Le lien entre Michael Jordan et Catalan

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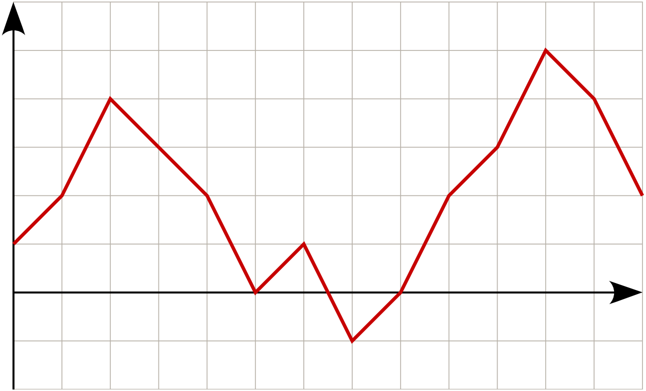
Basketball walks



Basketball walk: integer-valued walk with step-set $\{-2, -1, +1, +2\}$

Generating functions

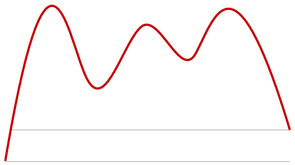
Theorem (Banderier & Krattenthaler & Krinik & Kruchinin & Kruchinin & Nguyen & Wallner '16)

The generating function G of basketball walks from 0 to 1 that are positive except at the origin, counted with weight z per step is given by

$$G(z) = -\frac{1}{2} + \frac{1}{2}\sqrt{\frac{2-3z-2\sqrt{1-4z}}{z}}.$$

$\mathcal{G} := \{$basketball walks from 0 to 1 that are positive except at the origin$\}$

$|\mathfrak{w}|$: number of steps of $\mathfrak{w}$



$$G(z) := \sum_{\mathfrak{w}\in\mathcal{G}} z^{|\mathfrak{w}|} = \sum_{n=0}^{\infty} \left|\{\mathfrak{w}\in\mathcal{G} : |\mathfrak{w}| = n\}\right| z^n$$

Catalan is everywhere!

The previous authors observed that

$$1 + G(z) + G^2(z) = \mathrm{Cat}(z) \tag{1}$$

where Cat is the Catalan generating function.

$\mathrm{Cat}(z) = \sum_{n=0}^{\infty} c_n z^n$ where $c_n := \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number

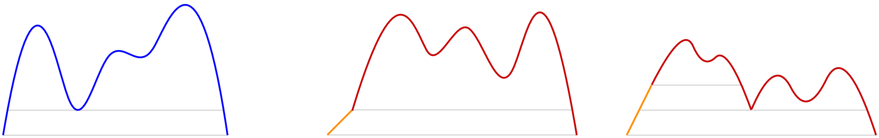
1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012

n-edge rooted trees, $n+1$-leaf binary rooted trees, $2n$-step Dyck walks, well-parenthesized words with n pairs of parentheses, rooted triangulations of the $n+2$-gon, noncrossing partitions of the n-set, etc.

C-walks

C-walk: basketball walk from 0 to 0 that visits 1 and is positive except at the extremities

$$\mathcal{C} := \{C\text{-walks}\} \qquad C(z) := \sum_{\mathfrak{w}\in\mathcal{C}} z^{|\mathfrak{w}|}$$



$$\mathcal{C} = \mathcal{Z}\,\mathcal{G} + \mathcal{Z}\,\mathcal{G}\,\mathcal{G}$$

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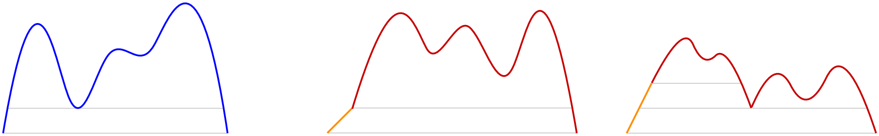


$$\mathcal{C} = \mathcal{Z}\ \mathcal{G} + \mathcal{Z}\ \mathcal{G}\ \mathcal{G}$$

$$C(z) = zG(z) + zG^2(z)$$

Equation (1) becomes $C(z) = z\left(\mathrm{Cat}(z) - 1\right)$, which is the generating function of nontrivial binary trees counted with weight z per leaf.

Refined enumeration

even step: step starting at even height
odd step: step starting at odd height

Proposition

The number of C-walks with $2d$ ± 1-steps, ℓ odd $+2$-steps or even -2-steps, and r odd -2-steps or even $+2$-steps is equal to

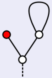
$$\frac{1}{d}\binom{2d-2}{d-1}\binom{\ell+r+2d-2}{\ell+r}\binom{\ell+r}{\ell}.$$

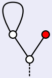
Matched statistics

Proposition

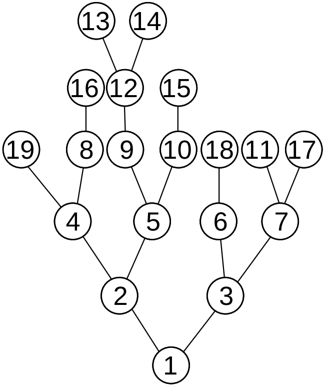
n-step *C*-walk $\longleftrightarrow$ *n*-leaf binary tree

± 1-steps  or  $\longleftrightarrow$  double leaves

odd $+2$ / *even* -2  or  $\longleftrightarrow$  left leaves

even $+2$ / *odd* -2  or  $\longleftrightarrow$  right leaves

Increasing unary-binary tree



increasing unary-binary tree of size n: plane tree with n vertices labeled 1, 2, . . . , n such that each vertex has at most 2 children, and all have larger labels

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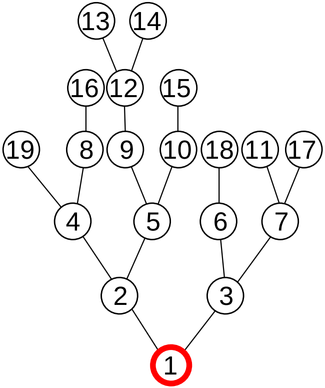

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We associate with it the permutation obtained by reading the labels of the tree in breadth-first search order

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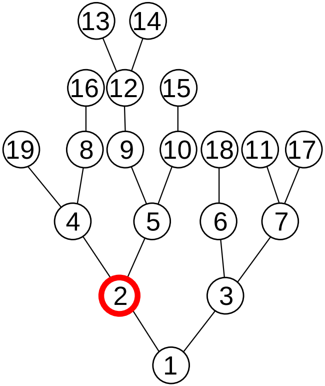

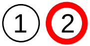

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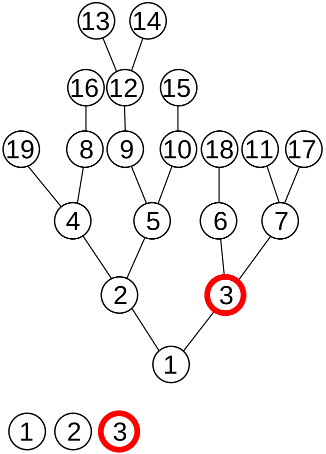

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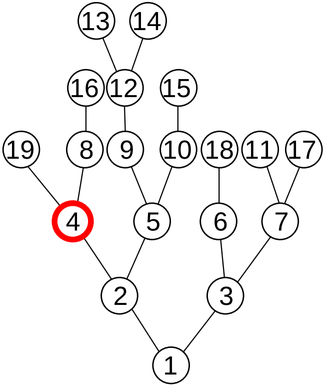

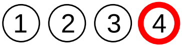

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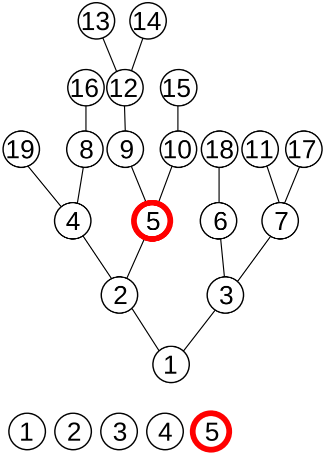

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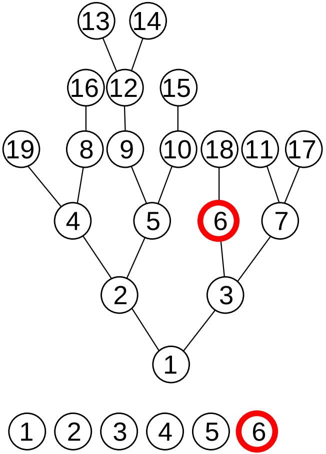

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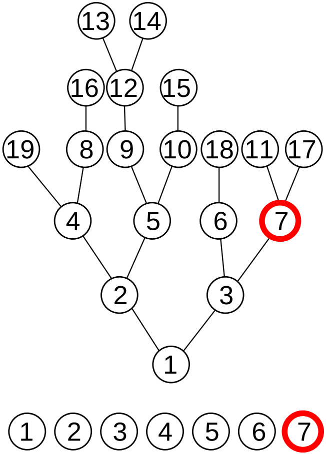

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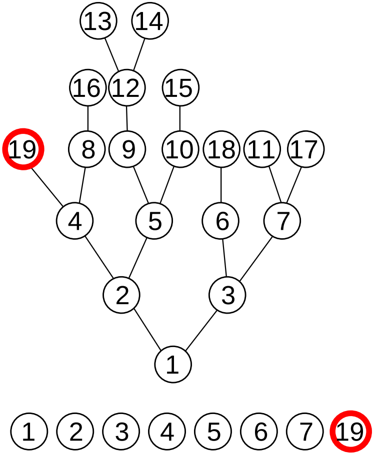

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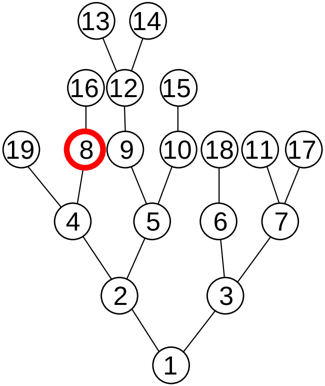

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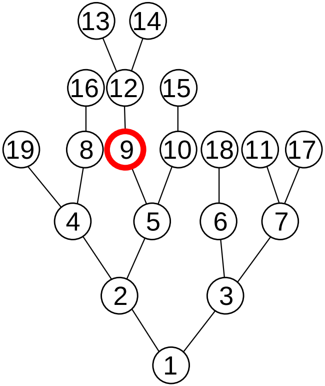

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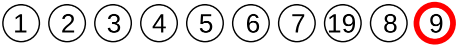

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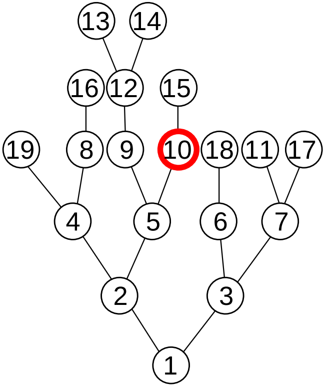

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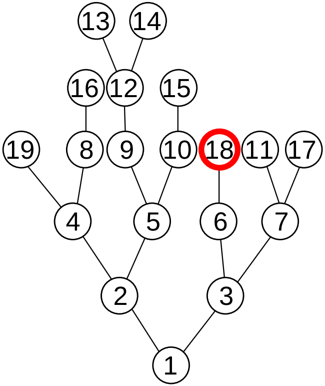

