Mixed twistor $\mathscr{D}$-modules and some examples

Takuro Mochizuki

RIMS, Kyoto University

2017 April

Introduction

Principle (Simpson's Meta Theorem)

Concepts and theorems concerned with Hodge structures should have their counterparts in the context of twistor structures.

Morihiko Saito **established the theory of mixed Hodge modules.**

Mixed twistor $\mathscr{D}$-modules should be a counterpart of mixed Hodge modules in the context of twistor structures.

Twistor structure

- **Twistor structure** $\overset{\text{def}}{\Longleftrightarrow}$ **holomorphic vector bundle of finite rank on** $\mathbb{P}^1$
- V: **pure of weight** $w \Longleftrightarrow V \simeq \mathscr{O}_{\mathbb{P}^1}(w)^{\oplus \operatorname{rank} V}$
- (V, W): **mixed twistor structure** $\Longleftrightarrow \operatorname{Gr}_m^W(V)$: **pure of weight** m

Mixed twistor structure is a structure on the vector space $\Xi_{\mathrm{DR}}(V) := V_{|1}$.

Complex Hodge structure $\Longrightarrow$ Twistor structure

Let $(H;F,G)$ be a complex Hodge structure.

- $\xi(H;F) := \sum_{j\in\mathbb{Z}} F^{-j}\lambda^j$ **is a free $\mathbb{C}[\lambda]$-module, i.e., a vector bundle on $\operatorname{Spec}\mathbb{C}[\lambda]$.**
- $\xi(H;G) := \sum_{j\in\mathbb{Z}} G^{j}\lambda^j$ **is a free $\mathbb{C}[\lambda^{-1}]$-module, i.e., a vector bundle on $\operatorname{Spec}\mathbb{C}[\lambda^{-1}]$.**
- **From $(H;F,G)$, by gluing $\xi(H;F)$ and $\xi(H;G)$, we obtain a vector bundle on $\mathbb{P}^1$.**

Simpson

- **Complex Hodge structures are equivalent to $\mathbb{C}^*$-equivariant twistor structures.**
- **Many concepts concerned with Hodge structures have their counterparts in the context of twistor structures.**
- *Harmonic bundles* **can be regarded as polarized variations of twistor structure.**

Harmonic bundles

Let Y be a complex manifold. Let $(E, \overline{\partial}_E)$ be a holomorphic vector bundle on Y. Let θ be a Higgs field of $(E, \overline{\partial}_E)$, i.e., θ is a holomorphic section of $\mathrm{End}(E) \otimes \Omega^1$ such that $\theta \wedge \theta = 0$.

For a Hermitian metric h of E, we have the Chern connection ∇_h determined by $(\overline{\partial}_E, h)$. We also have the adjoint $\theta_h^\dagger$ of θ with respect to h.

Definition

If $\mathbb{D}^1 := \nabla_h + \theta + \theta_h^\dagger$ is flat, h is called pluri-harmonic metric, and $(E, \overline{\partial}_E, \theta, h)$ is called a harmonic bundle.

Kobayashi-Hitchin correspondences

Higgs bundles		**harmonic bundles**		**flat bundles**
$(E, \overline{\partial}_E, \theta)$	$\Longleftarrow$	$(E, \overline{\partial}_E, \theta, h)$	$\Longrightarrow$	$(E, \nabla_h + \theta + \theta^{\dagger})$

Theorem (Corlette, Simpson, (1-dimensional case, Donaldson, Hitchin))

On smooth projective varieties, the following objects are equivalent.

- **Harmonic bundles**
- **Higgs bundles (polystable, trivial Chern class)**
- **Flat bundles (semisimple)**

Non-abelian Hodge theory

- **Moduli spaces (∞-stacks) of Higgs bundles and flat bundles**
- **Some deep theorems for polarized variation of Hodge structure can be generalized to the context of harmonic bundles.**

Simpson's Meta Theorem

From a harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ on a complex projective manifold X, we have the flat bundle $(E, \mathbb{D}^1) = (E, \nabla_h + \theta + \theta_h^\dagger)$.

Simpson

- $H^i_{\mathrm{DR}}(X, (E, \mathbb{D}^1)) := \mathbb{H}^i\big(X, (\Omega_X^\bullet, d) \otimes (E, \mathbb{D}^1)\big) \simeq \mathbb{H}^i\big(X, (\Omega_X^\bullet, 0) \otimes (E, \theta)\big)$.
- **Let L be an ample line bundle of X. Then,**

$$c_1(L)^j : H^{\dim X - j}_{\mathrm{DR}}(X, (E, \mathbb{D}^1)) \simeq H^{\dim X + j}_{\mathrm{DR}}(X, (E, \mathbb{D}^1)).$$

Principle (Simpson's Meta Theorem)

Objects and theorems concerned with Hodge structures are generalized to objects and theorems to the context of twistor structures.

Mixed Hodge modules

Very roughly, mixed Hodge modules are regular holonomic $\mathscr{D}$-modules with mixed Hodge structure.

For any smooth complex algebraic variety X, we have the category of algebraic graded polarizable mixed Hodge modules $\mathrm{MHM}^{\mathrm{alg}}(X)$ on X with the forgetful functor to the category of regular holonomic $\mathscr{D}_X$-modules. For each $w \in \mathbb{Z}$, we have the semisimple full subcategories $\mathrm{PHM}(X,w)$ of polarizable pure Hodge modules of weight w.

Theorem (M. Saito)

- **We have 6-operations ($f_*, f_!, f^*, f^!, \otimes$, $R\mathscr{H}om$) for algebraic graded polarizable mixed Hodge modules, compatible with 6-operations for regular holonomic $\mathscr{D}$-modules.**
- **For any projective morphism $f : X \longrightarrow Y$ and any $M \in \mathrm{PHM}(X,w)$, we obtain $f_\dagger^i(M) \in \mathrm{PHM}(Y,w+i)$.**

We say a mixed Hodge module is "locally free" if the underlying algebraic $\mathscr{D}$-module is a flat bundle.

Theorem (M. Saito)

Let X be a smooth algebraic variety.

- **"Locally free" polarizable pure Hodge modules of weight w on X correspond to polarizable variation of pure Hodge structure of weight $w - \dim X$ on X.**
- **"Locally free" algebraic mixed Hodge modules on X correspond to admissible graded polarizable variation of mixed Hodge structure on X.**

Mixed twistor $\mathscr{D}$-modules

Mixed twistor $\mathscr{D}$-modules are holonomic $\mathscr{D}$-modules with mixed twistor structure (studied by Sabbah and M).

For any smooth complex algebraic variety X, we have the category of graded polarizable algebraic mixed twistor D-modules $\mathrm{MTM}^{\mathrm{alg}}(X)$ with the forgetful functor $\Xi_{\mathrm{DR}} : \mathrm{MTM}^{\mathrm{alg}}(X) \longrightarrow \mathrm{Hol}^{\mathrm{alg}}(X)$. We have the semisimple full subcategories $\mathrm{PTM}^{\mathrm{alg}}(X,w)$ of algebraic polarizable pure twistor $\mathscr{D}$-modules of weight w.

Theorem

- **We have the 6-operations for algebraic graded polarizable mixed twistor $\mathscr{D}$-modules which are compatible with 6-operations for algebraic holonomic $\mathscr{D}$-modules.**
- **For any $\mathscr{T} \in \mathrm{PTM}^{\mathrm{alg}}(X,w)$ and any projective morphism $f : X \longrightarrow Y$, we obtain $f_\dagger^i \mathscr{T} \in \mathrm{PTM}^{\mathrm{alg}}(Y, w+i)$.**

A mixed twistor $\mathscr{D}$-module is called "locally free" if the underlying algebraic $\mathscr{D}$-module is a flat bundle.

