Rational solutions of linear Mahler equations

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Some Mahler equations

- Integers partitions:
  $b \geq 2$

  $$u(n) = \# \text{ of partitions} \quad n = \sum_{k \geq 0} \varepsilon_k b^k \quad \text{with } \varepsilon_k \in \mathbb{N}_{\geq 0}$$

  $$u(x) = \sum_{n \geq 0} u(n) x^n = \prod_{j \geq 0} \frac{1}{1 - x^{b^j}}$$

  $$(1 - x)\boldsymbol{u(x) = u(x^b)}$$

Mahler operator
$\boldsymbol{y(x) \longmapsto y(x^b)}$

Some Mahler equations

- Integers partitions
- Words:

$$u(n) = \# \text{ of 1's in the binary expansion of } n$$

$u = 0, 1, 1, 2, 1, 2, 2, 3, 1, 2, 2, 3, 2, 3, 3, 4, 1, 2, 2, 3, 2 \ldots$

$$\boldsymbol{u(x)} = (1+x)\boldsymbol{u(x^2)} + \frac{x}{1-x^2}$$

Mahler operator

$\boldsymbol{y(x)} \longmapsto \boldsymbol{y(x^2)}$

Some Mahler equations

- Integers partitions
- Words
- Divide-and-Conquer strategy:

$u(n) =$ # of ring multiplications used to multiply two degree-n polynomials in Karatsuba's algorithm

$u = 1, 3, 7, 9, 17, 21, 25, 27, 43, 51, 59, 63, 71, 75, 79, 81, 113, 129 \ldots$

$$\boldsymbol{u(x)} = (1+x)(2+x)\boldsymbol{u(x^2)} - 1$$

Mahler operator

$\boldsymbol{y(x)} \longmapsto \boldsymbol{y(x^2)}$

Some Mahler equations

- Integers partitions
- Words
- Divide-and-Conquer strategy
- Transcendence

Kurt Mahler
1903–1988

Some Mahler equations

- Integers partitions
- Words
- Divide-and-Conquer strategy
- Transcendence: Fredholm series
  - transcendental function
    $$u(x) = \sum_{k=0}^{+\infty} x^{2^k}$$
  - Mahler equation
    $$u(x) = x + u(x^2)$$
  - algebraic number $0 < |\alpha| < 1$

then $u(\alpha) = \sum_{k=0}^{+\infty} \alpha^{2^k}$ transcendental

Kurt Mahler
1903–1988

Mahler operator
$$y(x) \longmapsto y(x^2)$$

Motivation

- transcendence revival
  (Loxton and Van der Poorten 1977
  Adamczewski & Faverjon 2015, Bell & Coons 2015,
  Dreyfus, Hardouin & Roques 2015, Roques 2015, 2016)
- classical topic in computer algebra
  (Liouville 1833, Abramov 1989. . .
  PhD 1993)

Rational solutions

Classical approach:

- search for a denominator
- solution for numerator

We search for
(a common denominator of the)
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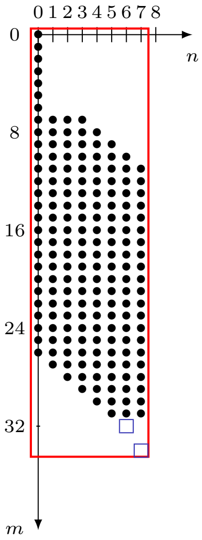

A geometric intuition

$$\ell_0(x)y(x) + \ell_1(x)My(x) + \cdots + \ell_r(x)M^r y(x) = 0$$

linear homogeneous equation with polynomial coefficients and $\ell_0\ell_r \neq 0$

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$$\left(\ell_0 \frac{p}{q} + \ell_1 \frac{Mp}{Mq} + \cdots + \ell_{r-1} \frac{M^{r-1}p}{M^{r-1}q}\right) + \ell_r \frac{M^r p}{M^r q} = 0$$