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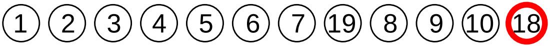

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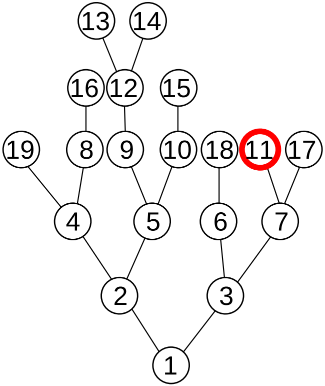

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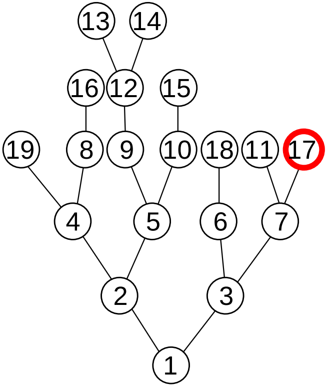

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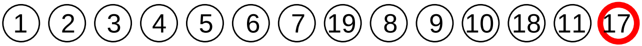

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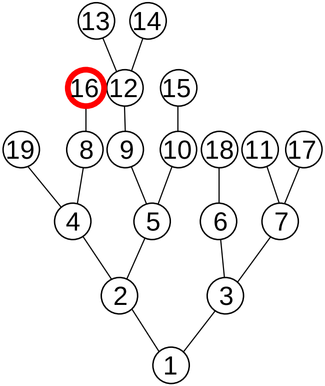

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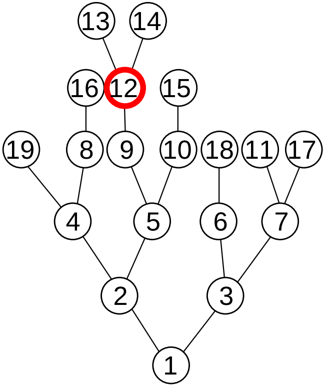

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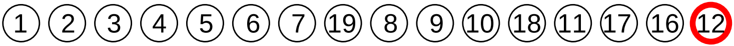

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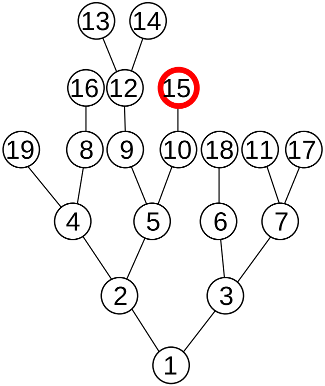

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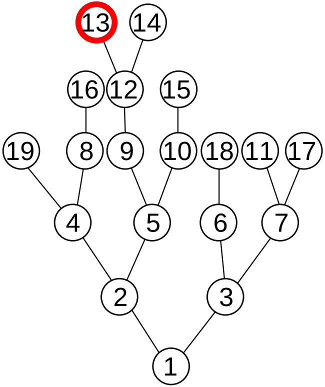

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1 2 3 4 5 6 7 19 8 9 10 18 11 17 16 12 15 13

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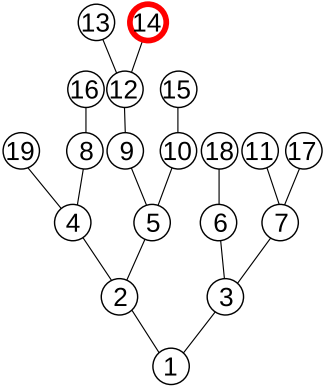

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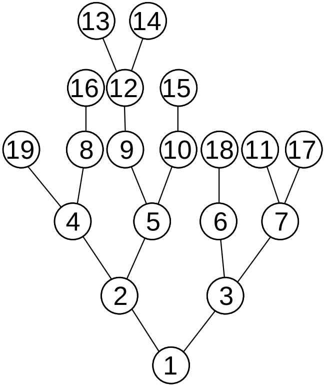

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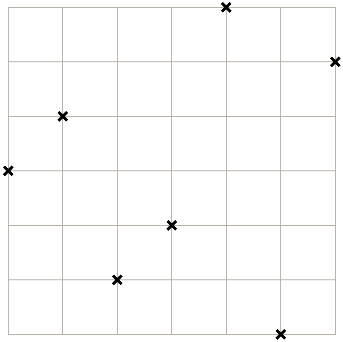
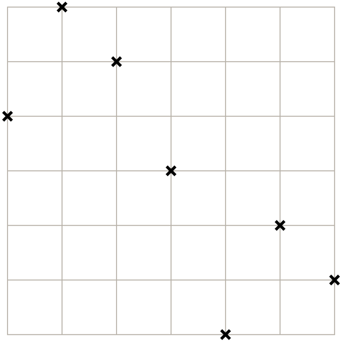
Permutation avoiding 213





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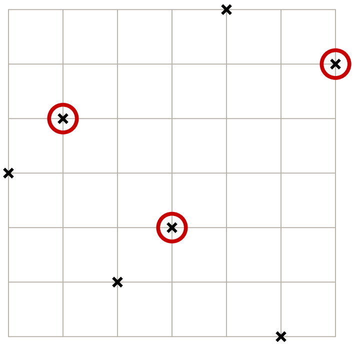
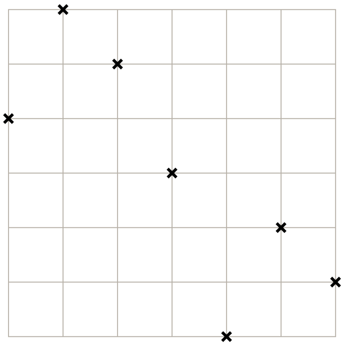


contains 213



avoids 213

Counting IUBTs

IUBT: increasing unary-binary tree with associated permutation avoiding 213

Theorem

IUBTs are counted by G-walks (basketball walks from 0 to 1 that are positive except at the origin).

Proposition

For $n \geq 1$ and $0 \leq k \leq \lfloor (n-1)/2 \rfloor$, the number of n-vertex IUBTs with exactly $n-1-2k$ unary nodes is

$$\frac{1}{n}\binom{2n}{k}\binom{n-k}{k+1}.$$

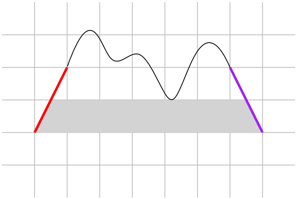
Matched statistics

IUBT: increasing unary-binary tree with associated permutation avoiding 213

Proposition

n-step G-walk $\longleftrightarrow$ *n-vertex IUBT*

staggered ± 2*-steps* $\longleftrightarrow$  *unary nodes*



The red and purple steps are paired.

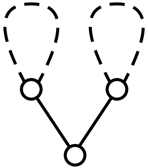
A staggered ± 2-step is a ± 2-step that is not paired with any other ± 2-step.

Decomposition of binary trees

$\mathcal{N}$: class of nontrivial binary trees counted by number of leaves

Goal

Understand bijectively that $\mathcal{C} = \mathcal{N}$



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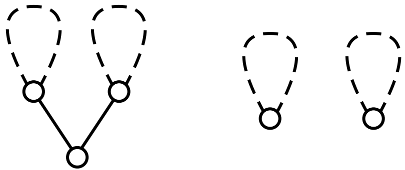

$\mathcal{N} \quad = \quad \times$

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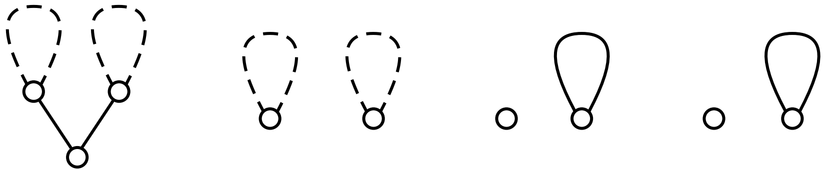
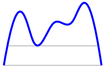


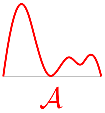
$$\mathcal{N} \quad = \quad \times \quad = (\mathcal{Z} + \mathcal{N}) \times (\mathcal{Z} + \mathcal{N})$$

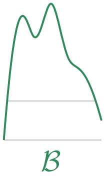
Elementary decomposition of basketball walks

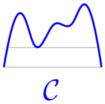


$\mathcal{C}$

Elementary decomposition of basketball walks







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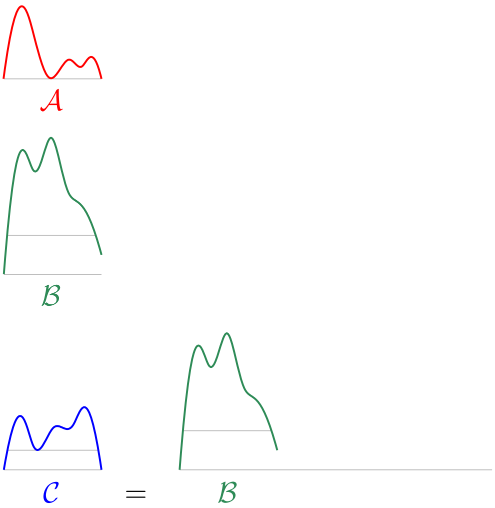

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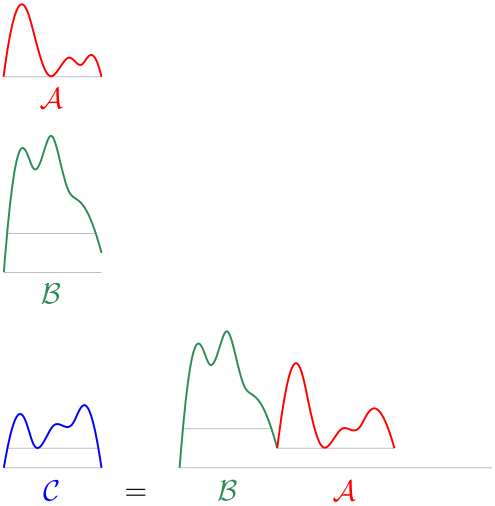

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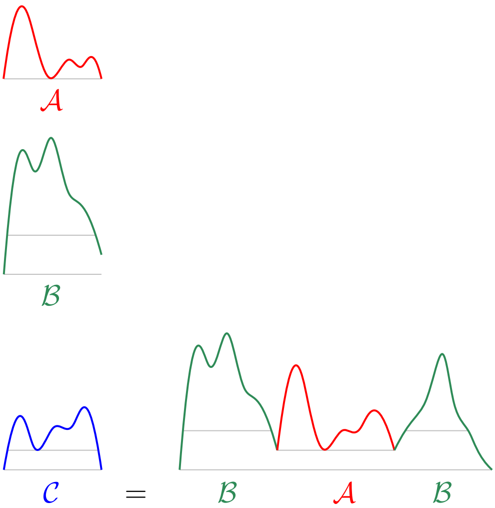