Theorem

Let X be a smooth algebraic variety. Let $\overline{X}$ be any algebraic smooth compactification.

- **"Locally free" algebraic pure twistor D-modules of weight w on X are wild harmonic bundles, up to shift of the weights.**
- **"Locally free" graded polarizable mixed twistor D-modules on X correspond to graded polarizable admissible variation of mixed twistor structure on $(\overline{X}, X)$.**

Weight

A polarizable variation of twistor structure of weight m on X naturally gives a pure twistor structure of weight m.

On d_X-dimensional smooth variety X, the following is commutative.

$$\begin{array}{ccc} \begin{pmatrix} \text{var. of pure Hodge str.} \\ \text{weight } m \end{pmatrix} & \longrightarrow & \mathrm{PHM}(X, m+d_X) \\ \downarrow & & \downarrow \\ \begin{pmatrix} \text{var. of pure twistor str.} \\ \text{weight } m \end{pmatrix} & \xrightarrow{\otimes(\lambda^{d_X}\mathscr{O}, \mathscr{O}, C_0)} & \mathrm{PTM}(X, m+d_X) \end{array}$$

Examples (easy case)

- **Let f be an algebraic function on X. We have the wild harmonic bundle $(\mathscr{O}_X, df, h_0)$ on $(\overline{X}, X)$, where $h_0(1,1) = 1$, and the associated $\mathscr{T}(f) \in \mathrm{PTM}^{\mathrm{alg}}(X, \dim X)$. The underlying $\mathscr{D}$-module is isomorphic to $(\mathscr{O}_X, d + df)$.**
- **If $M \in \mathrm{Hol}^{\mathrm{alg}}(X)$ is semisimple, we have $\mathscr{T}_M \in \mathrm{PTM}^{\mathrm{alg}}(X, w)$ such that $\Xi_{\mathrm{DR}}(\mathscr{T}) \simeq M$.**

We can construct many mixed twistor $\mathscr{D}$-modules by applying 6-operations.

General Issue

Many holonomic $\mathscr{D}$-modules are enhanced to mixed twistor $\mathscr{D}$-modules. But, in most cases, we know only the existence.

> **We would like to relate mixed twistor $\mathscr{D}$-modules with objects in other area of mathematics.**
> **We would like to describe some interesting classes of mixed twistor $\mathscr{D}$-modules as explicitly as possible.**

Plan of talk

- **$\mathscr{R}$-modules**
- **V-filtrations**

$\mathscr{R}$-modules

For any complex manifold X, let $p_\lambda : \mathbb{C}_\lambda \times X \longrightarrow X$ denote the projection.
Let $\mathscr{R}_X \subset \mathscr{D}_{\mathbb{C}_\lambda \times X}$ be the subalgebra generated by $\lambda p_\lambda^* \Theta_X$ over $\mathscr{O}_{\mathbb{C}_\lambda \times X}$.
For example, $\mathscr{R}_{\mathbb{C}^n} = \mathscr{O}_{\mathbb{C}_\lambda \times \mathbb{C}} \langle \lambda \partial_1, \dots, \lambda \partial_n \rangle$. We consider left $\mathscr{R}_X$-modules.

- An $\mathscr{R}_X$-module is equivalent to $\mathscr{O}_{\mathbb{C}_\lambda \times X}$-module $\mathscr{M}$ with a meromorphic relative flat connection

$$\nabla^{\mathrm{rel}} : \mathscr{M} \longrightarrow \lambda^{-1} p_\lambda^* \Omega_X^1 \otimes \mathscr{M}, \quad \nabla^{\mathrm{rel}} \circ \nabla^{\mathrm{rel}} = 0$$

- $\Xi_{\mathrm{DR}}(\mathscr{M}) = \iota_1^{-1}\big(\mathscr{M}/(\lambda - 1)\mathscr{M}\big)$ is called the underlying $\mathscr{D}$-module.
  ($\iota_1 : \{1\} \times X \longrightarrow \mathbb{C}_\lambda \times X$ denotes the inclusion.)
- $\mathscr{R}$-modules naturally appear in other areas of mathematics (Gromov-Witten theory, singularity theory, tt^*-geometry, WKB-analysis, etc.)

Mixed twistor $\mathscr{D}$-modules are $(\mathscr{M}_1, \mathscr{M}_2, C, W)$

- $\mathscr{M}_i$**:** $\mathscr{R}_X$**-modules**
- C**: sesqui-linear pairing**
- W**: weight filtration**
- **some conditions**

We focus on $\mathscr{R}_X$-modules in this talk.

In this talk, for a mixed twistor $\mathscr{D}$-module $\mathscr{T} = (\mathscr{M}_1, \mathscr{M}_2, C, W)$,

- $\mathscr{M}_2$ **is called the underlying $\mathscr{R}$-module of $\mathscr{T}$.**
- $\Xi_{\mathrm{DR}}(\mathscr{M}_2)$ **is called the underlying $\mathscr{D}$-module of $\mathscr{T}$.**

Issue

Describe $\mathscr{R}$-modules of interesting classes of MTM.

Example

Let X be a projective manifold with a hypersurface H. Let $f \in \mathscr{O}_X(*H)$. We have the $\mathscr{D}$-module

$$L_*(f,H) := \big(\mathscr{O}_X(*H), d + df\big)$$

on X. We naturally have the mixed twistor $\mathscr{D}$-module $\mathscr{T}_*(f,H)$ such that $\Xi_{\mathrm{DR}}(\mathscr{T}_*(f,H)) \simeq L_*(f,H)$. Let $\mathscr{L}_*(f,H)$ be the underlying $\mathscr{R}_X$-module.

If $(f)_0 \cap (f)_\infty = \emptyset$ and $|(f)_\infty| = H$,

$$\mathscr{L}_*(f,H) = \big(\mathscr{O}_{\mathbb{C}_\lambda \times X}(*(\mathbb{C}_\lambda \times H)), d_X + d_X(\lambda^{-1} f)\big)$$

But, in general, it is hard to describe $\mathscr{L}_*(f,H)$ explicitly.

Example

Let X be a smooth projective variety.
Let $(E, \overline{\partial}_E, \theta)$ be a stable Higgs bundle on X with $c_*(E) = 1$. We have a pluri-harmonic metric h. We obtain the Chern connection $\overline{\partial}_E + \partial_{E,h}$ and the adjoint $\theta_h^\dagger$ of θ.

Let $(\mathscr{E}, \nabla^{\mathrm{rel}})$ denote the $\mathscr{R}_X$-module underlying the corresponding pure twistor $\mathscr{D}$-module.

Let $p_\lambda : \mathbb{C}_\lambda \times X \longrightarrow X$ denote the projection.
Then, $\mathscr{E} = (p_\lambda^{-1}(E), \overline{\partial}_E + \lambda\theta_h^\dagger + \partial_{\overline{\lambda}})$, and

$$\nabla^{\mathrm{rel}} := \partial_{E,h} + \lambda^{-1}\theta : \mathscr{E} \longrightarrow \lambda^{-1}\mathscr{E} \otimes p_\lambda^* \Omega_X$$

It is difficult to explicitly describe h and hence $(\mathscr{E}, \nabla^{\mathrm{rel}})$.

It is significant to describe explicitly the $\mathscr{R}$-modules underlying some interesting classes of MTM.

