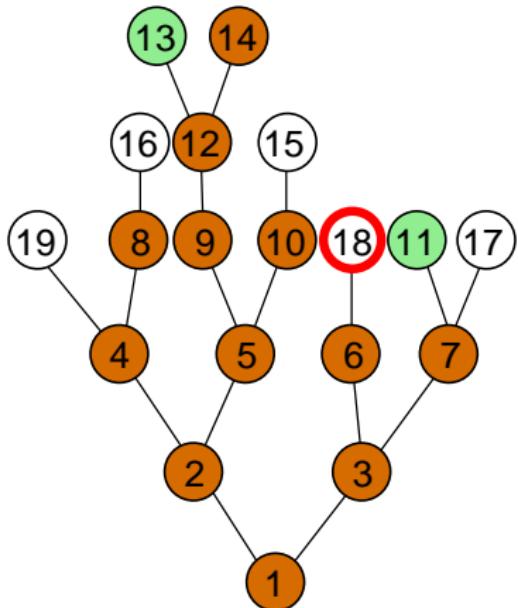


1 2 3 4 5 6 7 19 8 9

Encoding by decorated Motzkin paths

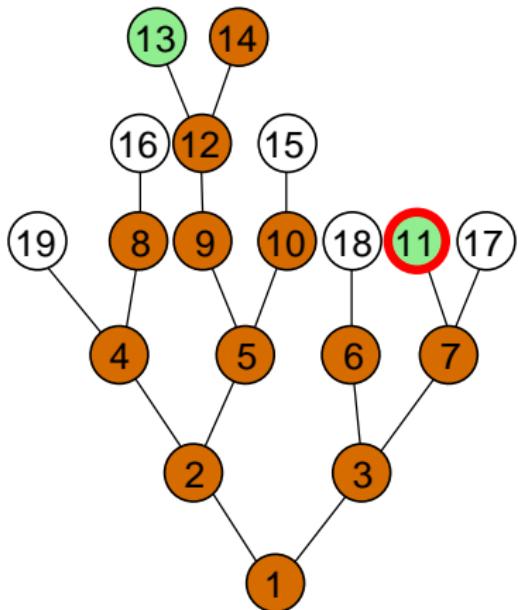


Encoding by decorated Motzkin paths



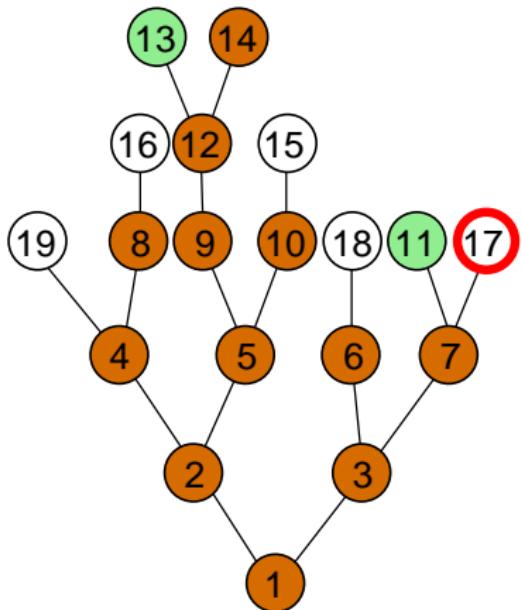
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Encoding by decorated Motzkin paths



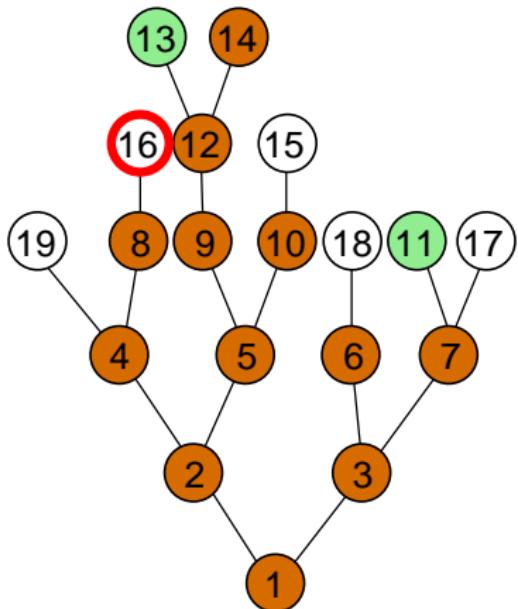
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Encoding by decorated Motzkin paths