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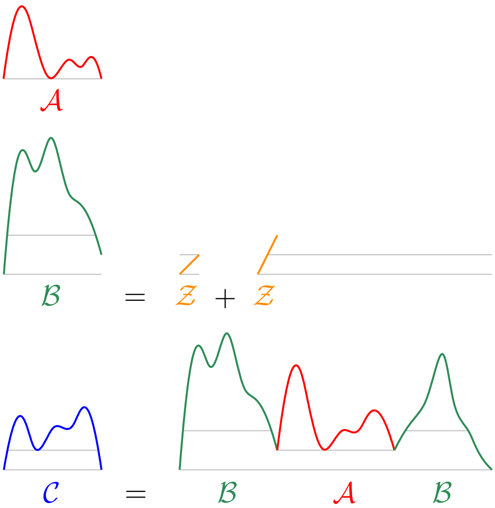
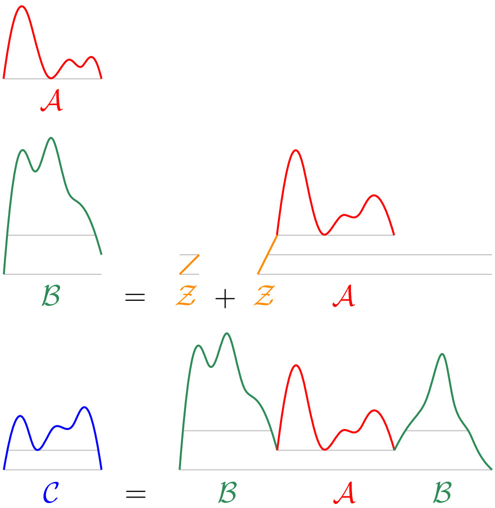
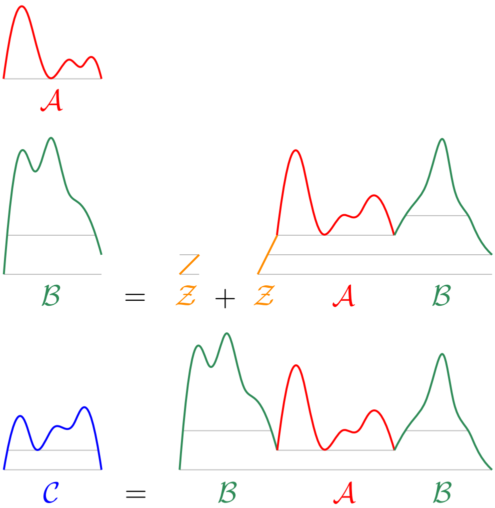


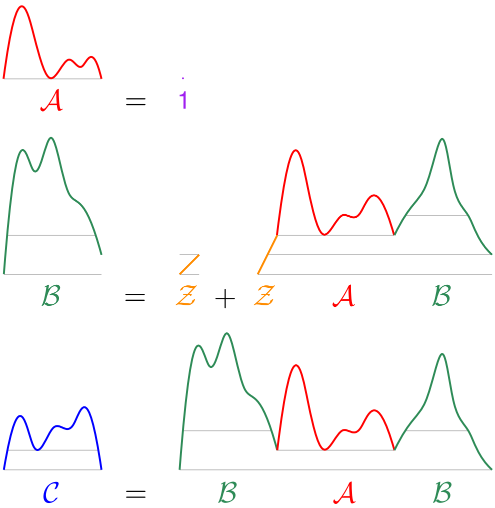
Elementary decomposition of basketball walks



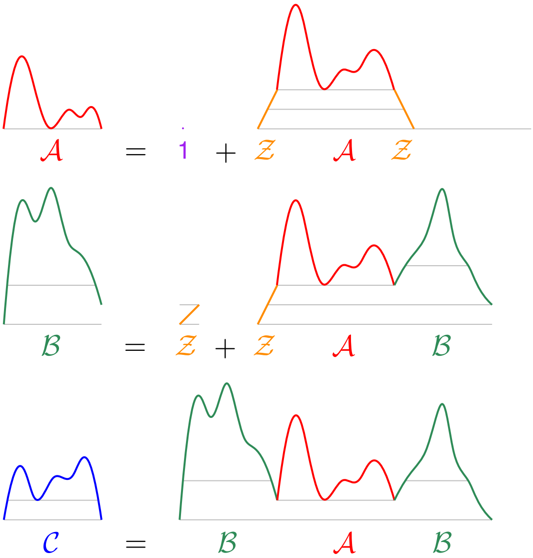
Elementary decomposition of basketball walks



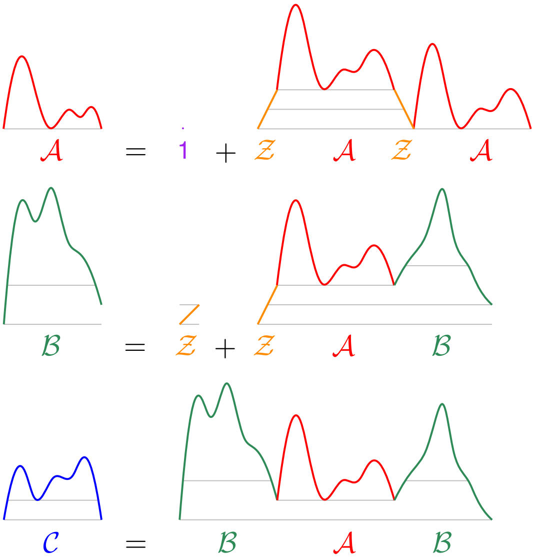
Elementary decomposition of basketball walks



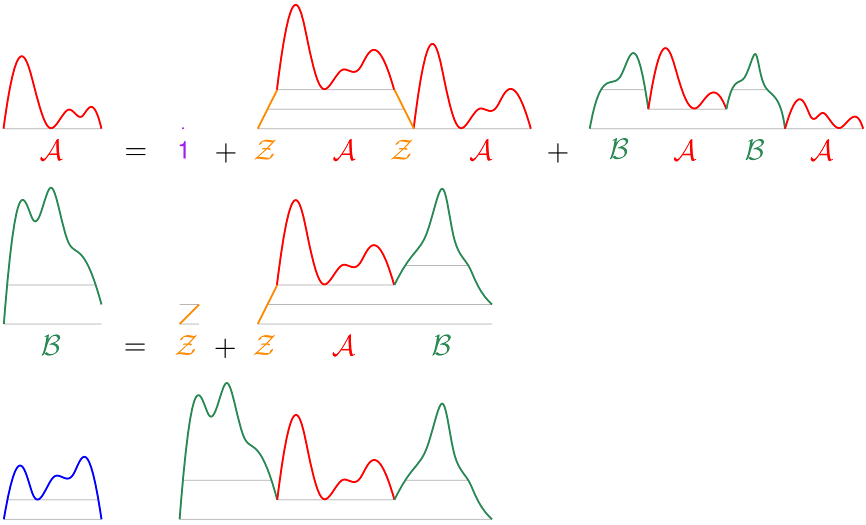
Elementary decomposition of basketball walks



Elementary decomposition of basketball walks



Elementary decomposition of basketball walks

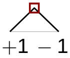


$$\mathcal{A} = 1 + \mathcal{Z}\,\mathcal{A}\,\mathcal{Z}\,\mathcal{A} + \mathcal{B}\,\mathcal{A}\,\mathcal{B}\,\mathcal{A}$$

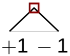
$$\mathcal{B} = \mathcal{Z} + \mathcal{Z}\,\mathcal{A}\,\mathcal{B}$$

$$\mathcal{C} = \mathcal{B}\,\mathcal{A}\,\mathcal{B}$$

The bijection

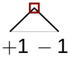


The bijection



$$\overset{\Phi}{\longmapsto} \quad (\varepsilon, \varepsilon)$$

The bijection



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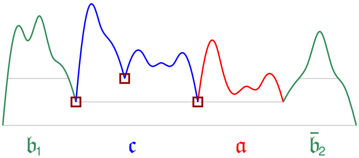
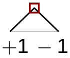


The bijection



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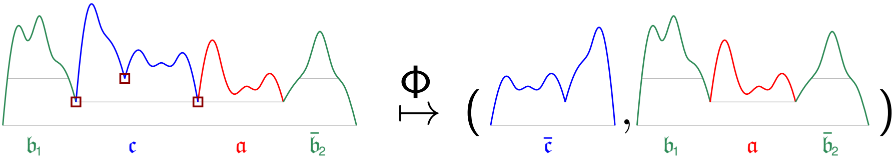

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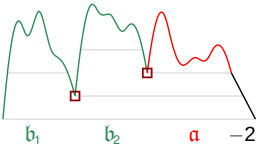

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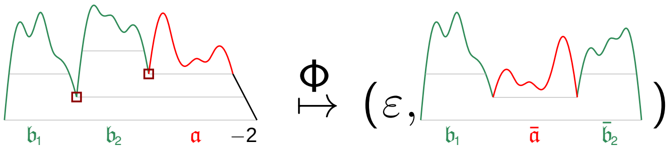

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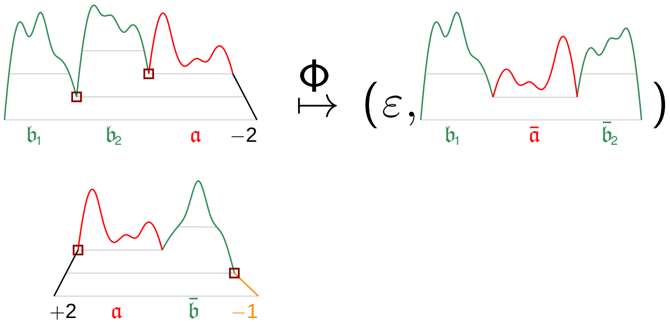

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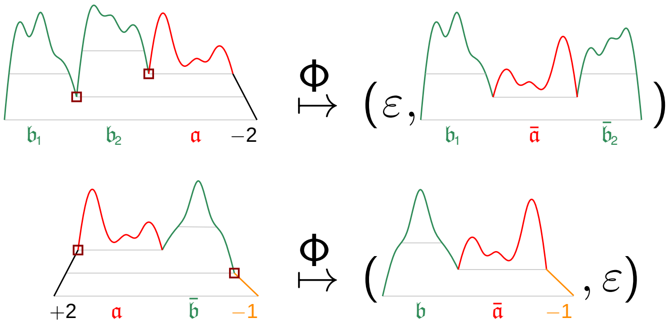

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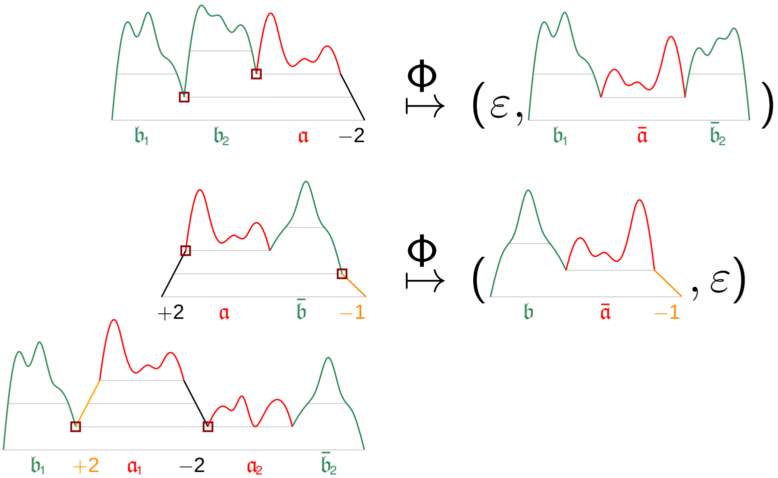