- **Better behaved GKZ-systems**
- **Toda-like harmonic bundles (tt^*-Toda equations)**

Better behaved GKZ-systems

Let $\mathscr{A} = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\} \subset \mathbb{Z}^n$ **be a subset generating** $\mathbb{Z}^n$**.**

$$K_{\mathbb{R}}(\mathscr{A}) := \left\{ \sum_{j=1}^{m} r_j \boldsymbol{a}_j \,\middle|\, r_j \in \mathbb{R}_{\geq 0} \right\}, \quad K(\mathscr{A}) := \mathbb{Z}^n \cap K_{\mathbb{R}}(\mathscr{A})$$

$$K(\mathscr{A})^{\circ} := \mathbb{Z}^n \cap (\textbf{the interior part of } K_{\mathbb{R}}(\mathscr{A})).$$

Let $\Gamma \subset K(\mathscr{A})$ **be any subset such that** $\Gamma + \boldsymbol{a} \subset \Gamma$ **for any** $\boldsymbol{a} \in \mathscr{A}$**.**
Let $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) \in \mathbb{C}^n$**.**

$\mathrm{GKZ}(\mathscr{A}, \Gamma, \boldsymbol{\beta})$**: the system of differential equations for** $(\Phi_{\boldsymbol{c}} | \boldsymbol{c} \in \Gamma)$

$$\partial_{x_j} \Phi_{\boldsymbol{c}} = \Phi_{\boldsymbol{c}+\boldsymbol{a}_j} \quad (\forall \boldsymbol{c} \in \Gamma, \forall j = 1, \ldots, m)$$

$$\Big(\sum_j a_{j,i} x_j \partial_{x_j} + c_i - \beta_i \Big) \Phi_{\boldsymbol{c}} = 0 \quad (\forall \boldsymbol{c} \in \Gamma, \forall i = 1, \ldots, n)$$

Let $M(\mathscr{A}, \Gamma, \boldsymbol{\beta})$ **denote the associated** $\mathscr{D}$**-module on** $\mathbb{C}^m$**.**

We are particularly interested in

$$M_{\mathscr{A},*} := M(\mathscr{A}, K(\mathscr{A}), 0), \quad M_{\mathscr{A},!} := M(\mathscr{A}, K(\mathscr{A})^\circ, 0).$$

We have the algebraic function $F_{\mathscr{A}} := \sum x_j t^{\boldsymbol{a}_j}$ **on** $(\mathbb{C}^*)^n \times \mathbb{C}^m$**. We obtain the algebraic** $\mathscr{D}$**-module** $L(F_{\mathscr{A}}) := (\mathscr{O}_{(\mathbb{C}^*)^n \times \mathbb{C}^m}, d + dF_{\mathscr{A}})$**.**
Let $\pi : (\mathbb{C}^*)^n \times \mathbb{C}^m \longrightarrow \mathbb{C}^m$ **be the projection.**

Lemma

$$M_{\mathscr{A},*} \simeq \pi^0_* L(F_{\mathscr{A}}), \quad M_{\mathscr{A},!} \simeq \pi^0_! L(F_{\mathscr{A}}).$$

By the description, we have $\mathfrak{T}_{\mathscr{A},\star} \in \mathrm{MTM}^{\mathrm{alg}}(\mathbb{C}^m)$ $(\star = *, !)$ **over** $M_{\mathscr{A},\star}$**.**
Namely, we have $\mathscr{T}(F_{\mathscr{A}}) \in \mathrm{MTM}^{\mathrm{alg}}((\mathbb{C}^*)^n \times \mathbb{C}^m)$ **associated to** $F_{\mathscr{A}}$**.**

$$\mathfrak{T}_{\mathscr{A},\star} := \pi^0_\star \mathscr{T}(F_{\mathscr{A}}) \quad (\star = *, !)$$

Because $\Xi_{\mathrm{DR}}(\mathscr{T}(F_{\mathscr{A}})) = L(F_{\mathscr{A}})$**, we obtain** $\Xi_{\mathrm{DR}}(\mathfrak{T}_{\mathscr{A},\star}) \simeq M_{\mathscr{A},\star}$**.**

Let $\mathscr{M}_{\mathscr{A},\star}$ **be the** $\mathscr{R}_{\mathbb{C}^m}$**-module underlying** $\mathfrak{T}_{\mathscr{A},\star}$**.**

Let us consider the $\mathscr{R}$-modules $\mathscr{M}^{\mathrm{GKZ}}(\mathscr{A},\Gamma,\boldsymbol{\beta})$ given by the following system $\mathrm{GKZ}_{\mathscr{R}}(\mathscr{A},\Gamma,\boldsymbol{\beta})$ for tuples of functions $(\Phi_{\boldsymbol{c}}\,|\,\boldsymbol{c}\in\Gamma)$

$$\lambda\partial_{x_j}\Phi_{\boldsymbol{c}}=\Phi_{\boldsymbol{c}+\boldsymbol{a}_j},\quad (\forall\boldsymbol{c}\in\Gamma, j=1,\ldots,m)$$

$$\Big(\lambda(c_i-\beta_i)+\sum_{j=1}^{m}a_{j,i}\lambda x_j\partial_{x_j}\Big)\Phi_{\boldsymbol{c}}=0,\quad (\forall\boldsymbol{c}\in\Gamma, i=1,\ldots,n)$$

Theorem (Reichelt-Sevenheck, M)

$$\mathscr{M}_{\mathscr{A},*}\simeq\mathscr{M}^{\mathrm{GKZ}}(\mathscr{A},K(\mathscr{A}),0),\quad \mathscr{M}_{\mathscr{A},!}\simeq\mathscr{M}^{\mathrm{GKZ}}(\mathscr{A},K(\mathscr{A})^{\circ},0)$$

- **$\mathscr{M}^{\mathrm{GKZ}}(\mathscr{A},K(\mathscr{A}),0)$, $\mathscr{M}^{\mathrm{GKZ}}(\mathscr{A},K(\mathscr{A})^{\circ},0)$ and $\mathscr{M}_{\mathscr{A},\star}$ $(\star=*,!)$ are enhanced to $\widetilde{\mathscr{R}}_{\mathbb{C}^m}$-modules, where $\widetilde{\mathscr{R}}_{\mathbb{C}^m}=\mathscr{R}_{\mathbb{C}^m}\langle\lambda^2\partial_\lambda\rangle$.**
  For $\mathscr{M}^{\mathrm{GKZ}}$, $\Big(\lambda^2\partial_\lambda+n\lambda+\sum_{j=1}^{m}\lambda x_j\partial_{x_j}\Big)\Phi_{\boldsymbol{c}}=0$.
- **The isomorphisms are also compatible with the actions of $\lambda^2\partial_\lambda$.**

- **According to Givental, Iritani, Mann, Mignon, Reichelt, Sevenheck, the $\widetilde{\mathscr{R}}_{\mathbb{C}^m}$-modules $\mathscr{M}^{\mathrm{GKZ}}(\mathscr{A}, K(\mathscr{A}), 0)$, $\mathscr{M}^{\mathrm{GKZ}}(\mathscr{A}, K(\mathscr{A})^{\circ}, 0)$ are important in the mirror symmetry for weak Fano manifolds, the mirror symmetry for hypersurfaces in toric Fano manifolds, and the local mirror symmetry for line bundles on toric Fano manifolds.**
- **By using the above isomorphisms, we can deduce an isomorphism of the quantum $\mathscr{D}$-modules for some local mirror symmetry from the mirror symmetry for toric weak Fano manifolds. We can also compare the weight filtration and pairings on the A-side studied by Konishi-Minabe, and the natural weight filtration and pairing on the B-side.**

$$\mathscr{A}_1 = \left\{ \begin{pmatrix} \boldsymbol{b}_1 \\ 1 \end{pmatrix}, \begin{pmatrix} \boldsymbol{b}_2 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} \boldsymbol{b}_N \\ 1 \end{pmatrix}, \begin{pmatrix} \boldsymbol{0} \\ -1 \end{pmatrix} \right\}$$

$$\Longrightarrow \mathscr{A}_2 = \left\{ \begin{pmatrix} \boldsymbol{b}_1 \\ 1 \end{pmatrix}, \begin{pmatrix} \boldsymbol{b}_2 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} \boldsymbol{b}_N \\ 1 \end{pmatrix} \right\}$$