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multiplication by $M^r q \times \mu_r(q)$, $\qquad \mu_r(q) = \operatorname{lcm}_{0 \leq j < r} M^j q$

$$(\ell_0 \tilde{\mu}_0 p + \ell_1 \tilde{\mu}_1 Mp + \cdots + \ell_{r-1} \tilde{\mu}_{r-1} M^{r-1} p) + \ell_r \mu_r(q) M^r p = 0.$$

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$$\left.\begin{array}{ll} M^r q \mid \tilde{\mu}_k, & 0 \leq k < r \\ p,\ q \text{ coprime} & \end{array}\right\} \Rightarrow M^r q \mid \ell_r\mu_r(q)$$

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$$q(\alpha) = 0 \text{ and } \beta^{b^r} = \alpha \implies \ell_r(\beta) = 0 \text{ or } \left(q(\alpha') = 0 \text{ for some } \alpha' = \beta^{b^j},\ j < r\right)$$

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$M^r q \mid \ell_r$ except if…

Action on polynomials

Mahler $\quad f(x) \xrightarrow{\;M\;} f(x^b)$

root α $\qquad$ roots $\alpha^{1/b}$

Action on polynomials

Mahler	$f(x)$	$\xrightarrow{M}$	$f(x^b)$
	root α		roots $\alpha^{1/b}$
Gräffe	$\mathrm{Res}_y(y^b - x, g(y))$	$\xleftarrow{G}$	$g(x)$
	root β^b		root β

Action on polynomials

oriented graph:

- node = prime polynomial
- edge from f to Gf
  (as a matter of fact $\sqrt{G}$)

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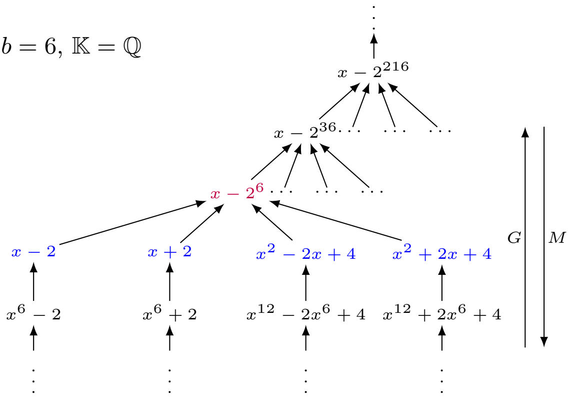
$b = 6$, $\mathbb{K} = \mathbb{Q}$



bi-infinite tree

Action on polynomials

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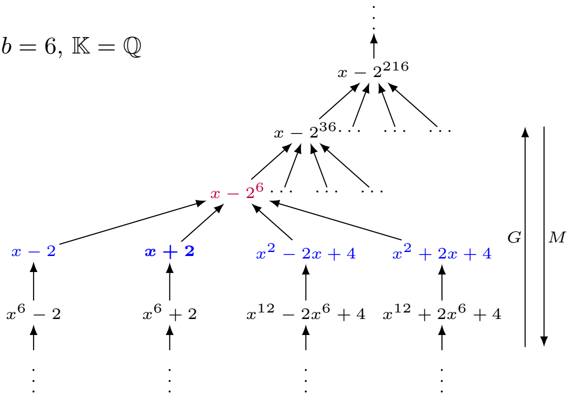

$G(x+2) = x - 2^6$

bi-infinite tree

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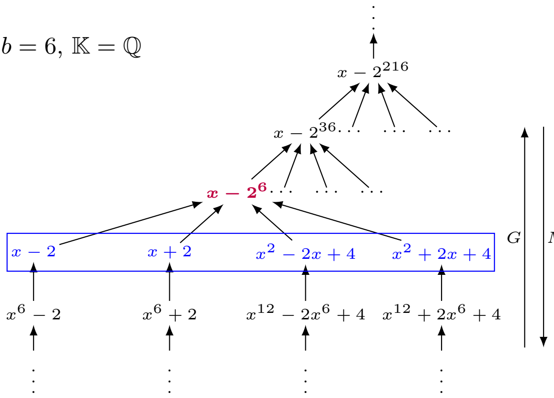