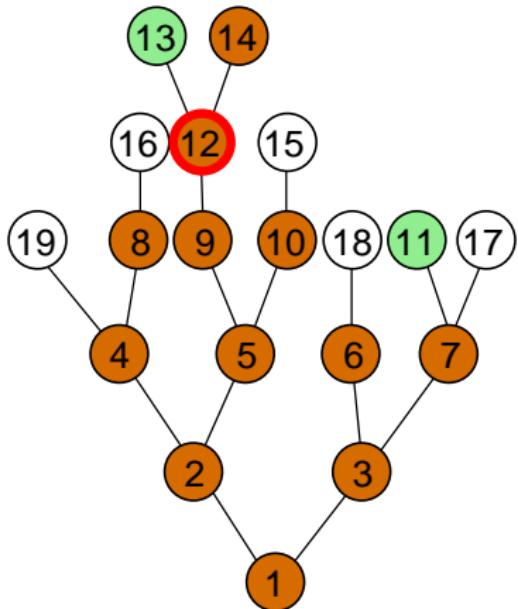
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Encoding by decorated Motzkin paths



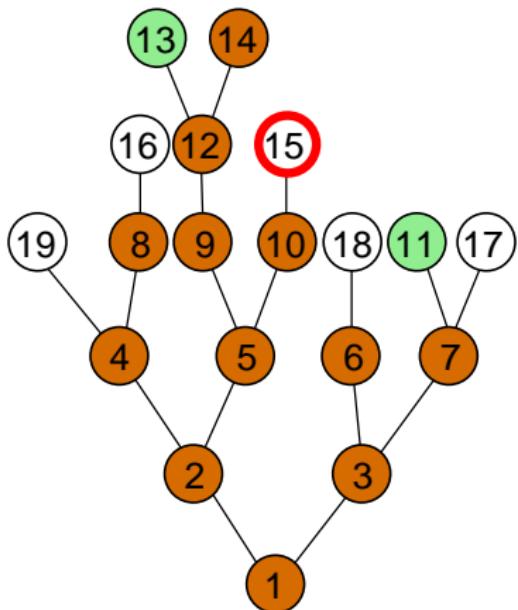
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Encoding by decorated Motzkin paths



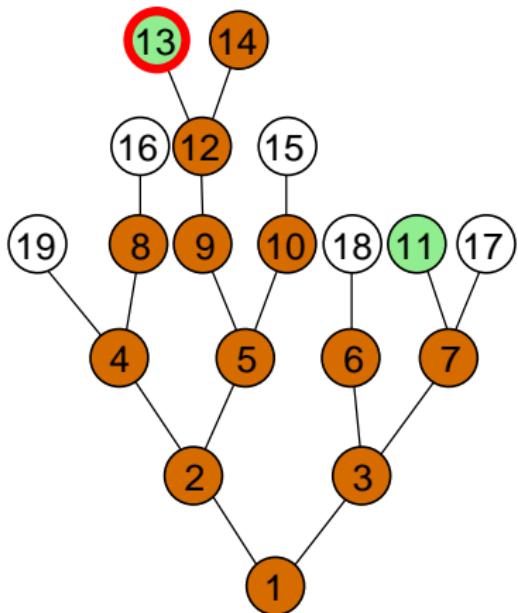
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Encoding by decorated Motzkin paths



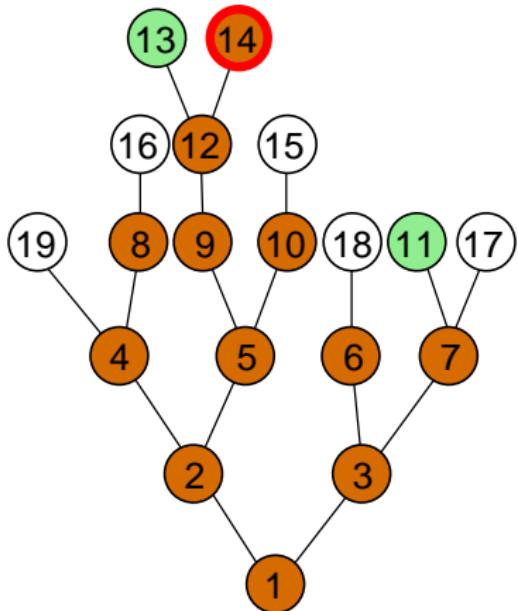
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Encoding by decorated Motzkin paths