For the proof $(\star = *)$, take an n-dimensional projective toric manifold X_Σ. Let $D_\Sigma := X_\Sigma \setminus (\mathbb{C}^*)^n$. Let $\iota : (\mathbb{C}^*)^n \times \mathbb{C}^m \longrightarrow X_\Sigma \times \mathbb{C}^m$ denote the inclusion. Let $\mathscr{L}_{*,\Sigma}(F_{\mathscr{A}})$ denote the $\mathscr{R}_{X_\Sigma \times \mathbb{C}^m}$-modules underlying $\iota_* \mathscr{T}(F_{\mathscr{A}}) \in \mathrm{MTM}^{\mathrm{alg}}(X_\Sigma \times \mathbb{C}^m)$.

$\mathscr{L}_{*,\Sigma}(F_{\mathscr{A}})$ is described explicitly and easily.

Let $\pi_\Sigma : X_\Sigma \times \mathbb{C}^m \longrightarrow \mathbb{C}^m$ denote the projection. Let $q : \mathbb{C}_\lambda \times X_\Sigma \times \mathbb{C}^m \longrightarrow X_\Sigma$ denote the projection. Set $\overline{\Omega}^j_{X_\Sigma} := \lambda^{-j} q^* \Omega^j_{X_\Sigma}$.

$$\mathscr{M}_{\mathscr{A},*} = \pi^0_{\Sigma +} \mathscr{L}_{*,\Sigma}(F_{\mathscr{A}}) \simeq R^n \pi_{\Sigma *} \left(\overline{\Omega}^\bullet_{X_\Sigma} \otimes \mathscr{L}_{*,\Sigma}(F_{\mathscr{A}}) \right)$$

Let $\overline{\Omega}^j_{X_\Sigma}(\log D_\Sigma) := \lambda^{-j} q^* \Omega^j_{X_\Sigma}(\log D_\Sigma)$.

Lemma **We have the following quasi-isomorphism:**

$$\overline{\Omega}^\bullet_{X_\Sigma} \otimes \mathscr{L}_{*,\Sigma}(F_{\mathscr{A}}) \simeq \left(\overline{\Omega}^\bullet_{X_\Sigma}(\log D_\Sigma) \otimes \mathscr{O}(*(F_{\mathscr{A}})_\infty), d_{X_\Sigma} + \lambda^{-1} d_{X_\Sigma} F_{\mathscr{A}} \right)$$

We can obtain

$$R^n \pi_{\Sigma *} \left(\overline{\Omega}^\bullet_{X_\Sigma}(\log D_\Sigma) \otimes \mathscr{O}(*(F_{\mathscr{A}})_\infty), d_{X_\Sigma} + \lambda^{-1} d_{X_\Sigma} F_{\mathscr{A}} \right) \simeq \mathscr{M}^{\mathrm{GKZ}}(\mathscr{A}, K(\mathscr{A}), 0).$$

A special case

In the case

$$\mathscr{A} = \left\{ \begin{pmatrix} \boldsymbol{b}_1 \\ 1 \end{pmatrix}, \begin{pmatrix} \boldsymbol{b}_2 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} \boldsymbol{b}_m \\ 1 \end{pmatrix} \right\},$$

the mixed twistor $\mathscr{D}$-modules $\mathfrak{T}_{\mathscr{A},\star}$ are naturally mixed Hodge modules. In particular, $\mathscr{M}_{\mathscr{A},\star}$ are the analytification of filtered $\mathscr{D}$-modules $(M_{\mathscr{A},\star}, F)$. For the computation of F, it is enough to see the action of $\lambda^2\partial_\lambda$.

We have the natural surjections

$$\bigoplus_{\boldsymbol{c}\in K(\mathscr{A})} \mathscr{O}_{\mathbb{C}^m} e(\boldsymbol{c}) \longrightarrow M_{\mathscr{A},*} \qquad \bigoplus_{\boldsymbol{c}\in K(\mathscr{A})^\circ} \mathscr{O}_{\mathbb{C}^m} e(\boldsymbol{c}) \longrightarrow M_{\mathscr{A},!}$$

$F_j\mathscr{M}_{\mathscr{A},\star}$ **are the image of**

$$\bigoplus_{\substack{\boldsymbol{c}\in K(\mathscr{A}) \\ c_n\leq j+n}} \mathscr{O}_{\mathbb{C}^m} e(\boldsymbol{c}), \qquad \bigoplus_{\substack{\boldsymbol{c}\in K(\mathscr{A})^\circ \\ c_n\leq j+n}} \mathscr{O}_{\mathbb{C}^m} e(\boldsymbol{c}).$$

If $K(\mathscr{A}) = \mathbb{Z}_{\geq 0}\cdot\mathscr{A}$, then $F_j(M_{\mathscr{A},*}) = F^{\mathrm{ord}}_{j+n}(M_{\mathscr{A},*})$.

Toda-like harmonic bundles and Toda equations

Let $\mathscr{K}(r,1)$ be $\mathbb{C}_q^* \longrightarrow \{r$-square matrix$\}$:

$$\mathscr{K}(r,1)_{i,j} := \begin{cases} 1 & (i = j+1) \\ q & ((i,j) = (1,r)) \\ 0 & (\textbf{otherwise}) \end{cases}$$

We have the Higgs field θ on $E = \bigoplus \mathscr{O}_{\mathbb{C}^*} e_i$ given by $\theta \boldsymbol{e} = \boldsymbol{e}\mathscr{K}(r,1)dq/q$.

Theorem

Harmonic metrics h on (E,θ) such that $\det(h) = A\log|q|$ are classified by $\boldsymbol{a} = (a_1,\dots,a_r) \in \mathbb{R}^r$ such that $a_1 \geq a_2 \geq \cdots \geq a_r \geq a_1 - 1$.

For $\boldsymbol{a} \in \mathbb{R}^r$, we have a unique harmonic metric $h_{\boldsymbol{a}}$ such that

- $\sum \log h_{\boldsymbol{a}}(e_i,e_i) = -\sum a_i \log|q|^2$.
- $\log h_{\boldsymbol{a}}(e_i,e_i) + a_i \log|q|^2 = O\Big(\log(\log|q|^{-1})\Big)$ **around** $q = 0$.
- **We have** $h_{\boldsymbol{a}}(e_i,e_j) = 0$ $(i \neq j)$.

For each $\boldsymbol{a} \in \mathbb{R}^r$, we have the pure twistor $\mathscr{D}$-module corresponding to $(E, \theta, h_{\boldsymbol{a}})$. Let $\mathscr{M}_{\boldsymbol{a}}$ denote the underlying $\mathscr{R}$-module.