$M(x - 2^6) = (x - 2)(x + 2)$
$\times(x^2 - 2x + 4)(x^2 + 2x + 4)$

bi-infinite tree

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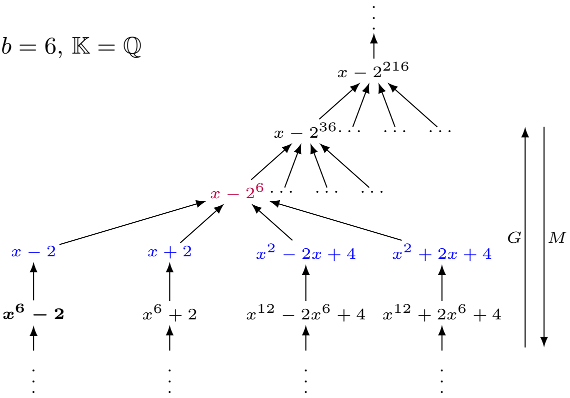

$G(x^6 - 2) = (x - 2)^6$

$\deg Gf = \deg f$

$\deg Mf = b \deg f$

bi-infinite tree

Action on polynomials

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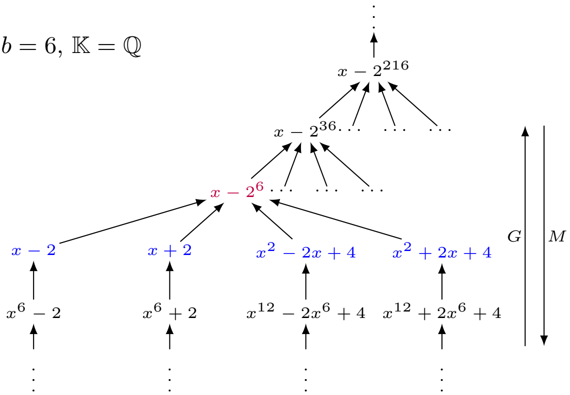

bi-infinite tree

fixed point

Action on polynomials

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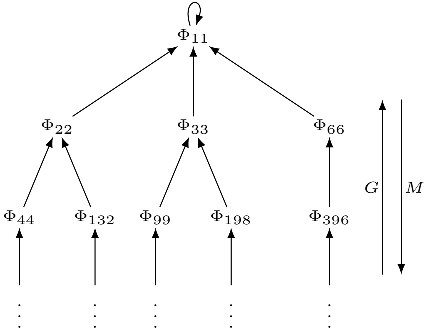
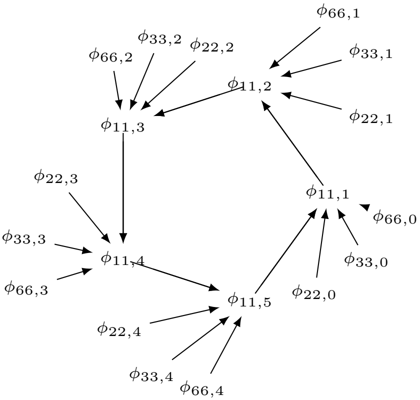


$b = 6$, $\mathbb{K} = \mathbb{Q}(\alpha)$



$$\alpha = \frac{\zeta + \zeta^{10}}{2}, \ \phi = x^2 - 2\omega x + 1$$

infinite tree with a loop at its root

infinite trees rooted on a cycle

cyclotomic case

Searching for a denominator

$$\ell_0(x)y(x) + \ell_1(x)My(x) + \cdots + \ell_r(x)M^r y(x) = 0$$

$$\mu_r(q) = \operatorname*{lcm}_{0 \leq j < r} M^j q$$

Proposition

Let q be a nonconstant polynomial, with $q(0) \neq 0$, such that

- $M^r q \mid \ell_r \, \mu_r(q)$.