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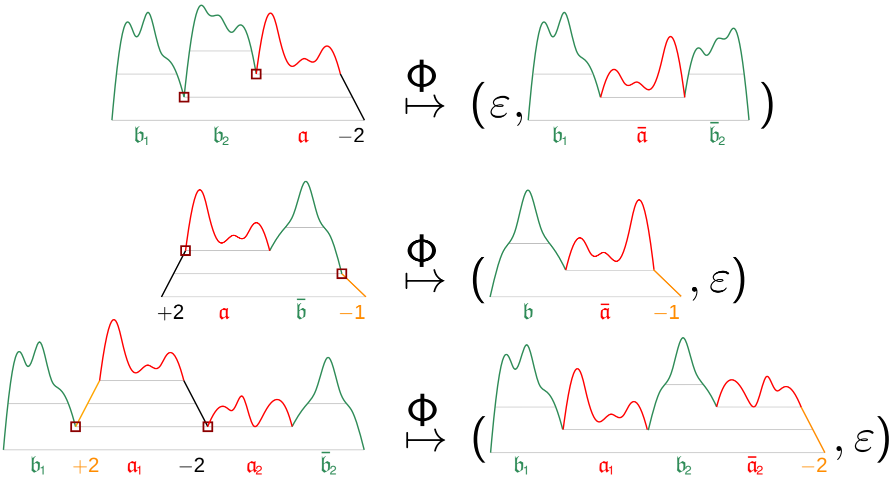

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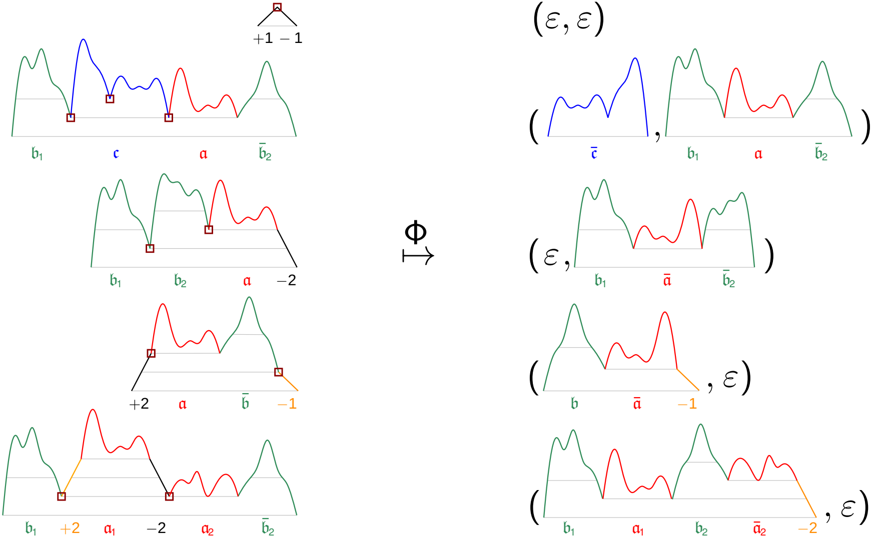

The bijection

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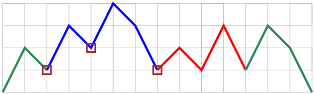

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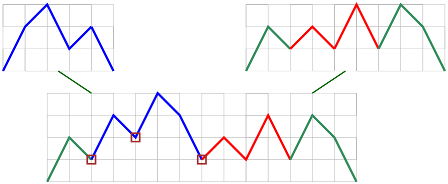

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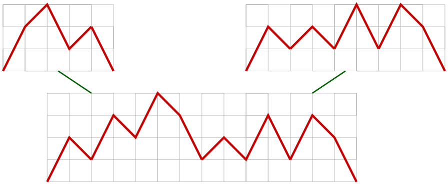

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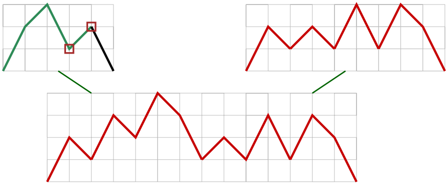

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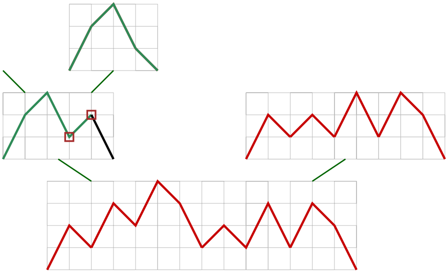

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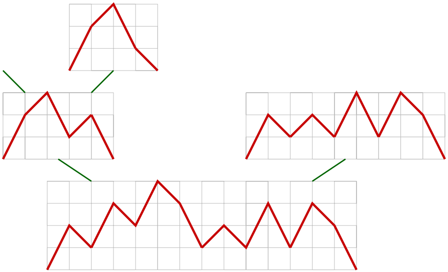

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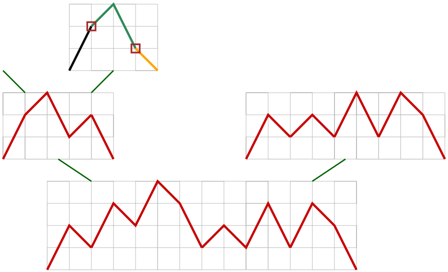

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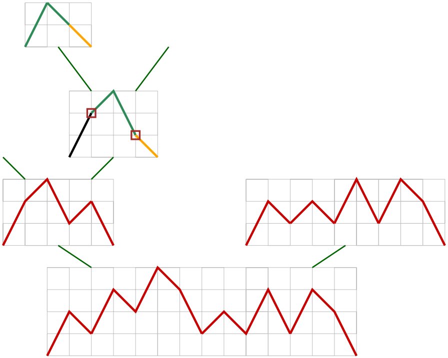

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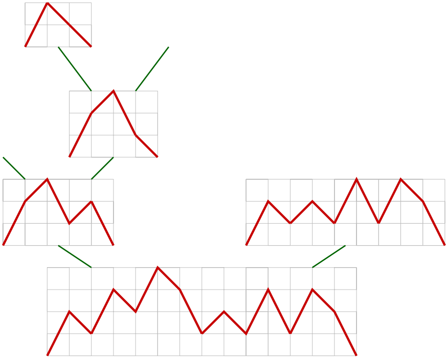

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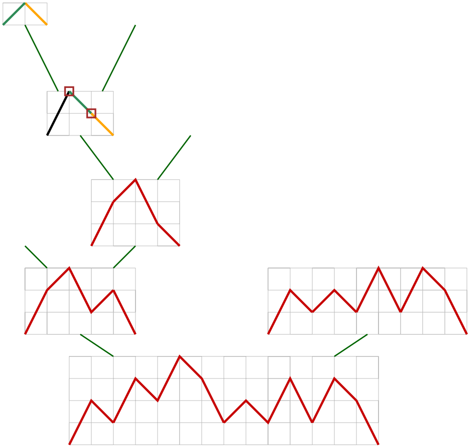

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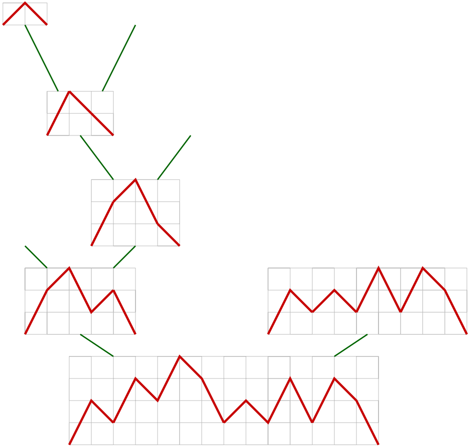

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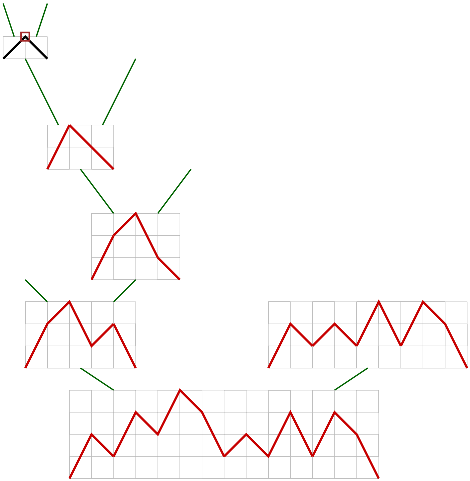

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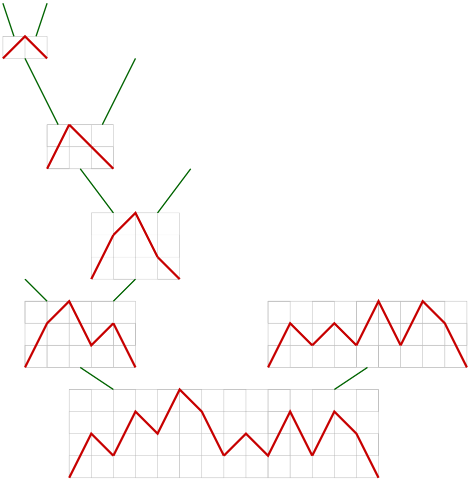

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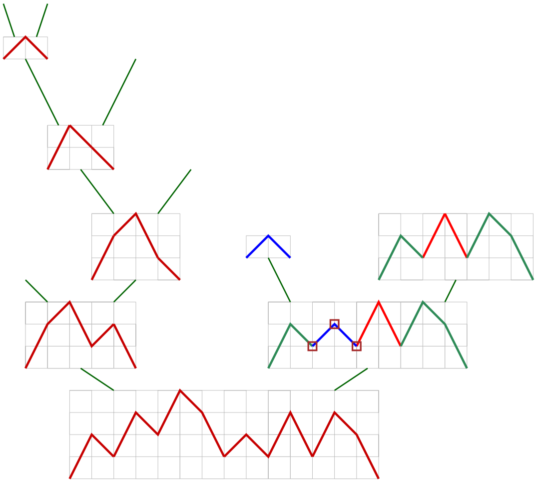