Theorem $\mathscr{M}_{\boldsymbol{a}}(*q) \simeq \left(\bigoplus_{i=1}^r \mathscr{O}_{\mathbb{C}_\lambda \times \mathbb{C}_q^*} \widetilde{e}_i, \nabla^{\mathrm{rel}}\right)$:

$$q\nabla_q^{\mathrm{rel}} \widetilde{\boldsymbol{e}} = \widetilde{\boldsymbol{e}}\left(-\operatorname{diag}[a_1, \ldots, a_r] + \frac{1}{\lambda}\mathscr{K}(r,1)\right)$$

In this case, ∇^{rel} is naturally extended to a meromorphic flat connection

$$\nabla : \mathscr{M}_{\boldsymbol{a}}(*q) \longrightarrow \mathscr{M}_{\boldsymbol{a}}(*q) \otimes \lambda^{-1}\Omega^1_{\mathbb{C}_\lambda \times \mathbb{C}_q^*}(\log \lambda)$$

by the following action of $\lambda\partial_\lambda$:

$$\lambda\nabla_\lambda \widetilde{\boldsymbol{e}} = \widetilde{\boldsymbol{e}}\left(\operatorname{diag}[1, \ldots, r] + r\operatorname{diag}[a_1, \ldots, a_r] - \frac{r}{\lambda}\mathscr{K}(r,1)\right)$$

The Stokes matrices at $q = \infty$ can be easily computed in terms of the parabolic weights. The Stokes structure at $\lambda = 0$ is essentially the same.

Outline of the proof

- **Let** $p_\lambda : \mathbb{C}_\lambda \times \mathbb{C}^* \longrightarrow \mathbb{C}^*$ **be the projection. We have** $\mathscr{E} = (p_\lambda^{-1}(E), \overline{\partial}_E + \lambda\theta^\dagger + \partial_{\overline{\lambda}})$ **and** $\nabla^{\mathrm{rel}} = \partial_E + \lambda^{-1}\theta : \mathscr{E} \longrightarrow \mathscr{E} \otimes \lambda^{-1} p_\lambda^* \Omega^1_{\mathbb{C}^*}$.
- **The bundle** $E = \bigoplus_{i=1}^r \mathscr{O}_{\mathbb{C}^*} e_i$ **is** S^1**-equivariant, where** $t^*q = t^r q$, $t^* e_i = t^i e_i$.

  We have $t^*\theta = t\theta$. **We obtain** $t^* h_{\boldsymbol{a}} = h_{\boldsymbol{a}}$.

  $\mathscr{E}$ **is naturally** S^1**-equivariant with respect to the action** $t(\lambda, q) = (t\lambda, t^r q)$, **for which** $t^*\nabla^{\mathrm{rel}} = \nabla^{\mathrm{rel}}$.
- **By using the** S^1**-action,** ∇^{rel} **is extended to the meromorphic connection** $\nabla : \mathscr{E} \longrightarrow \mathscr{E} \otimes \lambda^{-1}\Omega^1(\log\lambda)$.
- **We have a connection** ∇_1 **of** E **for which** $\nabla^{\mathrm{rel}} = \nabla_1 + \lambda^{-1}\theta$. **(***Key step***)**
- **The bundle** E **is** $\mu_r = \{\tau \in \mathbb{C} \mid \tau^r = 1\}$**-equivariant, where** $\tau^* q = q$, $\tau^* e_i = \tau e_i$.

  We have $\tau^*\theta = \theta$ **and** $\tau^* h_{\boldsymbol{a}} = h_{\boldsymbol{a}}$.

  $\mathscr{E}$ **is naturally** μ_r**-equivariant with respect to the action** $\tau(\lambda, q) = (\tau\lambda, q)$, **for which** $\tau^*\nabla^{\mathrm{rel}} = \nabla^{\mathrm{rel}}$.
- **It implies** ∇_1 **is** μ_r**-equivariant and** $\mathbb{C}^*$**-equivariant. and hence we have** $b_i \in \mathbb{C}$ **such that** $\nabla_1(e_i) = e_i b_i dq/q$.
- **The eigenvalues of** $\mathrm{Res}(\nabla^{\mathrm{rel}})$ **on** $\mathrm{Gr}^{(0)}_{a_i}(\mathscr{E})$ **are** $-a_i \Longrightarrow b_i = -a_i$.

Relation with tt^*-Toda equations

These harmonic bundles $(E,\theta,h_{\boldsymbol{a}})$ are closely related to solutions of the Toda equations studied by Cecotti-Vafa:

$$2\partial_{\overline{z}}\partial_z w_i - e^{2(w_i-w_{i-1})} + e^{2(w_{i+1}-w_i)} = 0, \quad w_{r+i} = w_i \quad (i \in \mathbb{Z})$$

From the harmonic metric $h_{\boldsymbol{a}}$, we obtain a solution

$$w_i(z) = -(i-1)\log|z/r| + \frac{1}{2}\log\bigl(h^{(1)}_{\boldsymbol{a}}(e_i,e_i)_{|q=(z/r)^r}\bigr) \quad (i=1,\ldots,r)$$

Any solution (w_i) satisfying $\sum_{i=1}^{r} w_i(z) + \frac{1}{2}r(r-1)\log|z/r| + a\log|z/r| = 0$ is obtained in this way.
Note that for any solution, we have $\sum w_i(z) = \mathrm{Re}(f) + A\log|z|$ for a holomorphic function f on $\mathbb{C}^*$ and $A \in \mathbb{R}$. We can normalize it as above.

Remark **Guest and Lin, with Its, studied these equations in a systematic way from a viewpoint of tt^*-geometry on the mathematical side: classification of solutions, the associated Stokes structures, etc.**

V-filtrations

V-filtrations with nilpotent Higgs residue

Let $\mathscr{T}$ be a mixed twistor $\mathscr{D}$-module on $Y = Y_0 \times \mathbb{C}_t$. Let $\mathscr{M}$ be the underlying $\mathscr{R}$-module of $\mathscr{T}$.

A V-filtration of $\mathscr{M}$ along t with nilpotent Higgs residue is an increasing filtration $V_a\mathscr{M}$ $(a \in \mathbb{R})$ satisfying the following conditions:

- $V_a\mathscr{M}$ $(a \in \mathbb{R})$ **are $\mathscr{O}_{\mathbb{C}_\lambda \times Y}$-submodules of $\mathscr{M}$. We have $\mathscr{M} = \bigcup_{a\in\mathbb{R}} V_a\mathscr{M}$ and $V_b\mathscr{M} = \bigcap_{b<a} V_a\mathscr{M}$.**
- **The relative flat connection ∇^{rel} is logarithmic with respect to $V_a\mathscr{M}$, i.e.,**

$$\nabla^{\mathrm{rel}}(V_a\mathscr{M}) \subset V_a\mathscr{M} \otimes \left(\lambda^{-1}\Omega^1_{\mathbb{C}\times Y/\mathbb{C}}(\log t)\right)$$

- $\lambda\nabla_t \cdot t + \lambda a$ **are nilpotent on** $\mathrm{Gr}^V_a(\mathscr{M}) = V_a(\mathscr{M})/V_{<a}(\mathscr{M})$ **for any** $a \in \mathbb{R}$**.**
- $t : V_a\mathscr{M} \longrightarrow V_{a-1}\mathscr{M}$ **are isomorphisms if** $a < 0$**, and** $\lambda\nabla_t : \mathrm{Gr}^V_a(\mathscr{M}) \longrightarrow \mathrm{Gr}^V_{a+1}(\mathscr{M})$ **are isomorphisms for** $a > -1$**.**
- $\mathrm{Gr}^V_a(\mathscr{M})$ **are flat over $\mathscr{O}_{\mathbb{C}_\lambda}$.**
- $V_a(\mathscr{M})$ **satisfy some "coherence condition".**

Remark

In general, the underlying $\mathscr{R}$-module of a MTM does not necessarily have V-filtration with nilpotent Higgs residue. Their V-filtrations are characterized by a more complicated condition.