Then there exists a nonconstant polynomial u such that:

1. *either* $M^r u \mid \ell_r$,
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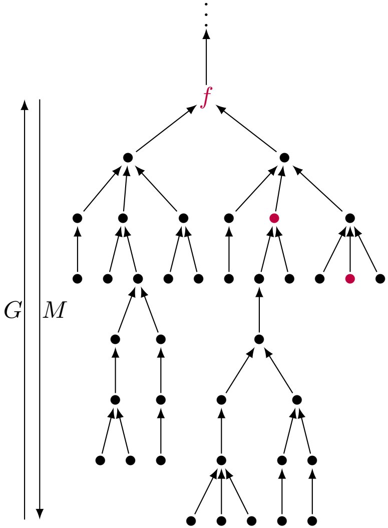
Sketch of proof

$M^r q \mid \ell_r \mu_r(q)$



f a prime factor of q

• prime factors of q

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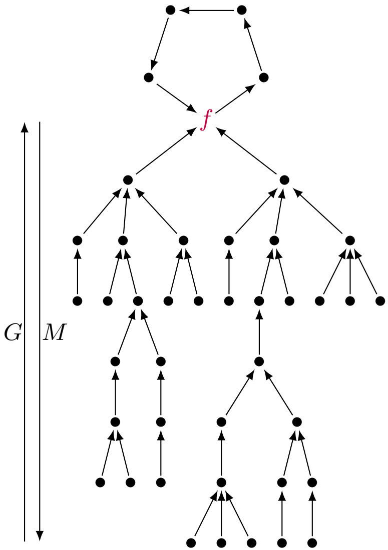

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Pure cyclic case:

(all prime factors in cycles)

$\forall f \; \exists \tilde{f}$

- $\forall j > 0,\ G^j \tilde{f} \neq \tilde{f}$
- no factor of q in the subtree rooted at $\tilde{f}$

then

- $M^{r-1} \tilde{f} \mid M^r f \mid M^r q$
- with $u = \tilde{f}$
  $M^{r-1} u \mid \ell_r$ and $G u = f^{\geq 1}$

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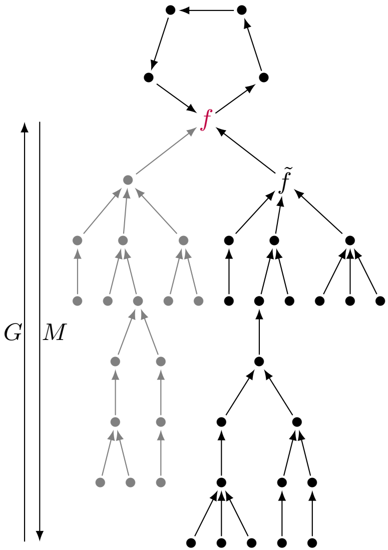

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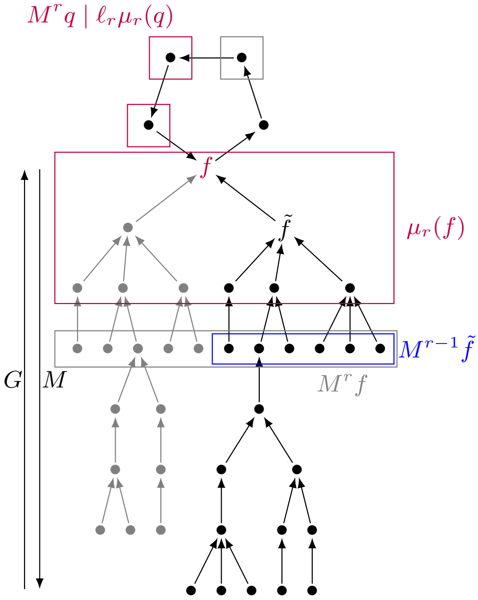

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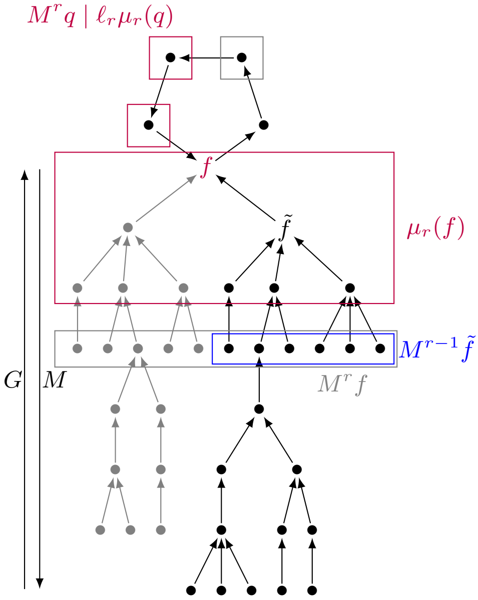

