A factorization theorem for rank-two irregular flat connections

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Flat rank 2 connections

Let X be a smooth projective variety over $\mathbb{C}$.

DEFINITION: A flat meromorphic connection (E, ∇) of rank 2 is

- $E \to X$ is a rank 2 vector bundle,
- $D > 0$ an effective (polar) divisor on X,
- $\nabla : E \to E \otimes \Omega^1_X(D)$ connection,
- flatness condition: $\nabla \cdot \nabla = 0$.

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- $E \to X$ is a rank 2 vector bundle, $E|_U \stackrel{\sim}{\to} \mathbb{C}^2$
- $D > 0$ an effective (polar) divisor on X, $A \in \mathfrak{gl}(\Omega^1_U(D))$
- $\nabla : E \to E \otimes \Omega^1_X(D)$ connection, $\nabla|_U : \; Y \mapsto dY + AY$
- flatness condition: $\nabla \cdot \nabla = 0$. $dA + A \wedge A = 0$

Projective and birational equivalence

DEFINITION: 1) Projective equivalence: $(E_1, \nabla_1) \sim_{\text{proj}} (E_2, \nabla_2)$
$\Leftrightarrow \exists (L, \zeta)$ rank 1 flat meromorphic connection s.t. $(E_2, \nabla_2) = (L, \zeta) \otimes (E_1, \nabla_1)$.

2) Birational equivalence: $(E_1, \nabla_1) \sim_{\text{bir}} (E_2, \nabla_2)$
$\Leftrightarrow \exists\, \phi : E_1 \dashrightarrow E_2$ birational bundle tranformation such that $\nabla_1 = \phi^* \nabla_2$.

$$\begin{array}{ccc} E_1 & \dashrightarrow & E_2 \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

3) Projective-birational equivalence: $(E_1, \nabla_1) \sim (E_2, \nabla_2)$
$\Leftrightarrow \exists (E_3, \nabla_3)$ s.t. $(E_1, \nabla_1) \sim_{\text{proj}} (E_3, \nabla_3) \sim_{\text{bir}} (E_2, \nabla_2)$.

Projective and birational equivalence

DEFINITION: 1) Projective equivalence: $(E_1,\nabla_1) \sim_{\text{proj}} (E_2,\nabla_2)$
$\Leftrightarrow \exists (L,\zeta)$ rank 1 flat meromorphic connection s.t. $(E_2,\nabla_2) = (L,\zeta)\otimes(E_1,\nabla_1)$.

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \sim_{\text{proj}} \begin{pmatrix} \alpha+\omega & \beta \\ \gamma & \delta+\omega \end{pmatrix}$$

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$$A \ \sim_{\text{bir}} \ M^{-1}dM + M^{-1}AM, \quad M \in \mathrm{GL}_2(\mathcal{M}_X)$$

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Motivation

Let $(E \to X, \nabla : E \to E \otimes \Omega^1_X(D))$ flat meromorphic connection of rank 2.

THEOREM: Corlette-Simpson
Assume
- D has simple normal crossing
- ∇ is regular-singular
- ∇ has rational residual eigenvalues along D
- its monodromy has Zariski dense image in $\mathrm{PSL}_2(\mathbb{C})$.

Then we are in one of the following cases:

$\exists\ f : X \dashrightarrow C$ (curve) such that $(E, \nabla) \sim f^*(E_0, \nabla_0)$
(for a meromorphic connection (E_0, ∇_0) on C);

$\exists\ \phi : X \to \mathfrak{H}$ Shimura polydisk and $(E, \nabla) \sim f^*(E_0, \nabla_0)$
for one of the tautological connections (E_0, ∇_0) on $\mathfrak{H}$.

Result

Let $(E \to X, \nabla : E \to E \otimes \Omega^1_X(D))$ flat meromorphic connection of rank 2.

THEOREM: Corlette-Simpson + Cousin-Pereira + L.-Pereira-Touzet
Assume nothing more

Then we are in one of the following cases:

(1) $\exists\ f : X \dashrightarrow C$ (curve) such that $(E,\nabla) \sim f^*(E_0,\nabla_0)$
(for a meromorphic connection (E_0,∇_0) on C);
(2) $\exists\ \phi : \tilde{X} \to X$ generically finite such that $\phi^*(E,\nabla) \sim (\mathcal{O}_{\tilde{X}} \oplus \mathcal{O}_{\tilde{X}}, \nabla_0)$
with $\nabla_0 = d + \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}$ or $\begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix}$, ω closed meromorphic 1-form;
(3) $\exists\ \phi : X \to \mathfrak{H}$ Shimura polydisk and $(E,\nabla) \sim f^*(E_0,\nabla_0)$
for one of the tautological connections (E_0,∇_0) on $\mathfrak{H}$.

Today:

Let $(E \to X, \nabla : E \to E \otimes \Omega^1_X(D))$ flat meromorphic connection of rank 2.

THEOREM: Corlette-Simpson + Cousin-Pereira + L.-Pereira-Touzet
Assume ∇ **irregular**

Then we are in one of the following cases:

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First reductions:

Can pass to generically finite maps $\phi : \tilde{X} \to X$ if needed.

Can assume X is a surface.

Sabbah's good formal model:

Now X is a smooth projective surface and (E, ∇) a meromorphic connection.

THEOREM: Up to a generically finite map and projective-birational equivalence, we can assume D simple normal crossing and at any $p \in |D|$ we have one of the following models:

(1) $d + \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}$ with $\omega = \lambda_x \frac{dx}{x} + \lambda_y \frac{dy}{y}$;

(2) $d + \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix}$ with $\omega = \lambda_x \frac{dx}{x} + \lambda_y \frac{dy}{y}$;

(3) $d + \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}$ with $\omega = \frac{df}{f^{\kappa+1}} + \lambda_x \frac{dx}{x} + \lambda_y \frac{dy}{y}$;

(4) $d + \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} + \underbrace{\begin{pmatrix} * & * \\ * & * \end{pmatrix}}_{\text{holomorphic}}$ with $\omega = \frac{df}{f^{\kappa+1}} + \lambda \frac{df}{f}$.

where (x, y) local coordinates, $f = x^p y^q$ and $\lambda_x, \lambda_y, \lambda \in \mathbb{C}$.