Functoriality

Let Z_0 be any complex manifold, and let W be any projective manifold. Set $Y = Z_0 \times \mathbb{C}_t \times W$ and $Z = Z_0 \times \mathbb{C}_t$. Let $\rho : Y \longrightarrow Z$ be the projection. We set $\overline{\Omega}^j_W := \lambda^{-j} q_W^* \Omega^j_W$, where $q_W : \mathbb{C}_\lambda \times Y \longrightarrow W$ denotes the projection. For any $\mathscr{R}_Y$-module $\mathscr{M}$, $\rho^j_\dagger \mathscr{M}$ are defined as

$$R^{j+\dim W}(\mathrm{id} \times \rho)_* \Big(\overline{\Omega}^\bullet_W \otimes \mathscr{M} \Big)$$

Proposition (Sabbah, Saito)

Let $\mathscr{M}$ be the $\mathscr{R}$-module underlying a mixed twistor $\mathscr{D}$-module on Y. Suppose that $\mathscr{M}$ has a V-filtration along t with nilpotent Higgs residue. Then, each $\rho^j_\dagger \mathscr{M}$ also has a V-filtration along t with nilpotent Higgs residue, and the following holds:

$$V_a(\rho^j_\dagger \mathscr{M}) \simeq R^{j+\dim W}(\mathrm{id} \times \rho)_* \Big(\overline{\Omega}^\bullet_W \otimes V_a \mathscr{M} \Big) \quad (a \in \mathbb{R}).$$

Locally free property

Let $\mathscr{M}$ be the $\mathscr{R}$-module underlying a MTM $\mathscr{T}$ on $Y \times \mathbb{C}_t$. Suppose

- $\Xi_{\mathrm{DR}}(\mathscr{T})_{|Y \times \mathbb{C}_t^*}$ **is a locally free $\mathscr{O}$-module. Moreover, it is regular singular along $t = 0$.**
- $\mathscr{M}$ **has a V-filtration along t with nilpotent Higgs residue.**

Proposition

$V_a\mathscr{M}$ $(a < 0)$ **are locally free $\mathscr{O}_{\mathbb{C}_\lambda \times Y \times \mathbb{C}_t}$-modules.**

Kontsevich complexes

Let X be any smooth projective variety. Let H be a normal crossing hypersurface of X. Let $f : X \longrightarrow \mathbb{P}^1$ be a morphism such that $f^{-1}(\infty) \subset H$. Let $(f)_\infty$ denote the pole divisor of f. Let Ω_f^j be the kernel of

$$df : \Omega_X^j(\log H) \longrightarrow \Omega_X^{j+1}(\log H) \otimes \mathscr{O}_X((f)_\infty) \Big/ \Omega_X^{j+1}(\log H)$$

Then, for any $(\lambda, \tau) \in \mathbb{C}^2$, we obtain the complex of sheaves $(\Omega_f^\bullet, \lambda d_X + \tau df)$.

Kontsevich complexes and MTM

Set $X^{(1)} := X \times \mathbb{C}_\tau$ and $H^{(1)} := H \times \mathbb{C}_\tau$.
Let $q_X : \mathbb{C}_\lambda \times X^{(1)} \longrightarrow X$ denote the projection. Set $\overline{\Omega}_f^j := \lambda^{-j} q_X^* \Omega_f^j$.

$$\mathfrak{K}(f) := \left(\overline{\Omega}_f^\bullet, d_X + \lambda^{-1}\tau d_X f\right) \simeq \left(q_X^*\Omega_f^\bullet, \lambda d_X + \tau d_X f\right)$$

We have the meromorphic function τf on $(X^{(1)}, H^{(1)})$. We have the $\mathscr{D}$-module $L_*(\tau f, H^{(1)}) := \left(\mathscr{O}_{X^{(1)}}(*H^{(1)}), d_X + d_X(\tau f)\right)$. We have the natural MTM $\mathscr{T}_*(\tau f, H^{(1)})$ over $L_*(\tau f, H^{(1)})$. Let $\mathscr{L}_*(\tau f, H^{(1)})$ denote the underlying $\mathscr{R}_{X^{(1)}}$-module.

Theorem

- **$\mathscr{L}_*(\tau f, H^{(1)})$ has a V-filtration along τ with nilpotent Higgs residue.**
- **We have a natural quasi-isomorphism**

$$\mathfrak{K}(f) \longrightarrow \overline{\Omega}_X^\bullet \otimes V_{-1}\mathscr{L}_*(\tau f, H^{(1)})$$

We may revisit the following theorem.

Theorem (Esnault-Sabbah-Yu, Kontsevich, M. Saito) **For any** $(\lambda,\tau)\in\mathbb{C}^2$, $\dim\mathbb{H}^j(X,(\Omega_f^\bullet,\lambda d+\tau df))$ **are independent of** (λ,τ).

Let $\pi: X^{(1)}\longrightarrow\mathbb{C}_\tau$ denote the projection.

- By the functoriality of V-filtrations,

$$R^{j+\dim X}(\mathrm{id}\times\pi)_*\mathfrak{K}(f)\simeq R^{j+\dim X}(\mathrm{id}\times\pi)_*\big(\overline{\Omega}_X^\bullet\otimes V_{-1}\mathscr{L}_*(\tau f,H^{(1)})\big)$$
$$\simeq V_{-1}\pi_\dagger^j\mathscr{L}_*(\tau f,H^{(1)})$$

- $\Xi_{\mathrm{DR}}(\pi_\dagger^j\mathscr{L}_*(\tau f,H^{(1)}))$ are regular singular along $\tau=0$, and locally free on $\mathbb{C}_\tau^*$.
- $V_{-1}\pi_\dagger^j\mathscr{L}*(\tau f,H^{(1)})$ are locally free $\mathscr{O}_{\mathbb{C}_\lambda\times\mathbb{C}_\tau}$-modules.
- $R^j(\mathrm{id}\times\pi)_*\mathfrak{K}(f)$ are locally free $\mathscr{O}$-modules.

In this way, we obtain an alternative proof of the theorem.

Nahm transform of harmonic bundles on $\mathbb{P}^1$

Szilard Szabo has been studying Nahm transform under some assumptions on the singularity of $(E, \overline{\partial}_E, \theta, h)$.

$$\begin{pmatrix} \text{Wild harmonic bundles} \\ (E, \overline{\partial}_E, \theta, h) \\ \text{on } (\mathbb{P}^1, D \cup \{\infty\}) \end{pmatrix} \longrightarrow \begin{pmatrix} \text{Wild harmonic bundles} \\ (\widehat{E}, \overline{\partial}_{\widehat{E}}, \widehat{\theta}, h) \\ \text{on } (\mathbb{P}^1, \widehat{D} \cup \{\infty\}) \end{pmatrix}$$

We have the Higgs bundle $L(\tau t) = \big(\mathscr{O}_{\mathbb{C}_t \times \mathbb{C}_\tau}, d(\tau t)\big)$ with the metric $h_0(1,1) = 1$.

We obtain the wild harmonic bundle on $\mathbb{P}^1 \times \mathbb{P}^1$:

$$p_1^*(E, \overline{\partial}_E, \theta, h) \otimes L(\tau t)$$

Here, p_1 is the projection onto the first component. By taking the push-forward to the second component, we obtain a wild harmonic bundle on $\mathbb{P}^1$.

It is also "Fourier transform" for pure twistor $\mathscr{D}$-modules on $\mathbb{P}^1$.

Recently, Szabo and I studied the Nahm transform of wild harmonic bundles $(E,\overline{\partial}_E,\theta,h)$ on $(\mathbb{P}^1,D\cup\{\infty\})$ satisfying the following conditions:

- **tame at each point of D,**
- **unramifiedly good wild of Poincaré rank 1 at ∞.**

We established the basic properties such as the comparison with the algebraic transforms of parabolic Higgs bundles, the involutivity, etc.