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- with $u = \tilde f$
  $M^{r-1}u \mid \ell_r$ and $Gu = f^{\geq 1}$
- with $u = \prod \tilde f_i^{m_i}$ if $q = \prod f_i^{m_i}$
  $M^{r-1}u \mid \ell_r$ and $q \mid Gu$

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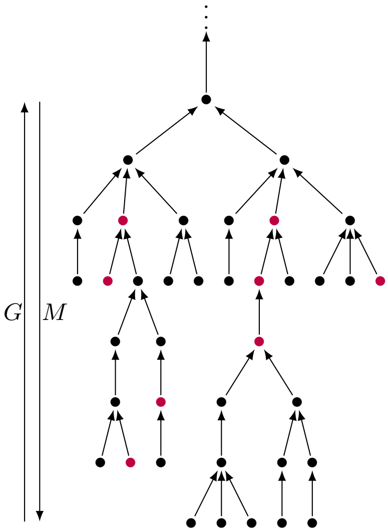
$M^r q \mid \ell_r \mu_r(q)$



f a prime factor of q

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Generic case:
(not all prime factors in cycles) then

- with f prime factor of q not in a cycle and without prime factor of q below it
- $M^r f \mid M^r q$
- with $u = f$, $M^r u \mid \ell_r$

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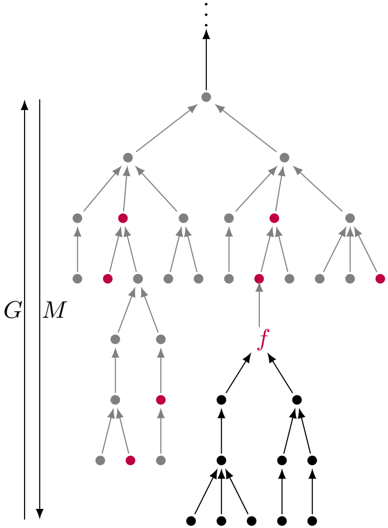

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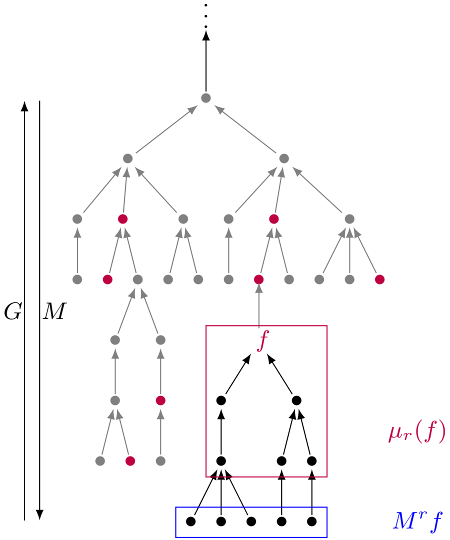

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Computing a denominator bound

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Let q be a nonconstant polynomial, with $q(0) \neq 0$, such that

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sections of ℓ: $$\ell(x) = \sum_{s=0}^{b-1} x^s T_{b,s} \ell(x^b)$$

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sections of ℓ: $\qquad \ell(x) = \sum_{s=0}^{b-1} x^s T_{b,s} \ell(x^b) \qquad \ell(x) = T_{2,0}\ell(x^2) + x T_{2,1}(x^2)$

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$$\text{sections of } \ell: \qquad \ell = \sum_{s=0}^{b-1} x^s M T_{b,s} \ell \qquad \ell = M T_{2,0} \ell + x M T_{2,1}$$

Lemma

The polynomial ℓ is divisible by Mu if and only if all its sections $T_{b,s}\ell$ are divisible by u, that is $\gcd_{s=0}^{b-1} T_{b,s}\ell$ is divisible by u .

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Algorithm

In A linear Mahler operator L of order r with $\ell_0 \neq 0$.