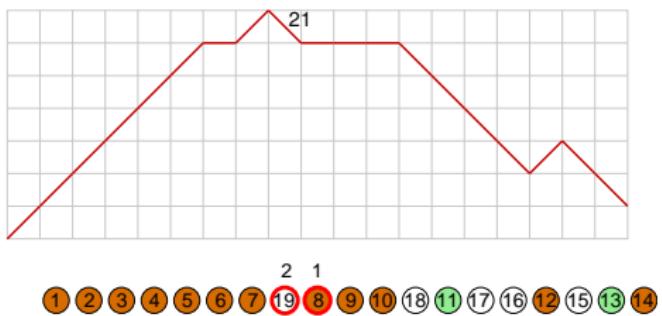
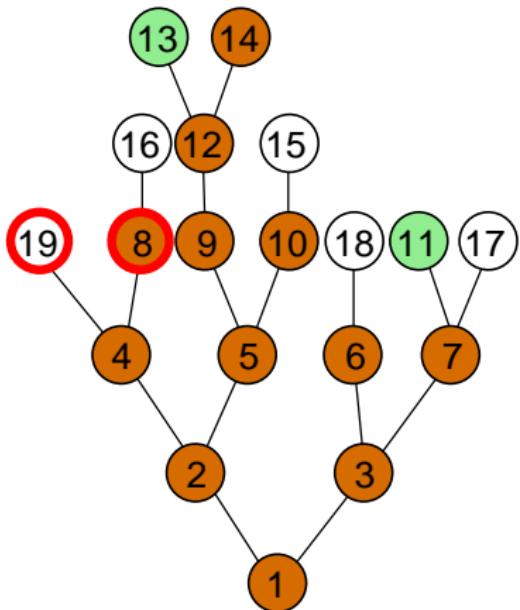
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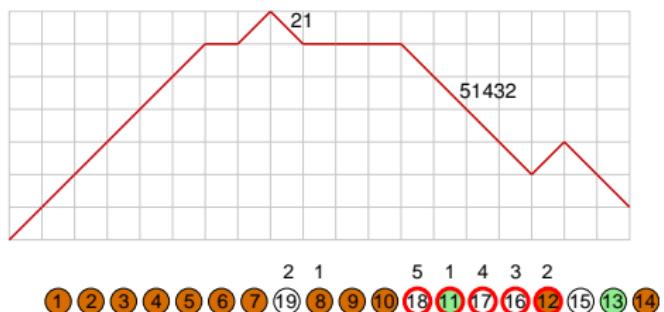
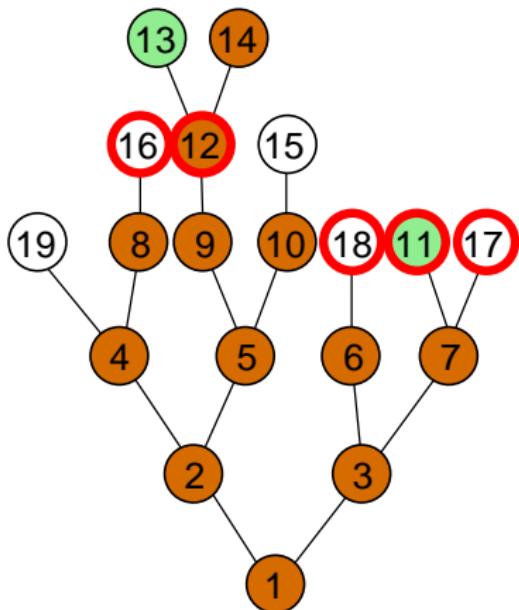