One of the key steps is the description of the parabolic structure of $(\widehat{E},\overline{\partial}_{\widehat{E}},\widehat{\theta},\widehat{h})$ at the points of $\widehat{D}\cup\{\infty\}$.

More recently, I obtained an alternative proof to use the functoriality of V-filtrations.
We explain it in the case of the induced parabolic for $(\widehat{E},\overline{\partial}_{\widehat{E}},\widehat{h})$ at ∞, under the assumptions (i) $D=\{0\}$, (ii) $\mathrm{Res}_0\,\theta$ is nilpotent.

The associated parabolic bundles

For any $(a,b)\in\mathbb{R}^2$, we extend E to the bundle $\mathscr{P}_{a,b}E$ on $\mathbb{P}^1$ by considering the local sections s satisfying $|s|_h=O(|z|^{-a-\varepsilon})$ $(\forall\varepsilon>0)$ around 0, and $|s|_h=O(|z|^{b+\varepsilon})$ $(\forall\varepsilon>0)$ around ∞.

Let $(\widehat{E},\overline{\partial}_{\widehat{E}},\widehat{\theta},\widehat{h})$ be the Nahm transform of $(E,\overline{\partial}_E,\theta,h)$.
Let $\widehat{U}_\infty$ be a neighbourhood of ∞ in $\mathbb{P}^1$. For each $a\in\mathbb{R}$, $(\widehat{E},\overline{\partial}_{\widehat{E}})_{|\widehat{U}_\infty\setminus\{\infty\}}$ is extended to a holomorphic vector bundle $\mathscr{P}_a^{(\infty)}\widehat{E}_{\widehat{U}_\infty}$.

In particular, we have the vector bundle $\mathscr{P}_0^{(\infty)}\widehat{E}_{\widehat{U}_\infty}$. The fiber $\mathscr{P}_0^{(\infty)}\widehat{E}_{\widehat{U}_\infty|\infty}$ is equipped with the filtration $\mathfrak{F}$ indexed by $-1<a\le 0$:

$$\mathfrak{F}_a\big(\mathscr{P}_0^{(\infty)}\widehat{E}_{\widehat{U}_\infty|\infty}\big):=\mathrm{Im}\Big(\mathscr{P}_a^{(\infty)}\widehat{E}_{\widehat{U}_\infty|\infty}\longrightarrow\mathscr{P}_0^{(\infty)}\widehat{E}_{\widehat{U}_\infty|\infty}\Big)$$

We would like to describe $\mathscr{P}_0^{(\infty)}\widehat{E}_{\widehat{U}_\infty}$ and the parabolic filtration $\mathfrak{F}$ in terms of $(\mathscr{P}_{a,b}E\,|\,(a,b)\in\mathbb{R}^2)$ and θ.

Algebraic description of the parabolic filtration at ∞

We have the endomorphism $\mathrm{Gr}_0^{(0)}(\mathrm{Res}_0(\theta))$ of $\mathrm{Gr}_0^{(0)}(E) := \mathscr{P}_{0,0}(E)/\mathscr{P}_{<0,0}(E)$. Let $\mathscr{G}_a$ be the kernel of

$$\mathscr{P}_{0,a}E \longrightarrow \frac{\mathrm{Gr}_0^{(0)}(E)}{\mathrm{Im}\,\mathrm{Gr}_0^{(0)}\,\mathrm{Res}(\theta)} \text{ (skyscraper sheaf at } 0)$$

Let $\widehat{U}_\infty$ be a small neighbourhood of ∞ in $\mathbb{P}^1$. Let $\tau := \zeta^{-1}$. Let p_i be the projection of $\mathbb{P}^1 \times \widehat{U}_\infty$ onto the i-th component. We have the following complex $\mathscr{C}_a^\bullet$:

$$p_1^*\mathscr{P}_{0,a}E \xrightarrow{\tau p_1^*(\theta)+p_1^*(dz)} p_1^*\big(\mathscr{G}_a \otimes \Omega^1(\{0\}+2\{\infty\})\big)$$

In the case $a=0$, we obtain a holomorphic bundle $\widehat{E}^{\mathrm{alg}}_{\widehat{U}_\infty} := Rp_{2*}\mathscr{C}_0^\bullet$ on $\widehat{U}_\infty$. Moreover, for $-1<a<0$, we set

$$\mathfrak{F}_a(\widehat{E}^{\mathrm{alg}}_{\widehat{U}_\infty|\infty}) := \mathrm{Im}\Big(Rp_{2*}(\mathscr{C}_a^\bullet)_{|\infty} \longrightarrow \widehat{E}^{\mathrm{alg}}_{\widehat{U}_\infty|\infty}\Big)$$

We obtain the induced filtration $\mathfrak{F}$ on $\widehat{E}^{\mathrm{alg}}_{\widehat{U}_\infty|\infty}$.

Theorem (M-Szabo)

The parabolic bundle $(\widehat{E}^{\mathrm{alg}}_{\widehat{U}_\infty}, \mathfrak{F})$ is isomorphic to $(\mathscr{P}_0^{(\infty)}\widehat{E}_{\widehat{U}_\infty}, \mathfrak{F})$.

Revisit from a twistor viewpoint

We have the algebraic pure twistor $\mathscr{D}$-module $\mathscr{T}$ on $\mathbb{P}^1$ associated to $(E, \overline{\partial}_E, \theta, h)$. Let $\mathscr{M}$ denote the $\mathscr{R}$-modules underlying $\mathscr{T}$.

We have the algebraic function $z\zeta$ on $\mathbb{C}_z \times \mathbb{C}_\zeta$. We have the associated pure twistor $\mathscr{D}$-module $\mathscr{T}(z\zeta)$. Let $\mathscr{L}(z\zeta)$ be the underlying $\mathscr{R}$-module.

Proposition

We have the following quasi-isomorphism

$$\mathscr{C}_0^\bullet \longrightarrow \Big(\overline{\Omega}^\bullet_{\mathbb{P}^1} \otimes V_{-1}\big(p_1^* \mathscr{M} \otimes \mathscr{L}(z\zeta)\big)\Big)_{|\{0\}\times\mathbb{P}^1\times\widehat{U}_\infty}$$

As a result, we obtain $\widehat{E}^{\mathrm{alg}}_{\widehat{U}_\infty} \simeq V_{-1} p^0_{2\dagger}\big(p_1^*\mathscr{M} \otimes \mathscr{L}(z\zeta)\big)_{|\{0\}\times\widehat{U}_\infty} \simeq \mathscr{P}_0^{(\infty)} \widehat{E}_{\widehat{U}_\infty}$.

How to describe $V_{-1+a} p_{2\dagger}\big(p_1^* \mathscr{M} \otimes \mathscr{L}(z\zeta)\big)_{|\{0\}\times\widehat{U}_\infty} \simeq \mathscr{P}_a^{(\infty)} \widehat{E}_{\widehat{U}_\infty}$?

In general, $\mathscr{C}_a^\bullet$ and $\Big(\overline{\Omega}^\bullet_{\mathbb{P}^1} \otimes V_{a-1}\big(p_1^*\mathscr{M} \otimes \mathscr{L}(z\zeta)\big)\Big)_{|\{0\}\times\mathbb{P}^1\times\widehat{U}_\infty}$ are not quasi-isomorphic.