Out A denominator q^* for the rational functions solutions of $Ly = 0$.

1. Set $\ell = \ell_r$, then repeat for $k = 1, 2, \ldots$:

   a set $u_k = \gcd_{s=0}^{b^r-1} T_{b^r,s} \ell$
   b set $\ell = (\ell / M^r u_k) \mu_r(u_k)$

   until $\deg u_k = 0$.

2. Set $\tilde{u} = \gcd_{s=0}^{b^{r-1}-1} T_{b^{r-1},s} \ell$.
3. Return $q^* = u_1 \cdots u_{k-1} \, G\tilde{u}$.

Computing a denominator bound

$$L = \ell_0 + \ell_1 M + \cdots + \ell_r M^r$$

$$\deg_x L = d \qquad\qquad \deg_M L = r$$

Theorem

Assume $Ly = 0$ *with* $y = \dfrac{p}{x^v q}$ *and* $q(0) \neq 0$. *Then,* $q \mid q^*$.

Computing a denominator bound

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Theorem

With $D = d$ *if* $b = 2$, $D = d/b^{r-1}$ *if* $b \geq 3$,

- $\deg q^* \leq D \leq d$
- *algorithm runs in* $\mathrm{O}(D\ \mathrm{M}(d) \log d)$ *ops*

Example

$$L = \qquad\qquad b = 2, \quad \mathbb{K} = \mathbb{Q}$$

$$(x^{15} - 3x^{14} - 3x^{13} - x^{12} - 3x^{11} + 9x^{10} + 9x^{9} + 3x^{8} + x^{7} - 3x^{6} - 3x^{5} - x^{4} - 3x^{3} + 9x^{2} + 9x + 3)M^2$$
$$+ (-x^{15} + 2x^{14} + 5x^{13} - 3x^{12} - 4x^{11} - 3x^{10} - 14x^{8} - 15x^{7} - 8x^{6} - 5x^{5} - 11x^{4} - 12x^{3} - 3x^{2})M$$
$$+ (x^{14} - 2x^{13} - 5x^{12} + 4x^{11} + 12x^{9} + 14x^{8} + 8x^{7} + 11x^{6} + 2x^{5} + 3x^{4}).$$

$$L\frac{x}{x^2+1} = 0, \qquad L\frac{x}{x-3} = 0$$

$$q = (x^2+1)(x-3)$$

Example

$$r = 2 \qquad \ell_2 = (x^8 + 1)(x^4 - 3)(x^3 - 3x^2 - 3x - 1) \qquad q = (x^2 + 1)(x - 3)$$

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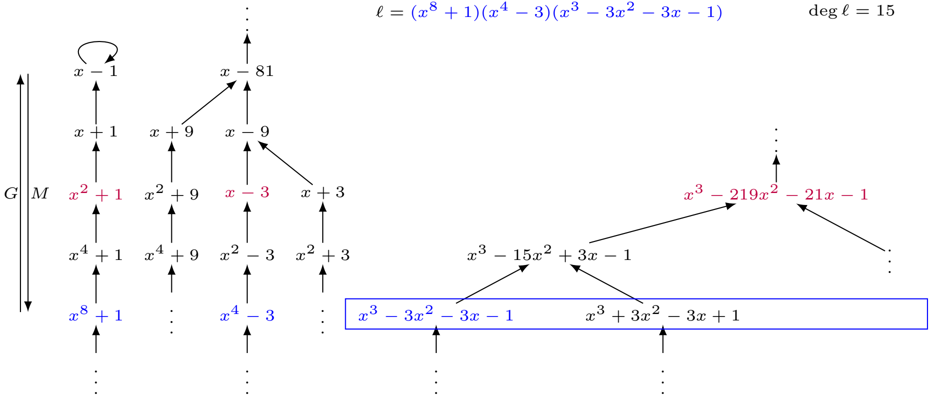
$\ell = (x^8+1)(x^4-3)(x^3-3x^2-3x-1) \qquad \deg \ell = 15$