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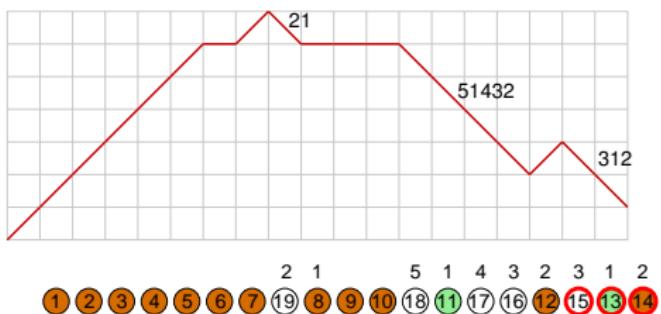
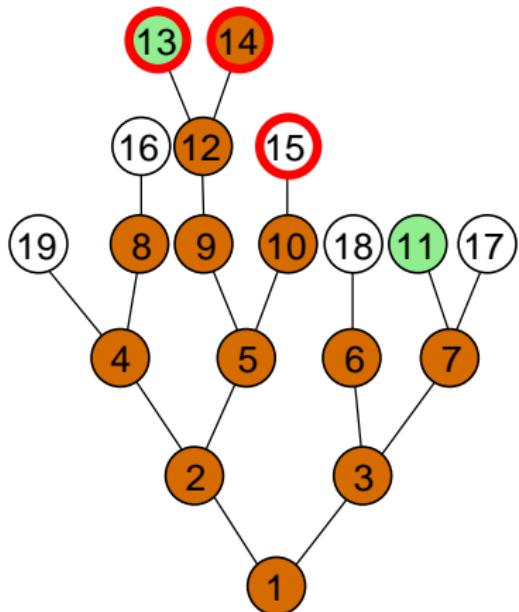
Encoding by decorated Motzkin paths



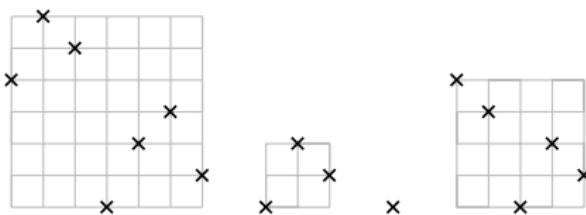
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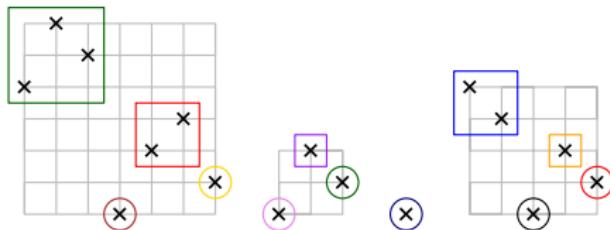
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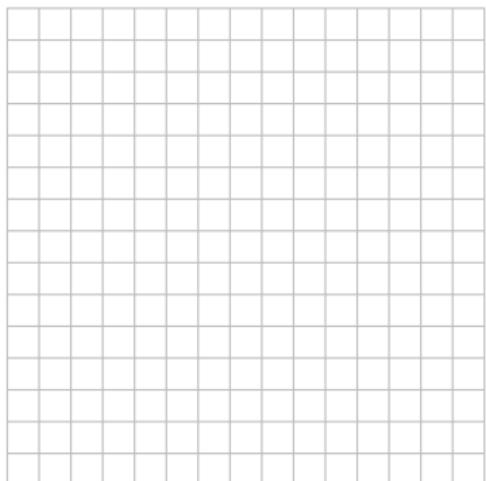
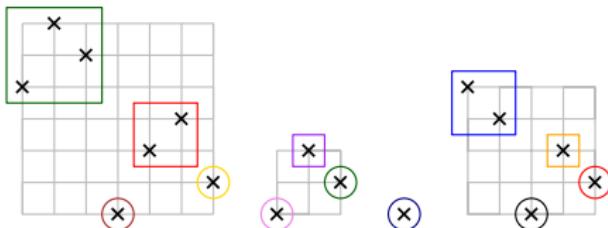
How to reconstruct the permutation