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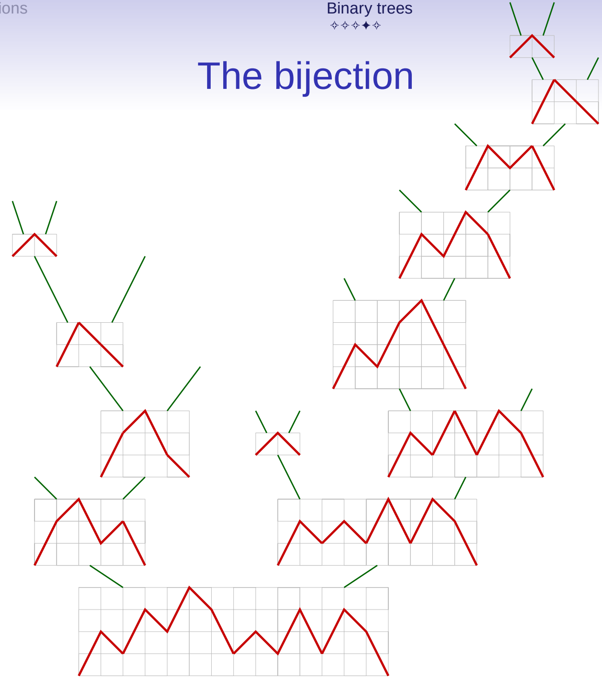
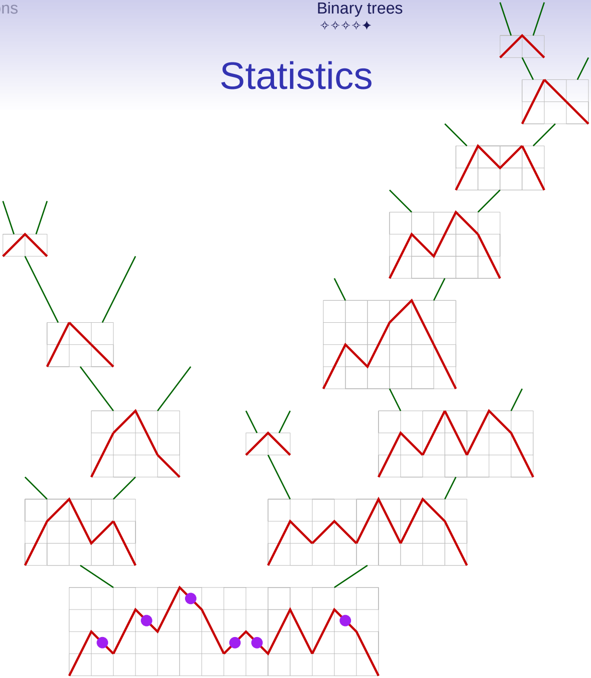
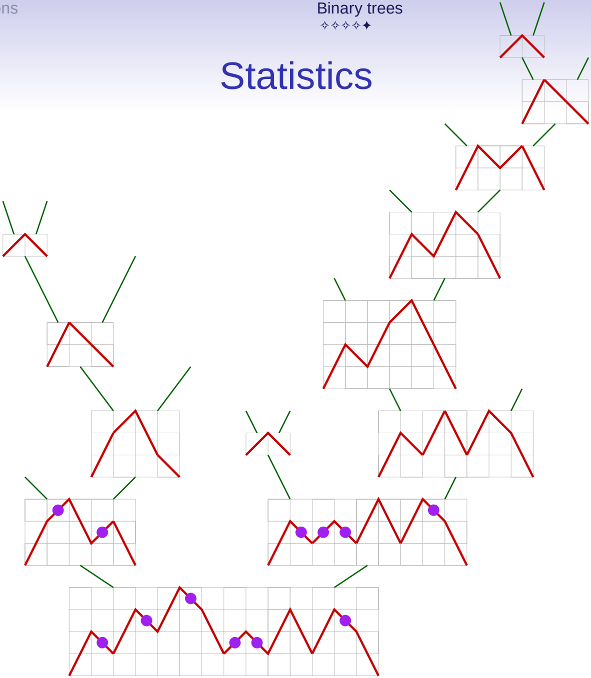


Statistics



Statistics



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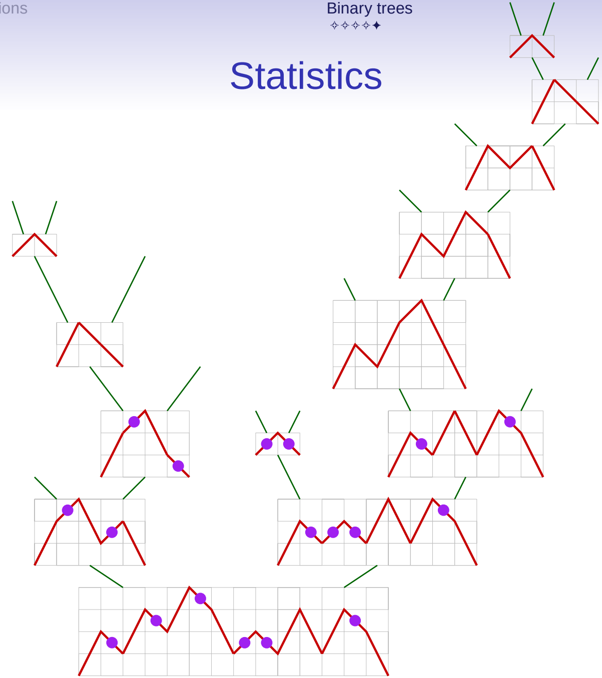

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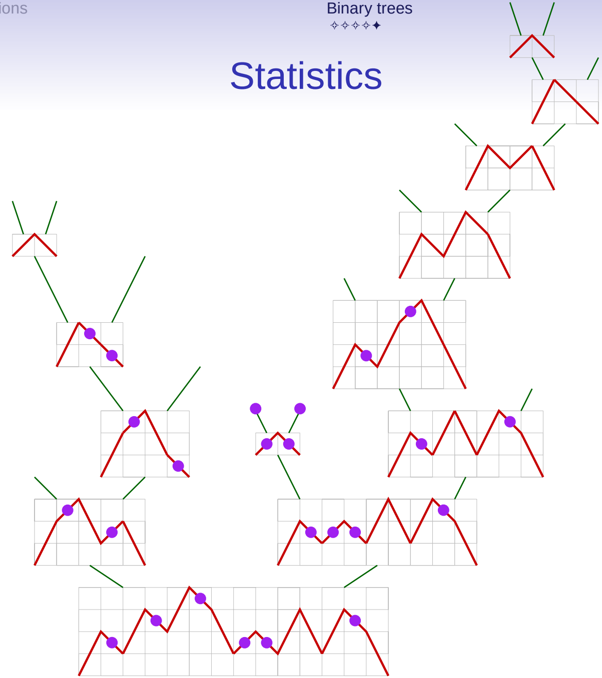

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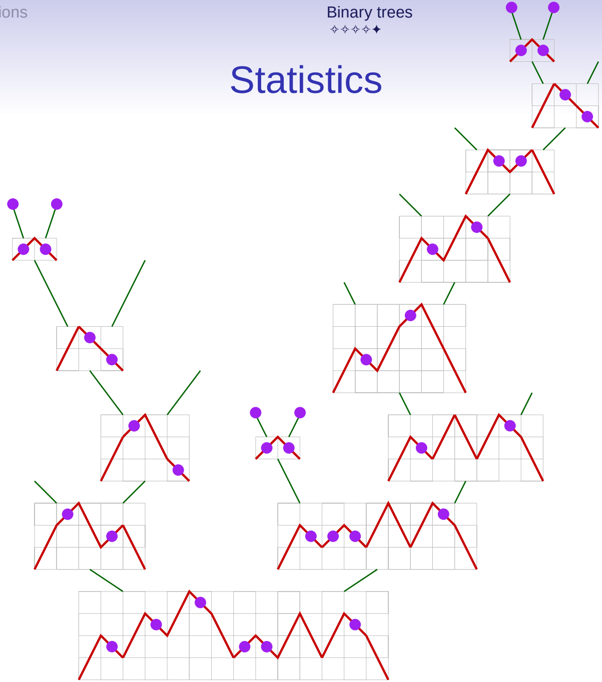

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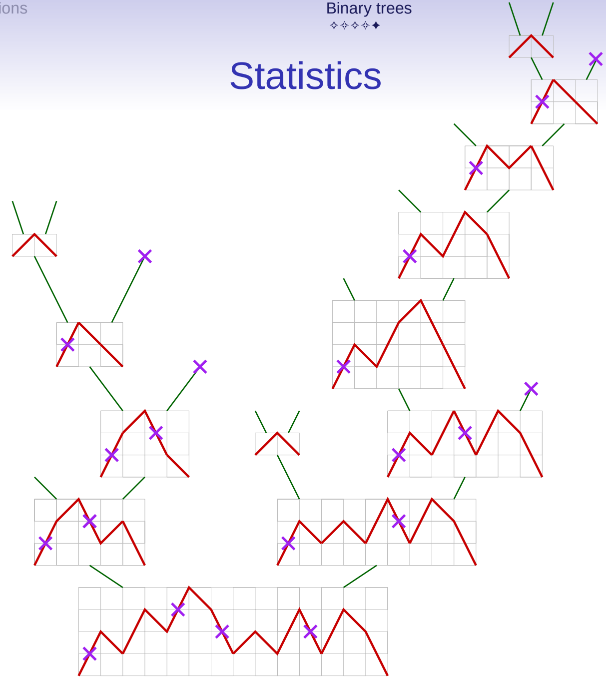
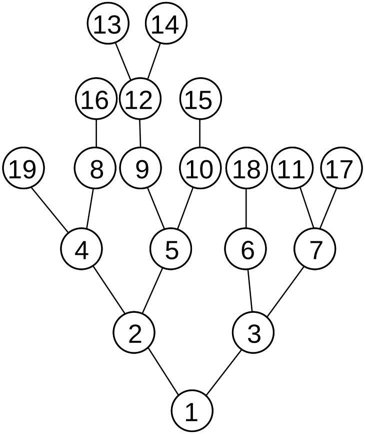


Valid permutations on a unary-binary tree

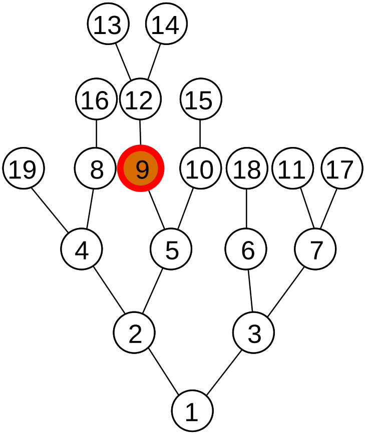


Lemma

A permutation is valid for a tree if and only if it avoids 213 and, the value taken at each node is a right-to-left minimum.

1 2 3 4 5 6 7 19 8 9 10 18 11 17 16 12 15 13 14

Valid permutations on a unary-binary tree



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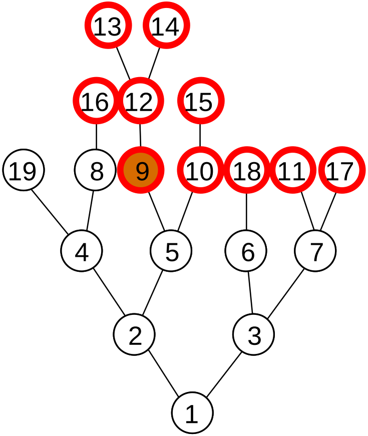
Valid permutations on a unary-binary tree



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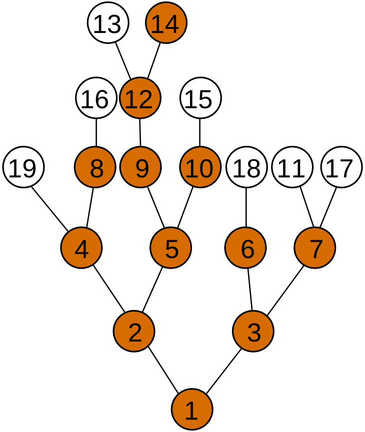

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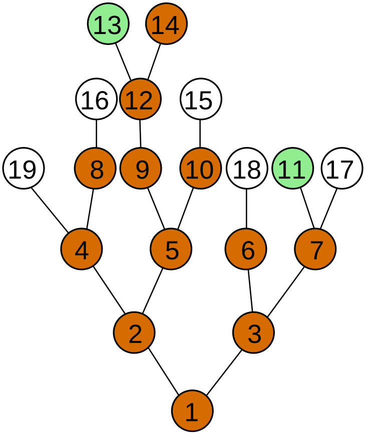