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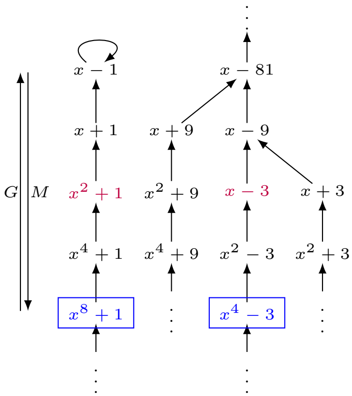


$\ell = (x^8 + 1)(x^4 - 3)(x^3 - 3x^2 - 3x - 1) \qquad \deg \ell = 15$

$u_1 = (x^2 + 1)(x - 3) = x^3 - 3x^2 + x - 3$

$$\ell = x^{15} - 3x^{14} - 3x^{13} - x^{12} - 3x^{11} + 9x^{10} + 9x^9 + 3x^8 + x^7 - 3x^6 - 3x^5 - x^4 - 3x^3 + 9x^2 + 9x + 3$$

$T_{4,0}\ell = -x^3 + 3x^2 - x + 3$

$T_{4,1}\ell = -3x^3 + 9x^2 - 3x + 9$

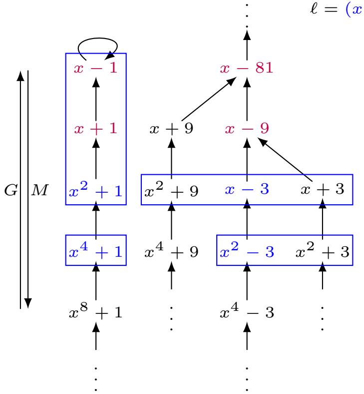
$T_{4,2}\ell = -3x^3 + 9x^2 - 3x + 9$

$T_{4,3}\ell = x^3 - 3x^2 + x - 3$

Example

$r = 2 \qquad \ell_2 = (x^8+1)(x^4-3)(x^3-3x^2-3x-1) \qquad q = (x^2+1)(x-3)$

$\ell = (x^2+1)(x^4+1)(x-3)(x^2-3)(x^3-3x^2-3x-1) \quad \deg \ell = 12$



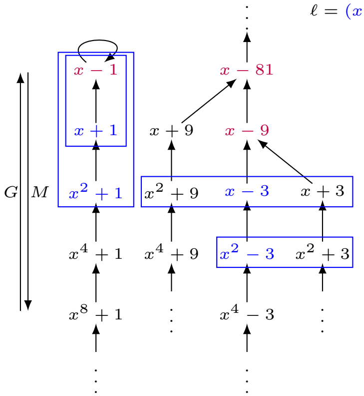
$u_1 = (x^2+1)(x-3)$

$u_2 = (x+1)$

Example

$$r = 2 \qquad \ell_2 = (x^8+1)(x^4-3)(x^3-3x^2-3x-1) \qquad q = (x^2+1)(x-3)$$

$$\ell = (x+1)(x^2+1)^2(x-3)(x^2-3)(x^3-3x^2-3x-1) \qquad \deg \ell = 11$$



$$u_1 = (x^2+1)(x-3)$$

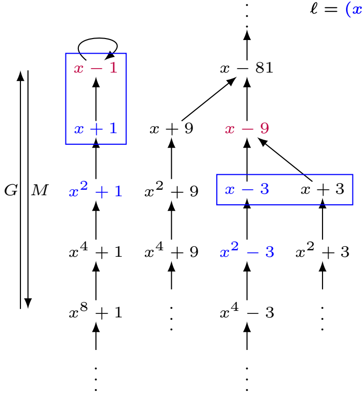
$$u_2 = (x+1)$$

$$u_3 = 1$$

Example

$r = 2$ $\ell_2 = (x^8+1)(x^4-3)(x^3-3x^2-3x-1)$ $q = (x^2+1)(x-3)$

$\ell = (x+1)(x^2+1)^2(x-3)(x^2-3)(x^3-3x^2-3x-1)$ $\deg \ell = 11$



$u_1 = (x^2+1)(x-3)$

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$u_3 = 1$

$\tilde{u} = (x+1)^2(x-3)$

Example

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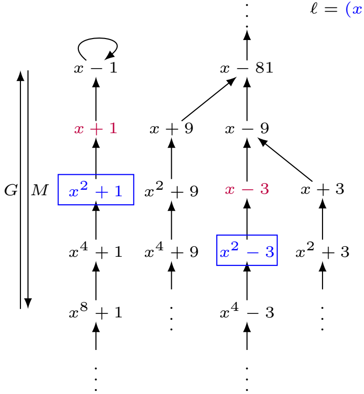