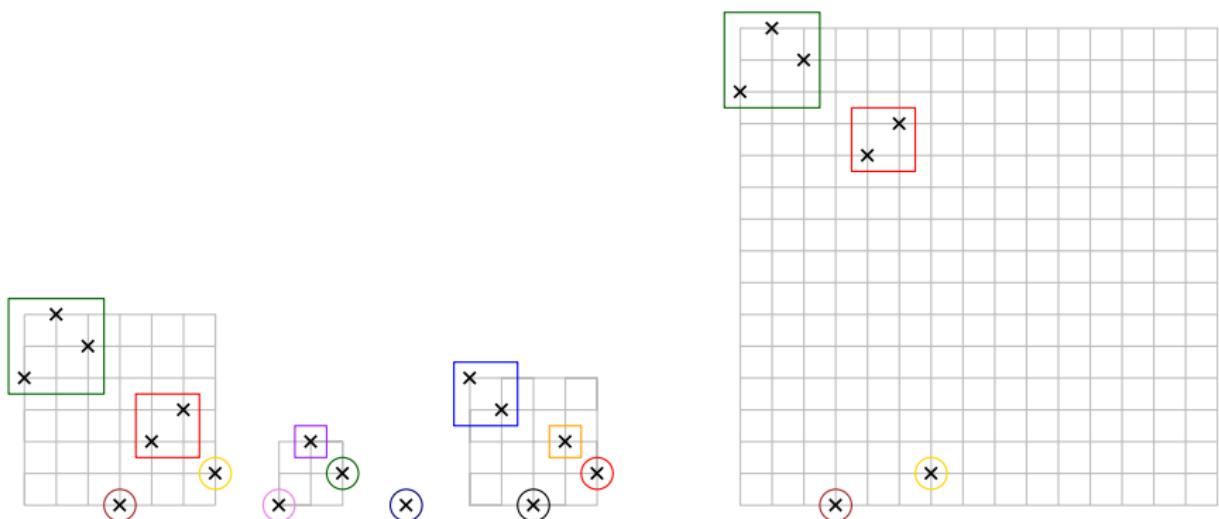
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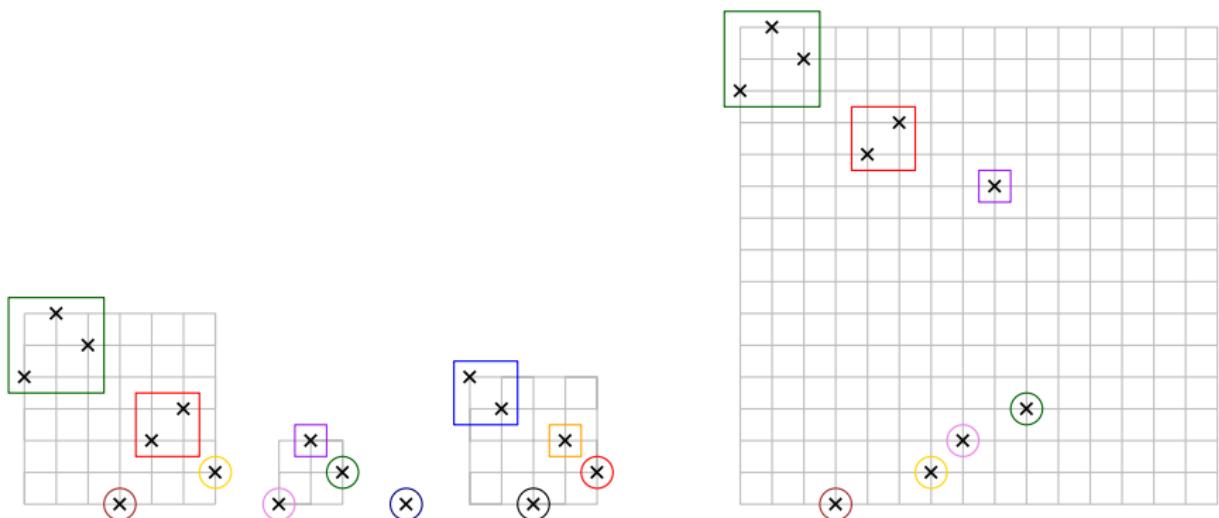
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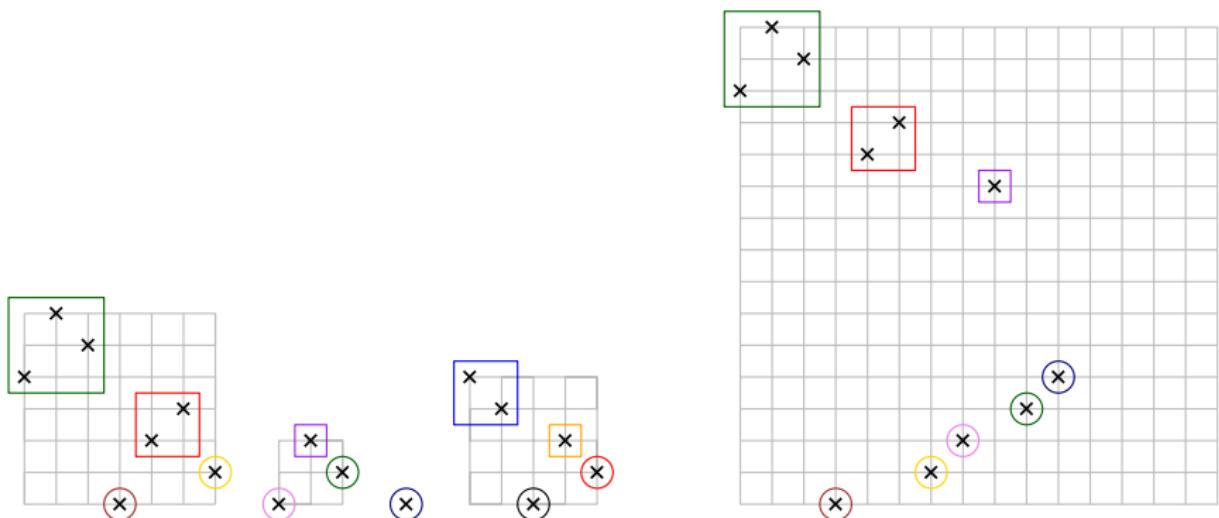