To compare the parabolic filtrations, it is convenient to consider the mixed twistor $\mathscr{D}$-module $\mathscr{T}[!0]$ (i.e., $\iota_! \iota^* \mathscr{T}$ for $\iota : \mathbb{P}^1 \setminus \{0\} \longrightarrow \mathbb{P}^1$).
Let $\mathscr{M}[!0]$ denote the $\mathscr{R}$-modules underlying $\mathscr{T}[!0]$. We have the morphisms:

$$V_{a-1} p_{2\dagger}\Big(p_1^*(\mathscr{M}[!0]) \otimes \mathscr{L}(z\zeta)\Big) \longrightarrow V_{a-1} p_{2\dagger}\Big(p_1^*(\mathscr{M}) \otimes \mathscr{L}(z\zeta)\Big)$$

For $-1 < a < 0$, the induced morphisms

$$\mathrm{Gr}^V_{a-1}\, p_{2\dagger}\Big(p_1^*(\mathscr{M}[!0]) \otimes \mathscr{L}(z\zeta)\Big) \longrightarrow \mathrm{Gr}^V_{a-1}\, p_{2\dagger}\Big(p_1^*(\mathscr{M}) \otimes \mathscr{L}(z\zeta)\Big)$$

are isomorphisms.

Proposition

For $-1 < a < 0$, we have the following quasi-isomorphism:

$$\mathscr{C}_a^\bullet \longrightarrow \Big(\overline{\Omega}^\bullet_{\mathbb{P}^1} \otimes V_{a-1}(\mathscr{M}[!0] \otimes \mathscr{L}(z\zeta))\Big)_{|\{0\}\times\mathbb{P}^1\times\widehat{U}_\infty}.$$

$$
\begin{aligned}
&\mathfrak{F}_a\big(\mathscr{P}_0^{(\infty)}\widehat{E}_{\widehat{U}_\infty|\infty}\big)\\
&\qquad= \operatorname{Im}\Big(V_{-1+a}\,p_{2\dagger}\big(p_1^*\mathscr{M}\otimes\mathscr{L}(z\tau)\big)_{|(0,\infty)}\longrightarrow V_{-1}\,p_{2\dagger}\big(p_1^*\mathscr{M}\otimes\mathscr{L}(z\tau)\big)_{|(0,\infty)}\Big)\\
&\qquad= \operatorname{Im}\Big(V_{-1+a}\,p_{2\dagger}\big(p_1^*\mathscr{M}[!0]\otimes\mathscr{L}(z\tau)\big)_{|(0,\infty)}\longrightarrow V_{-1}\,p_{2\dagger}\big(p_1^*\mathscr{M}\otimes\mathscr{L}(z\tau)\big)_{|(0,\infty)}\Big)\\
&\qquad\qquad= \operatorname{Im}\Big(R^2p_{2*}(\mathscr{C}_a^\bullet)_{|\infty}\longrightarrow R^2p_{2*}(\mathscr{C}_0^\bullet)_{|\infty}\Big) = \mathfrak{F}\big(\widehat{E}^{\mathrm{alg}}_{\widehat{U}_\infty|\infty}\big)
\end{aligned}
$$

Theorem of Donagi-Pantev-Simpson

Let $\Delta := \{t \in \mathbb{C} \mid |t| < 1\}$. Let X be a complex surface with a projective morphism $F : X \longrightarrow \Delta$. Assume the following:

- **$F^{-1}(\Delta \setminus \{0\}) \longrightarrow \Delta \setminus \{0\}$ is smooth.**
- **The divisor $F^{-1}(0)$ is normal crossing.**

Let $F^{-1}(0) = \bigcup_{j \in \Lambda_0} C_j$ be the irreducible decomposition.

Let $(E, \overline{\partial}_E, \theta, h)$ be a harmonic bundle on $X \setminus F^{-1}(0)$ which is "tame" along $F^{-1}(0)$. We also assume that $(E, \overline{\partial}_E, \theta, h)_{|F^{-1}(P)}$ are irreducible for any $P \in \Delta \setminus \{0\}$ with $\operatorname{rank} E > 1$.

- **We obtain vector bundle $^{\diamond}E$ on X by considering local holomorphic sections s such that $|s|_h = O(|F|^{-\varepsilon})$ for any $\varepsilon > 0$. We also have locally free sheaves $\mathscr{P}_{\boldsymbol{a}}E$ for any $\boldsymbol{a} \in \mathbb{R}^{\Lambda_0}$.**
- **Under the tamenees assumption, θ is logarithmic with respect to $^{\diamond}E$. We have the endomorphisms $\operatorname{Res}_{C_i}(\theta)$ of $^{\diamond}E_{|C_i}$. We assume that $\operatorname{Res}_{C_i}(\theta)$ are nilpotent.**

Set $\Delta^* := \Delta \setminus \{0\}$. On $X^* := X \setminus F^{-1}(0)$, we have the relative Higgs complex $E \otimes \Omega^\bullet_{X^*/\Delta^*}$.

- $E_1 := R^1 F_*(E \otimes \Omega^\bullet_{X^*/\Delta^*})$ **is a locally free $\mathscr{O}_{\Delta^*}$-module.**
- **It is naturally equipped with Higgs field θ_1, Hermitian metric h_1, so that (E_1, θ_1, h_1) is a harmonic bundle on Δ^*.**
- **We have the associated parabolic bundle. Namely, for any $a \in \mathbb{R}$, we have the locally free sheaf $\mathscr{P}_a E_1$ obtained as the extension of E_1 across 0, whose sections are characterized by the condition $|s|_{h_1} = O(|z|^{-a-\varepsilon})$ for any $\varepsilon > 0$.**

Donagi, Pantev and Simpson asked how $\mathscr{P}_a E_1$ are obtained directly from the tuple $(\mathscr{P}_{\boldsymbol{b}} E \,|\, \boldsymbol{a} \in \mathbb{R}^{\Lambda_0})$ and θ.

They introduced the following complex.
For each $a \in \mathbb{R}$, we have the following vector bundle $\mathscr{P}_{F,a}E$ obtained as the extension across $F^{-1}(0)$ whose local sections s are characterized by the growth condition $|s|_h = O(|F|^{-a-\varepsilon})$ for any $\varepsilon > 0$. We have $^{\diamond}E = \mathscr{P}_{F,0}E$.
We have the following complex $\mathrm{Dol}_a(E,\theta,h)$:

$$\mathscr{P}_{F,a}E \xrightarrow{\quad\theta\quad} \mathscr{P}_{F,a}E \otimes \Omega^1_{X/\Delta}(\log F)$$

Theorem (Donagi-Pantev-Simpson)

$$\mathscr{P}_a E_1 \simeq R^1 F_* \,\mathrm{Dol}_a(E,\theta,h).$$

Let $\mathfrak{T}$ be the pure twistor $\mathscr{D}$-module associated to $(E, \overline{\partial}_E, \theta, h)$. Let $\mathfrak{E}$ denote the underlying $\mathscr{R}$-module.
Let $\iota_F : X \longrightarrow X \times \Delta$ be the graph embedding.

> *Theorem (Donagi-Pantev-Simpson)*
>
> **For any $a < 1$, we have a natural quasi-isomorphism:**
>
> $$\iota_{F*} \mathrm{Dol}_a^\bullet(E, \theta, h) \longrightarrow \overline{\Omega}_X^\bullet \otimes \left(V_{a-1} \iota_{F\dagger} \mathfrak{E}\right)_{|\{0\}\times X \times \Delta}.$$

- **We obtain $R^1 F_*\big(\mathrm{Dol}_a^\bullet(E, \theta, h)\big) \simeq V_{a-1} F_\dagger^0(\mathfrak{E})_{|\{0\}\times\Delta}$.**
- **The theorem of Donagi-Pantev-Simpson follows from the relation between V-filtrations and the parabolic filtration.**