$u_1 = (x^2+1)(x-3)$

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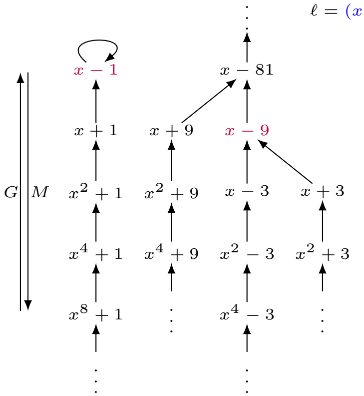

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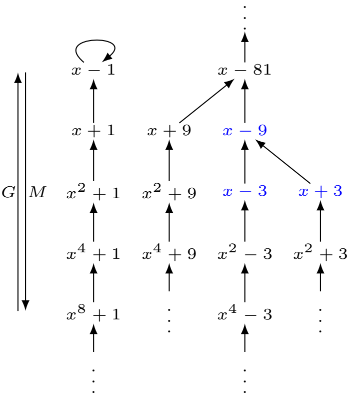
$\ell = (x-3)(x+3)(x-9)(x^3-3x^2-3x-1) \qquad \deg \ell = 6$



$u_1 = (x^2+1)(x-3)$

$u_2 = (x+1)$

$u_3 = 1$

$\tilde{u} = (x+1)^2(x-3) \qquad G\tilde{u} = (x-1)^2(x-9)$

$q^* = (x-1)^2(x+1)(x^2+1)(x-3)(x-9)$

Conclusion

Solutions type	Output size	Complexity

Text available on `arXiv:1612.05518`.

Computing solutions of linear Mahler equations

Frédéric Chyzak, Thomas Dreyfus, Philippe Dumas, Marc Mezzarobba

Conclusion

Solutions type	Output size	Complexity
rational functions denominator		
presented algorithm ($2^r = 1$)	$\mathrm{O}(d/b^r)$	$\mathrm{O}(d\,\mathrm{M}(d)\log(d)/b^r)$
variant	$\mathrm{O}(d\log d)$	$\mathrm{O}(d\log d)$

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series		
formal series at precision x^n		$O(rd^2 + r^2\,M(d) + (r+d)n)$
polynomials	$O(d/b^r)$	$\tilde{O}(d^2/b^r + M(d))$
Puiseux series in $\mathbb{K}((x^{1/N}))$ at precision x^n		$\tilde{O}(r^2 N d(d+n))$

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rational functions		
$b = 2$		$\tilde{\mathrm{O}}(2^r d^2) = \tilde{\mathrm{O}}(d^3)$
$b \geq 3$		$\tilde{\mathrm{O}}(d^2/b^r)$

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GCRD	$\tilde{d} \leq d$	

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future work		
infinite-products		
rational solutions of Riccati-type equation		

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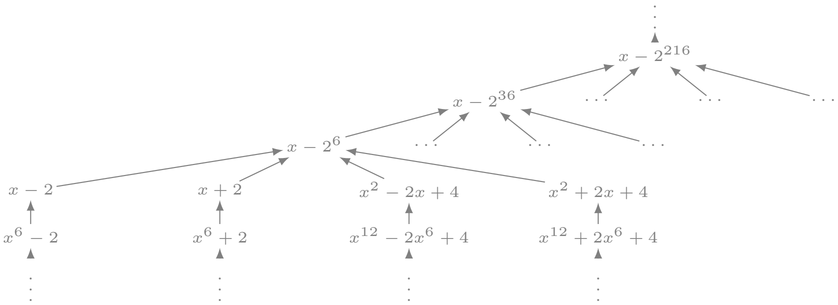

Thanks for your attention!

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