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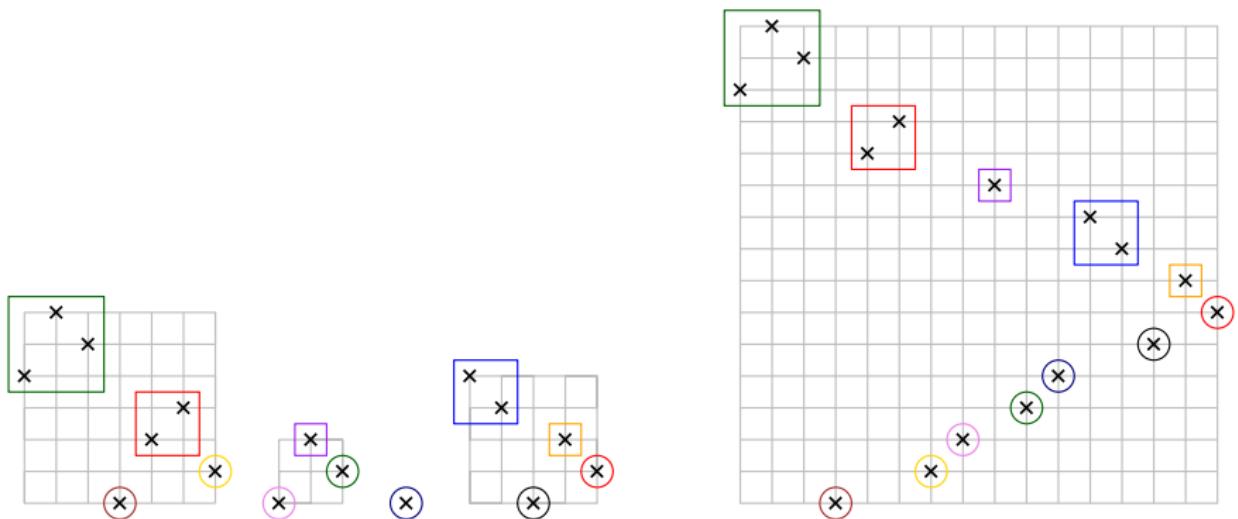
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Counting decorated Motzkin walks