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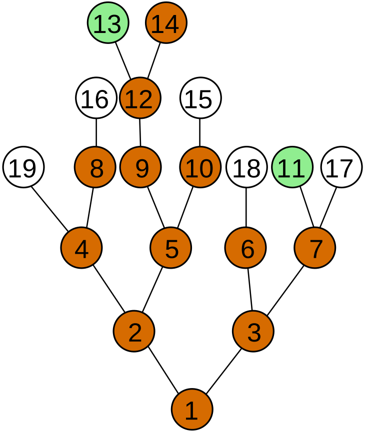
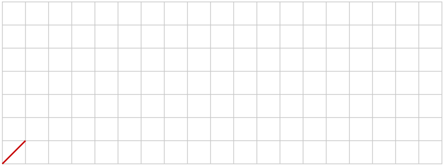
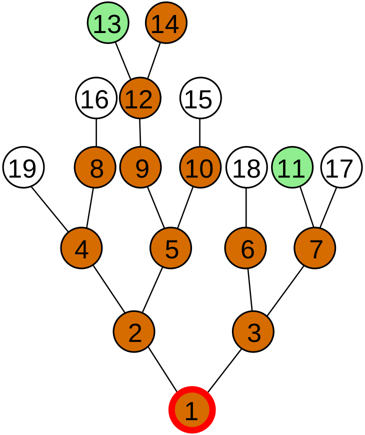
Encoding by decorated Motzkin paths





Encoding by decorated Motzkin paths



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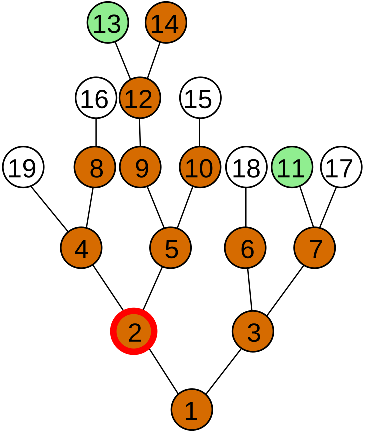

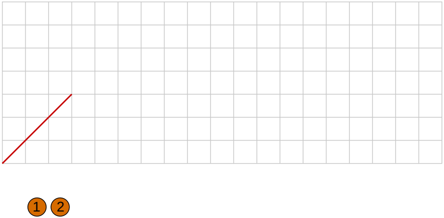

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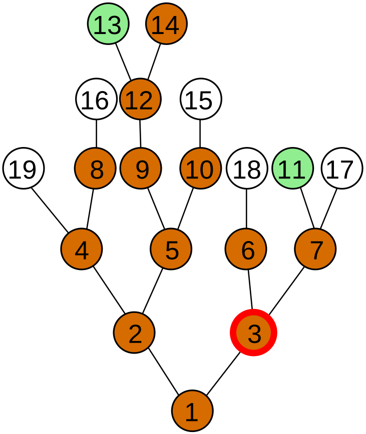

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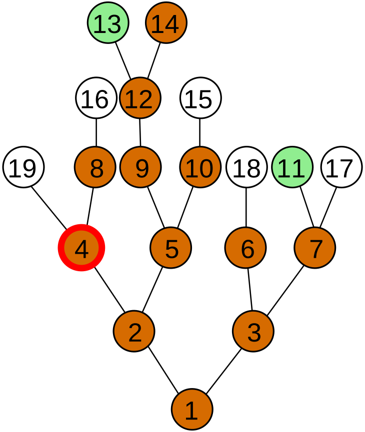

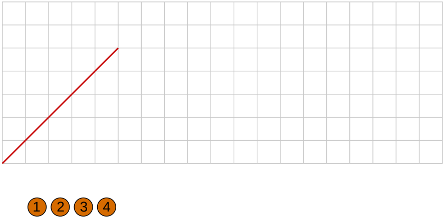

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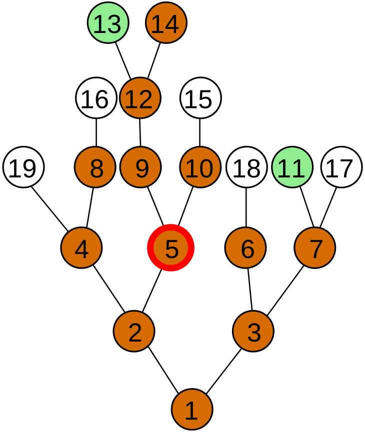

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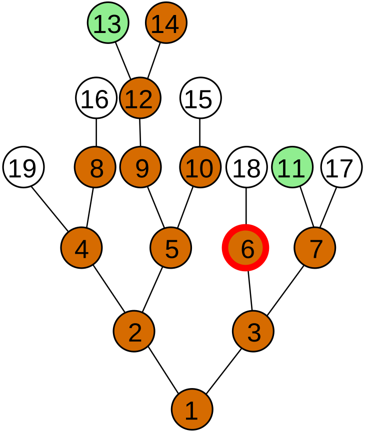

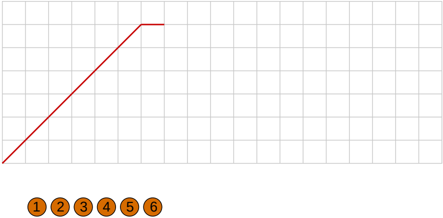

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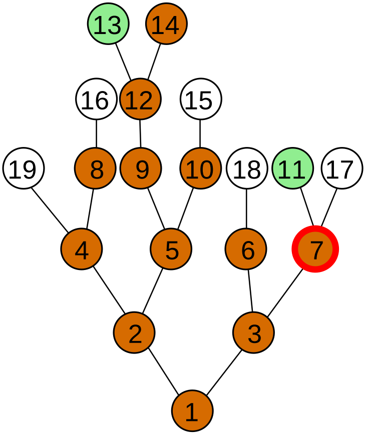

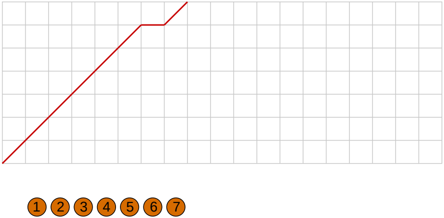

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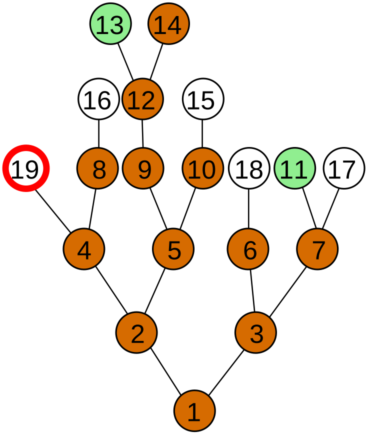

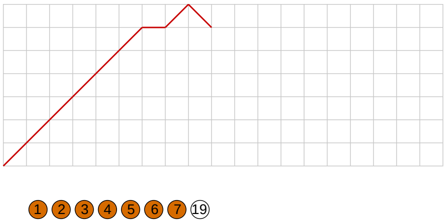

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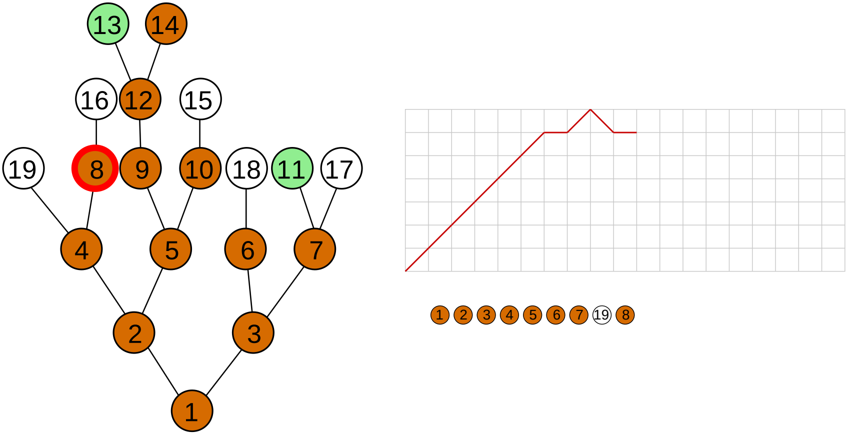

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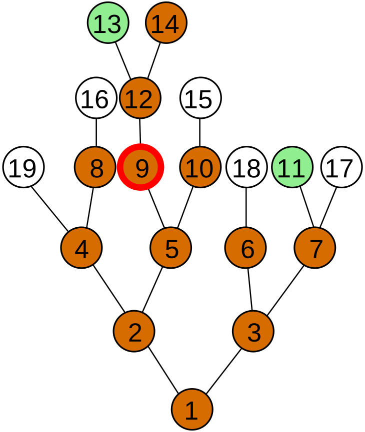

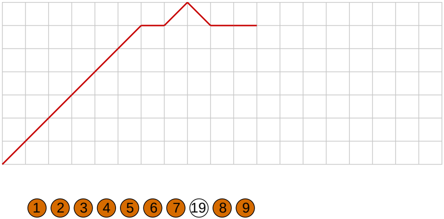

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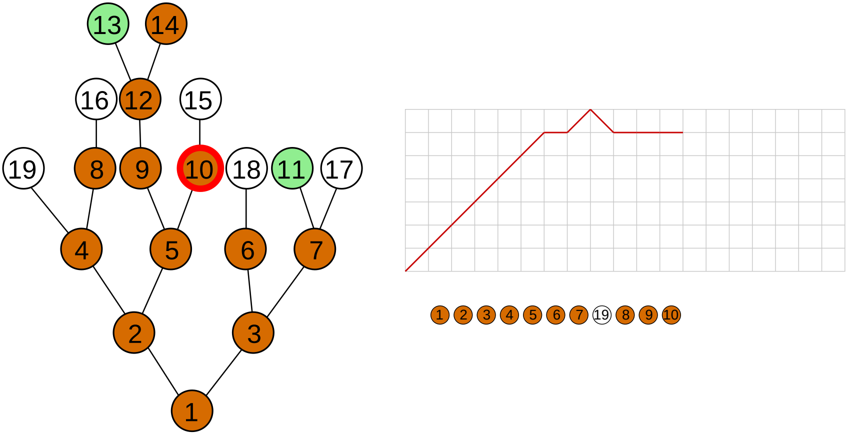

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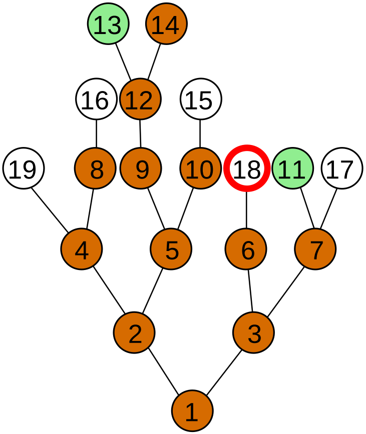

