

# **A factorization theorem for rank-two irregular flat connections**

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## Flat rank 2 connections

Let  $X$  be a smooth projective variety over  $\mathbb{C}$ .

**DEFINITION:** A flat meromorphic connection  $(E, \nabla)$  of rank 2 is

- $E \rightarrow X$  is a rank 2 vector bundle,
- $D > 0$  an effective (polar) divisor on  $X$ ,
- $\nabla : E \rightarrow E \otimes \Omega_X^1(D)$  connection,
- flatness condition:  $\nabla \cdot \nabla = 0$ .

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$$E|_U \xrightarrow{\sim} \mathbb{C}^2$$

- $D > 0$  an effective (polar) divisor on  $X$ ,

$$A \in \mathfrak{gl}(\Omega_U^1(D))$$

- $\nabla : E \rightarrow E \otimes \Omega_X^1(D)$  connection,

$$\nabla|_U : Y \mapsto dY + AY$$

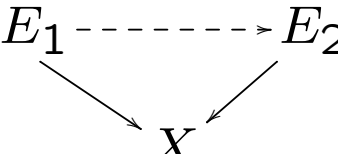
- flatness condition:  $\nabla \cdot \nabla = 0$ .

$$dA + A \wedge A = 0$$

## Projective and birational equivalence

**DEFINITION:** 1) **Projective** equivalence:  $(E_1, \nabla_1) \sim_{\text{proj}} (E_2, \nabla_2)$   
 $\Leftrightarrow \exists (L, \zeta)$  rank 1 flat meromorphic connection s.t.  $(E_2, \nabla_2) = (L, \zeta) \otimes (E_1, \nabla_1)$ .

2) **Birational** equivalence:  $(E_1, \nabla_1) \sim_{\text{bir}} (E_2, \nabla_2)$   
 $\Leftrightarrow \exists \phi : E_1 \dashrightarrow E_2$  birational bundle transformation such that  $\nabla_1 = \phi^* \nabla_2$ .



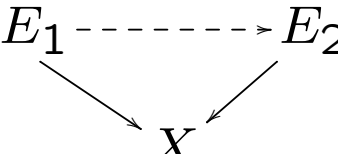
3) **Projective-birational** equivalence:  $(E_1, \nabla_1) \sim (E_2, \nabla_2)$   
 $\Leftrightarrow \exists (E_3, \nabla_3)$  s.t.  $(E_1, \nabla_1) \sim_{\text{proj}} (E_3, \nabla_3) \sim_{\text{bir}} (E_2, \nabla_2)$ .

## Projective and birational equivalence

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$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \sim_{\text{proj}} \begin{pmatrix} \alpha + \omega & \beta \\ \gamma & \delta + \omega \end{pmatrix}$$

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$$A \sim_{\text{bir}} M^{-1}dM + M^{-1}AM, \quad M \in \text{GL}_2(\mathcal{M}_X)$$

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## Motivation

Let  $(E \rightarrow X, \nabla : E \rightarrow E \otimes \Omega_X^1(D))$  flat meromorphic connection of rank 2.

### THEOREM: Corlette-Simpson

Assume

- $D$  has simple normal crossing
- $\nabla$  is regular-singular
- $\nabla$  has rational residual eigenvalues along  $D$
- its monodromy has Zariski dense image in  $\mathrm{PSL}_2(\mathbb{C})$ .

Then we are in one of the following cases:

$\exists f : X \dashrightarrow C$  (curve) such that  $(E, \nabla) \sim f^*(E_0, \nabla_0)$   
(for a meromorphic connection  $(E_0, \nabla_0)$  on  $C$ );

$\exists \phi : X \rightarrow \mathfrak{H}$  Shimura polydisk and  $(E, \nabla) \sim f^*(E_0, \nabla_0)$   
for one of the tautological connections  $(E_0, \nabla_0)$  on  $\mathfrak{H}$ .

**Result**

Let  $(E \rightarrow X, \nabla : E \rightarrow E \otimes \Omega_X^1(D))$  flat meromorphic connection of rank 2.

**THEOREM: Corlette-Simpson + Cousin-Pereira + L.-Pereira-Touzet**

Assume nothing more

Then we are in one of the following cases:

(1)  $\exists f : X \dashrightarrow C$  (curve) such that  $(E, \nabla) \sim f^*(E_0, \nabla_0)$

(for a meromorphic connection  $(E_0, \nabla_0)$  on  $C$ );

(2)  $\exists \phi : \tilde{X} \rightarrow X$  generically finite such that  $\phi^*(E, \nabla) \sim (\mathcal{O}_{\tilde{X}} \oplus \mathcal{O}_{\tilde{X}}, \nabla_0)$

with  $\nabla_0 = d + \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}$  or  $\begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix}$ ,  $\omega$  closed meromorphic 1-form;

(3)  $\exists \phi : X \rightarrow \mathfrak{H}$  Shimura polydisk and  $(E, \nabla) \sim f^*(E_0, \nabla_0)$

for one of the tautological connections  $(E_0, \nabla_0)$  on  $\mathfrak{H}$ .

**Today:**

Let  $(E \rightarrow X, \nabla : E \rightarrow E \otimes \Omega_X^1(D))$  flat meromorphic connection of rank 2.

**THEOREM: Corlette-Simpson + Cousin-Pereira + L.-Pereira-Touzet**

Assume  $\nabla$  **irregular**

Then we are in one of the following cases:

(1)  $\exists f : X \dashrightarrow C$  (curve) such that  $(E, \nabla) \sim f^*(E_0, \nabla_0)$

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**First reductions:**

Can pass to generically finite maps  $\phi : \tilde{X} \rightarrow X$  if needed.

Can assume  $X$  is a surface.

**Sabbah's good formal model:**

Now  $X$  is a smooth projective surface and  $(E, \nabla)$  a meromorphic connection.

**THEOREM:** Up to a generically finite map and projective-birational equivalence, we can assume  $D$  simple normal crossing and at any  $p \in |D|$  we have one of the following models:

$$(1) \ d + \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} \text{ with } \omega = \lambda_x \frac{dx}{x} + \lambda_y \frac{dy}{y};$$

$$(2) \ d + \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix} \text{ with } \omega = \lambda_x \frac{dx}{x} + \lambda_y \frac{dy}{y};$$

$$(3) \ d + \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} \text{ with } \omega = \frac{df}{f^{\kappa+1}} + \lambda_x \frac{dx}{x} + \lambda_y \frac{dy}{y};$$

$$(4) \ d + \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} + \underbrace{\begin{pmatrix} * & * \\ * & * \end{pmatrix}}_{\text{holomorphic}} \text{ with } \omega = \frac{df}{f^{\kappa+1}} + \lambda \frac{df}{f}.$$

where  $(x, y)$  local coordinates,  $f = x^p y^q$  and  $\lambda_x, \lambda_y, \lambda \in \mathbb{C}$ .