\mathcal{T} : decorated Motzkin walks

Aim

We want to show that $\mathcal{T} = \mathcal{G}$.



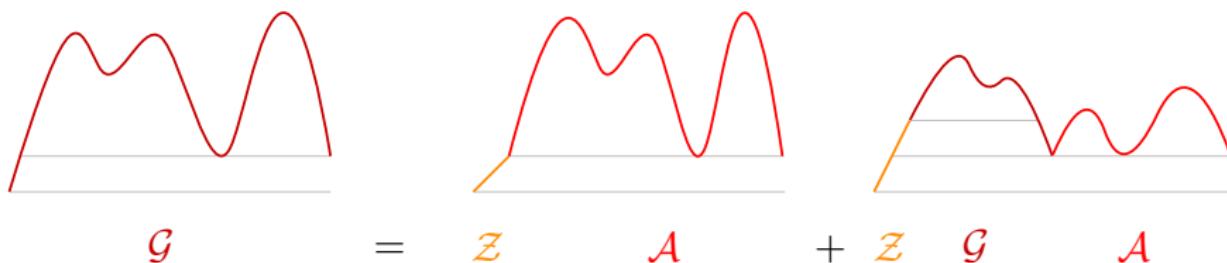
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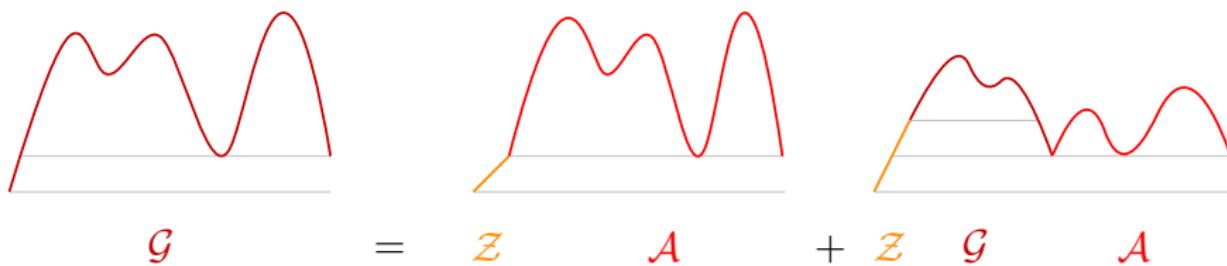


Counting decorated Motzkin walks

\mathcal{T} : decorated Motzkin walks

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$$\mathcal{G} = (\mathcal{Z}\mathcal{A}) + (\mathcal{Z}\mathcal{A})^2 + (\mathcal{Z}\mathcal{A})^3 + \dots = \text{Seq}_{\geq 1}(\mathcal{Z}\mathcal{A})$$

Step 1: $\mathcal{T} = \text{Seq}_{\geq 1}(\mathcal{M})$



\mathcal{M} : decorated Motzkin walks whose last 0-step or +1-step is a +1-step

Claim

$$\mathcal{T} = \text{Seq}_{\geq 1}(\mathcal{M})$$

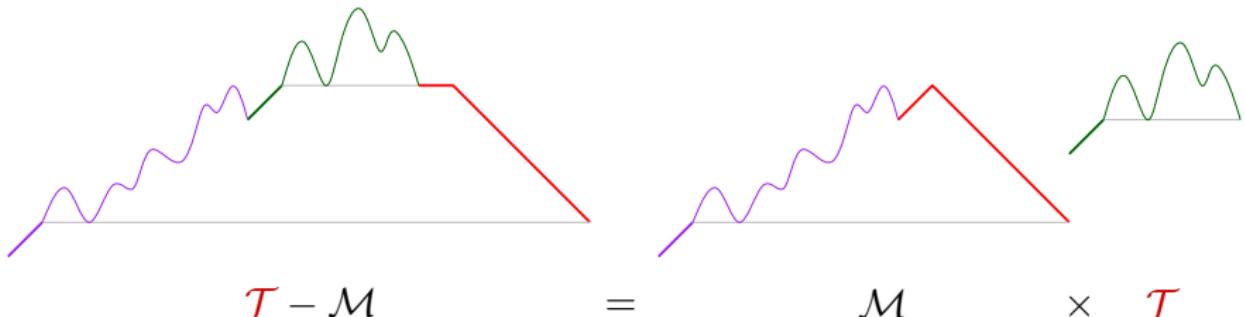
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Step 2: $\mathcal{M} = \mathcal{ZA}$

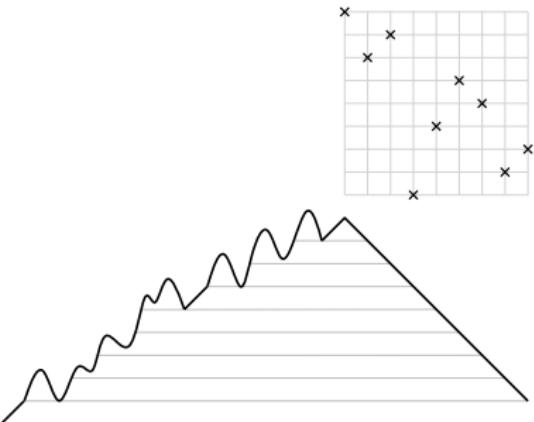
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$$zA = z \left(1 + (zA)^2 + \frac{(zA)^2}{(1 - zA)^2} \right)$$