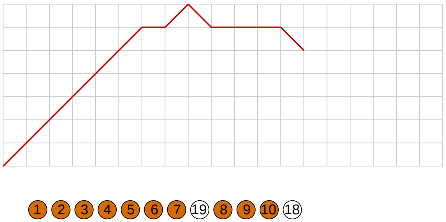

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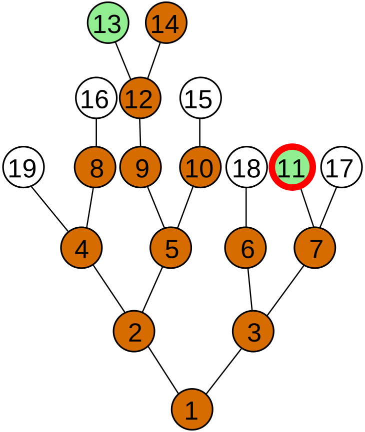

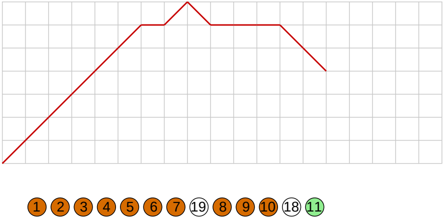

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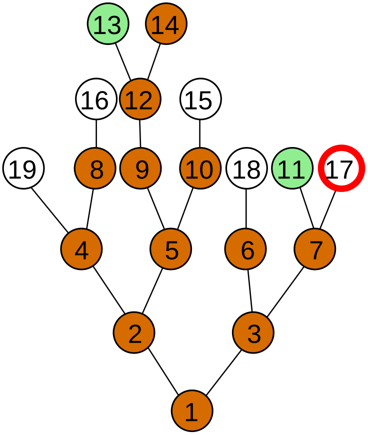
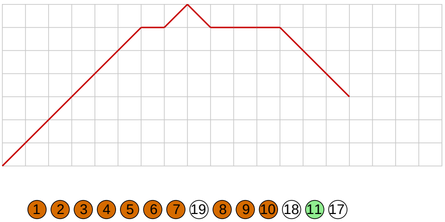
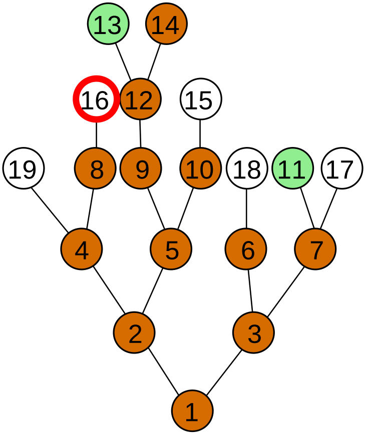
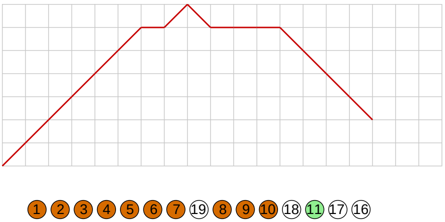




Encoding by decorated Motzkin paths





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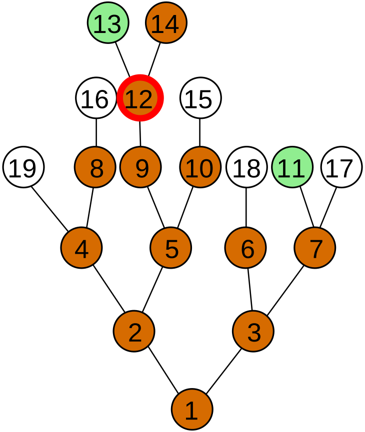

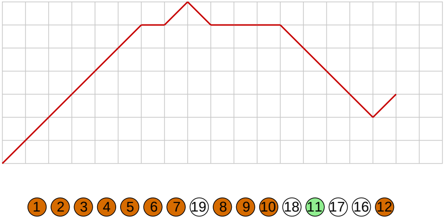

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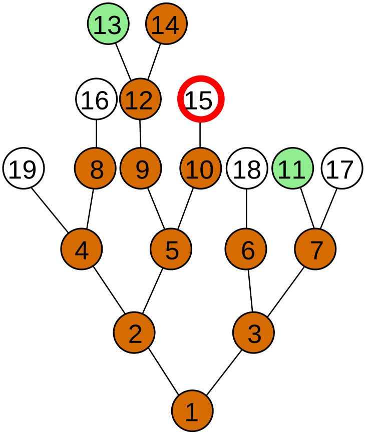

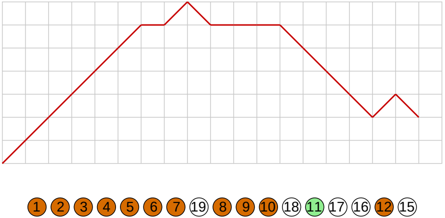

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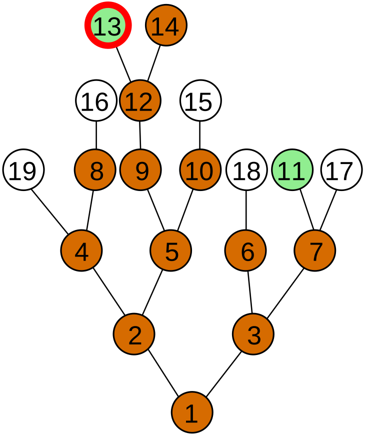
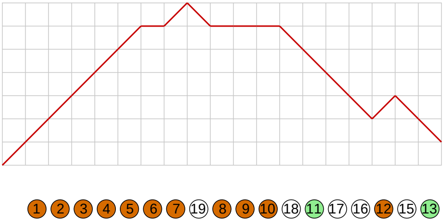
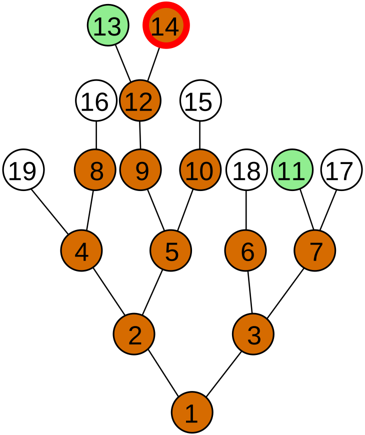
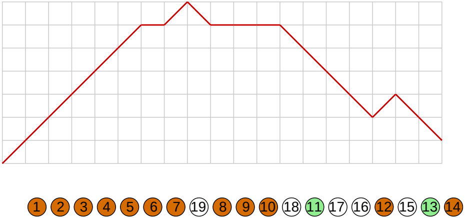




Encoding by decorated Motzkin paths





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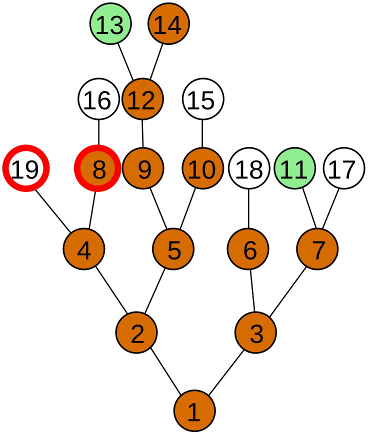

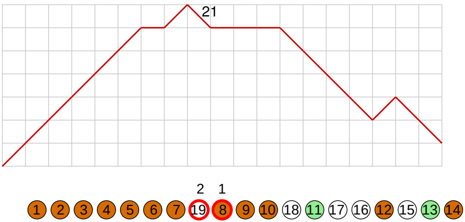

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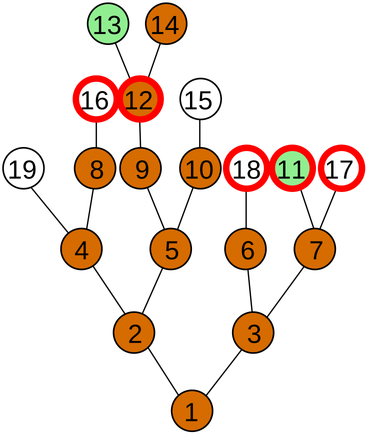

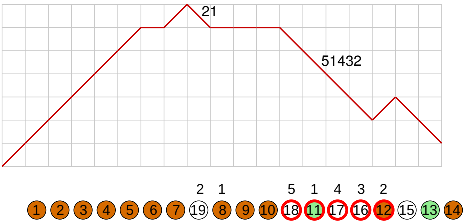

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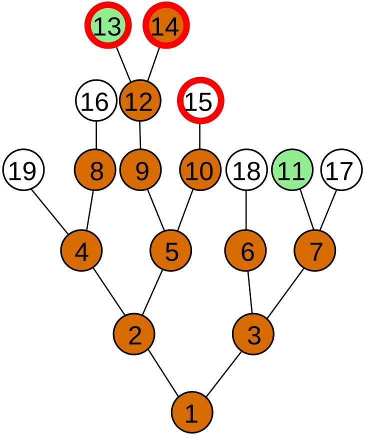
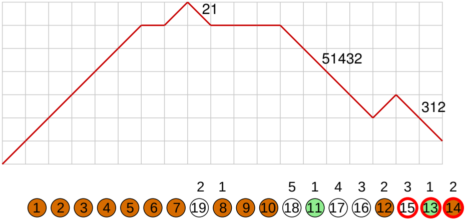
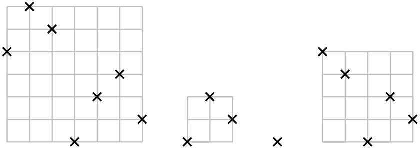




How to reconstruct the permutation



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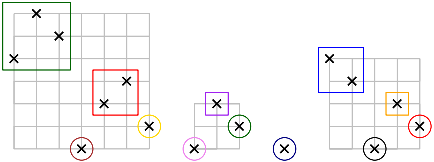

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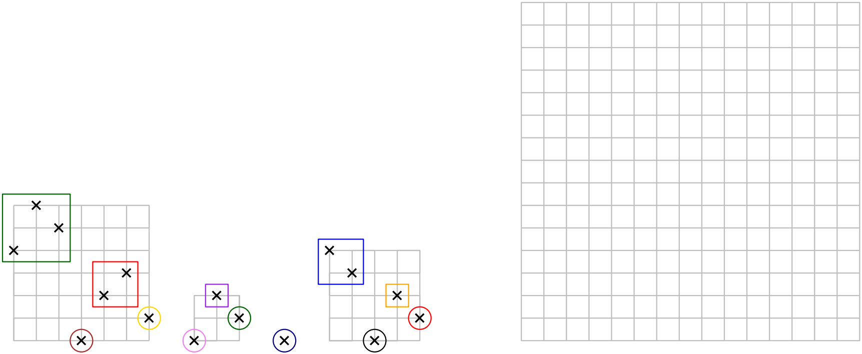

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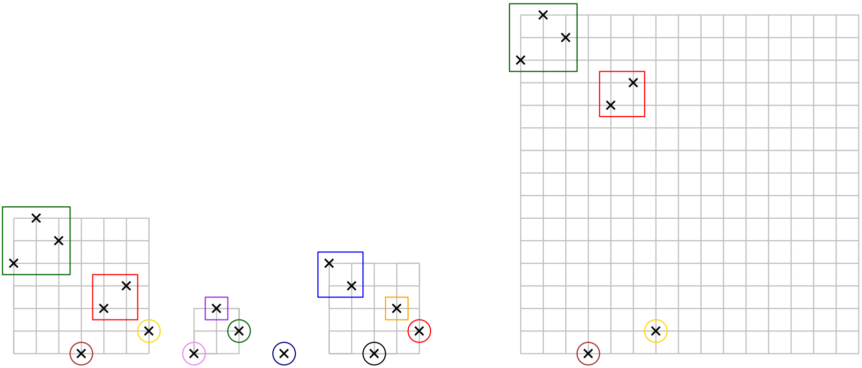