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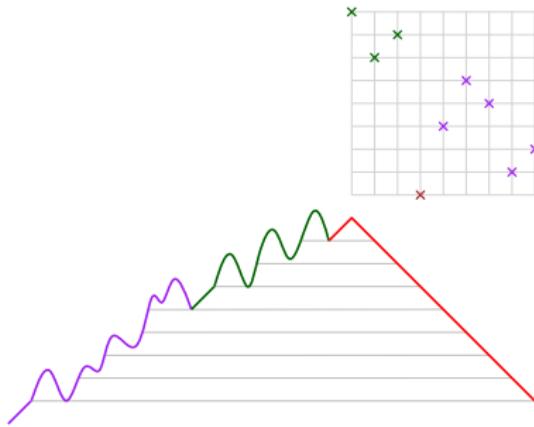
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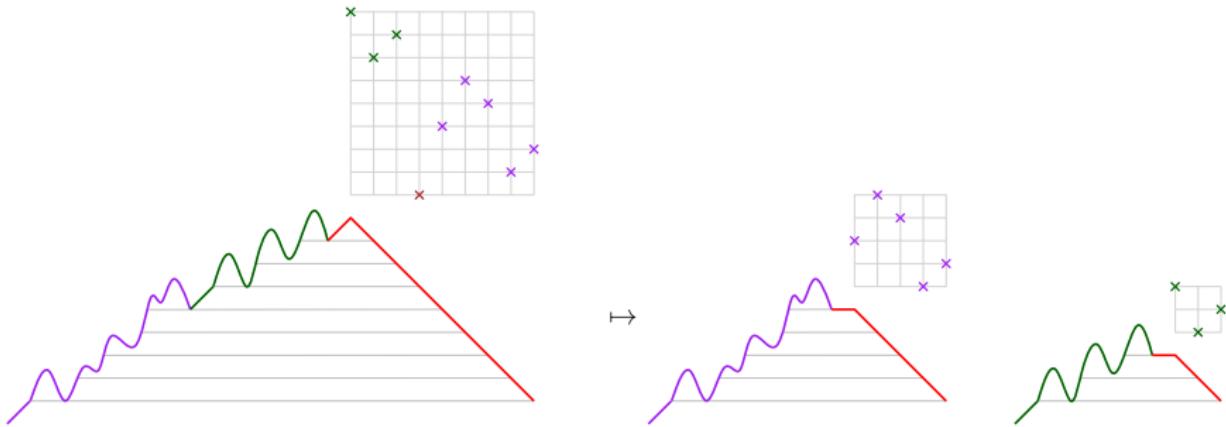




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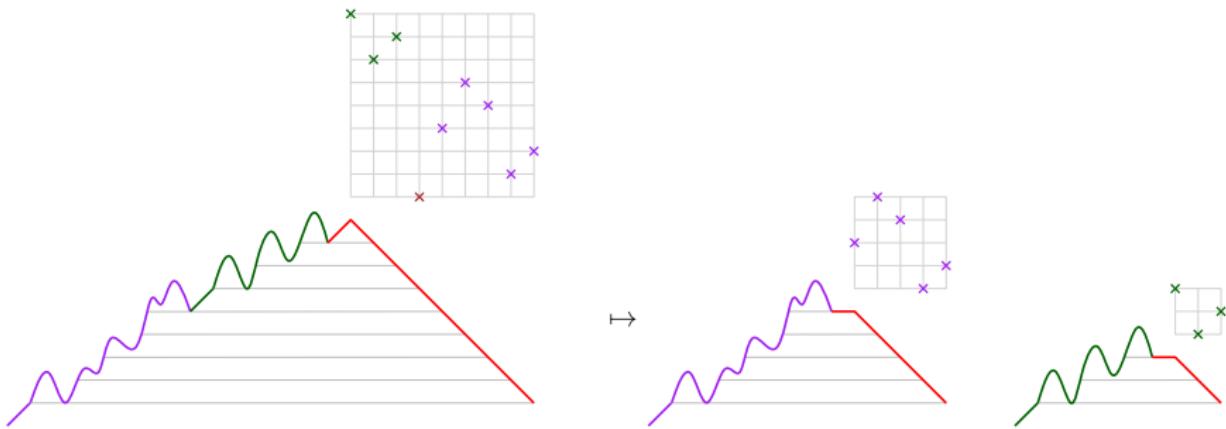
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$$M = z(1 + T - M)^2 = z \left(1 - M + \frac{M}{1 - M} \right)^2 = z \left(1 + M^2 + \frac{M^2}{(1 - M)^2} \right)$$