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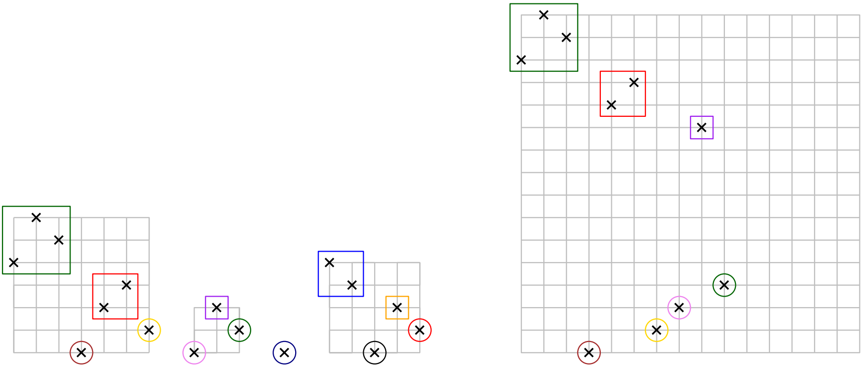

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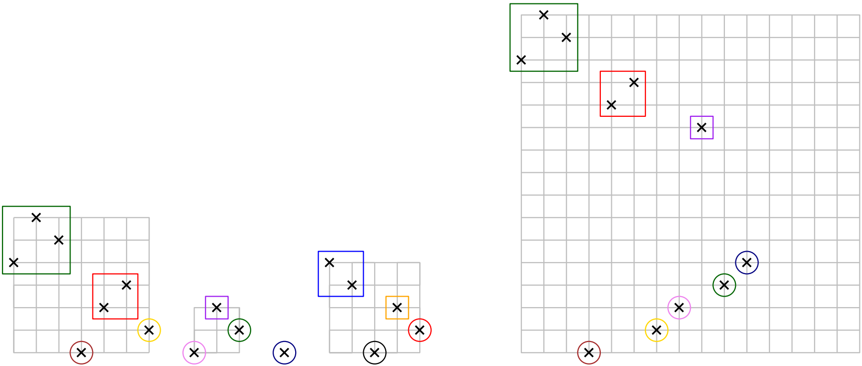

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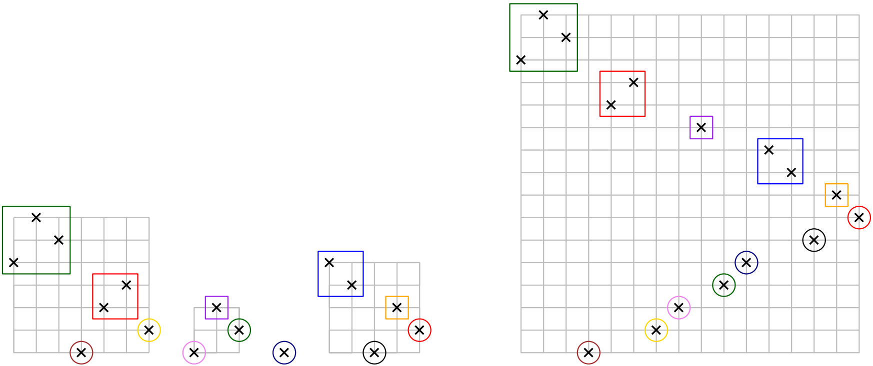
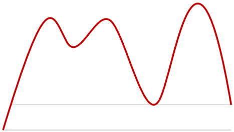


Counting decorated Motzkin walks

$\mathcal{T}$: decorated Motzkin walks

Aim

We want to show that $\mathcal{T} = \mathcal{G}$.



$\mathcal{G}$

Counting decorated Motzkin walks

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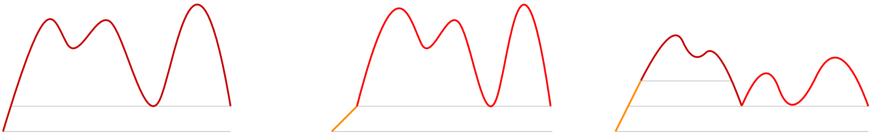

$$\mathcal{G} = \mathcal{Z}\ \mathcal{A} + \mathcal{Z}\ \mathcal{G}\ \mathcal{A}$$

Counting decorated Motzkin walks

$\mathcal{T}$: decorated Motzkin walks

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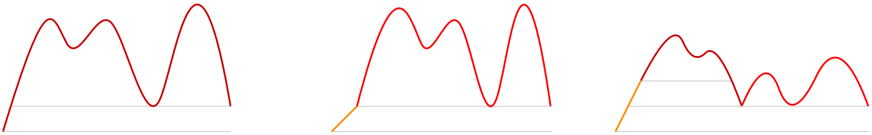
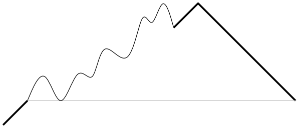


$$\mathcal{G} = \mathcal{Z} \quad \mathcal{A} + \mathcal{Z} \quad \mathcal{G} \quad \mathcal{A}$$

$$\mathcal{G} = (\mathcal{Z}\mathcal{A}) + (\mathcal{Z}\mathcal{A})^2 + (\mathcal{Z}\mathcal{A})^3 + \ldots = \mathrm{Seq}_{\geq 1}(\mathcal{Z}\mathcal{A})$$

Step 1: $\mathcal{T} = \mathrm{Seq}_{\geq 1}(\mathcal{M})$



$\mathcal{M}$: decorated Motzkin walks whose last 0-step or $+1$-step is a $+1$-step

Claim

$$\mathcal{T} = \mathrm{Seq}_{\geq 1}(\mathcal{M})$$

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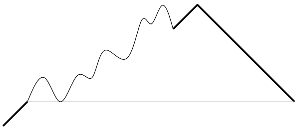

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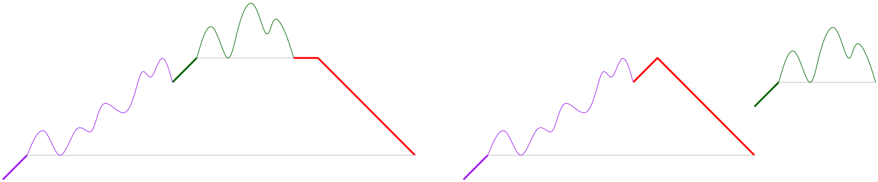

$$\mathcal{T} - \mathcal{M} \quad = \quad \mathcal{M} \quad \times \quad \mathcal{T}$$

Step 2: $\mathcal{M} = \mathcal{Z}\mathcal{A}$

We observed that $B = z/(1 - zA)$ and $A = 1 + (zA)^2 + (BA)^2$. Thus,

$$zA = z\left(1 + (zA)^2 + \frac{(zA)^2}{(1 - zA)^2}\right)$$

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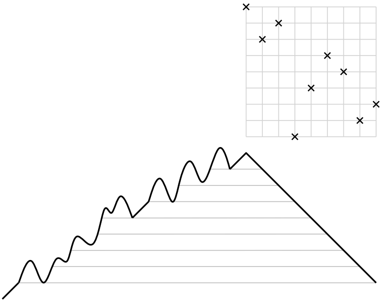

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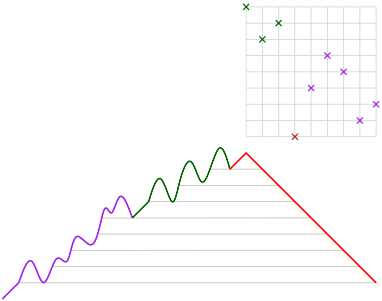

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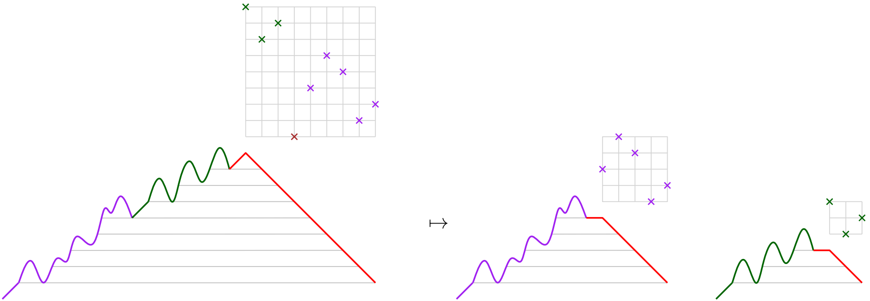

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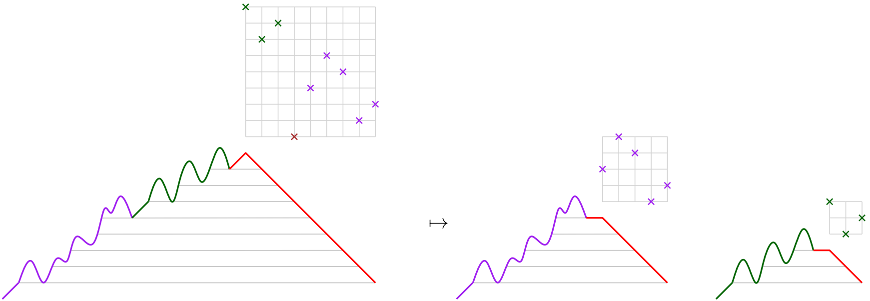
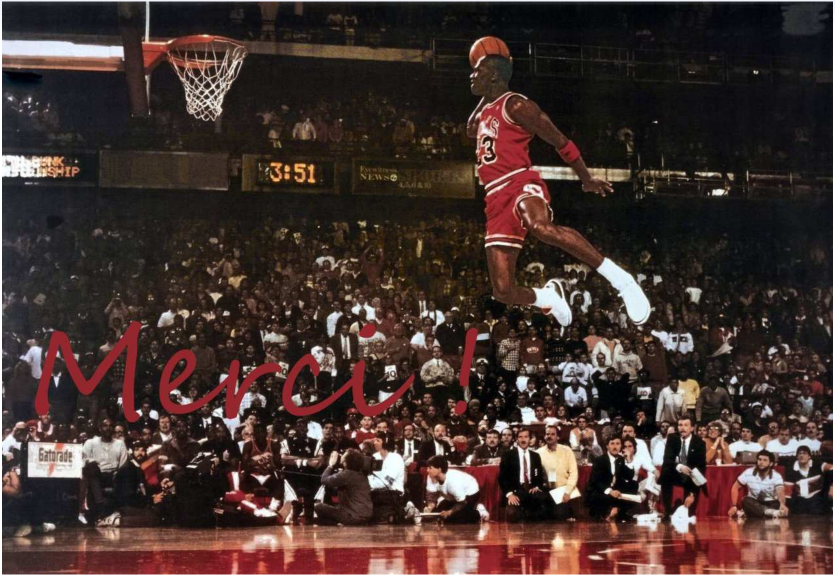


$$M = z\,(1 + T - M)^2 = z\left(1 - M + \frac{M}{1 - M}\right)^2 = z\left(1 + M^2 + \frac{M^2}{(1 - M)^2}\right)$$


Merci !
3:51
Gatorade