A New Efficient Algorithm for Validating Chebyshev Approximations of LODE Solutions

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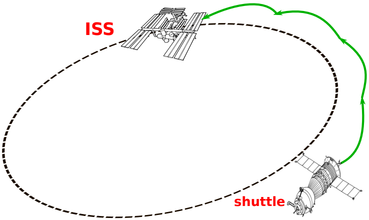
The Spacecraft Rendez-Vous Problem



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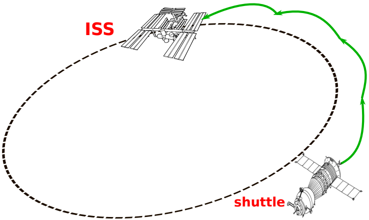

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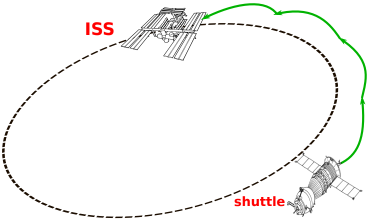

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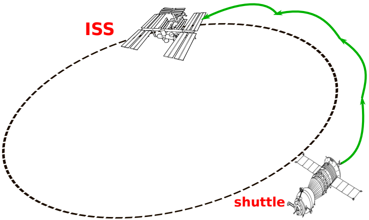

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- Approximating solutions with polynomials.
- Validating approximate solutions with certified error bounds.

Linear Ordinary Differential Equations

- Linear Ordinary Differential Equation over compact interval I:

$$f^{(r)}(t) + a_{r-1}(t)f^{(r-1)}(t) + \cdots + a_1(t)f'(t) + a_0(t)f(t) = g(t).$$

The functions a_i and g are supposed to be at least continuous.

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- This is an infinite-dimensional linear problem:

$$\mathbf{L} = \partial^r + a_{r-1}\partial^{r-1} + \cdots + a_1\partial + a_0 \qquad : \mathcal{C}^r(I) \to \mathcal{C}^0(I),$$
$$\mathbf{B}_{t_0} : f \mapsto \left(f(t_0), f'(t_0), \ldots, f^{(r-1)}(t_0)\right) \qquad : \mathcal{C}^r(I) \to \mathbb{R}^r.$$

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Example

$$\mathbf{L} = \partial^2 + 4 - \frac{3}{1 + e\cos t}$$
$$\mathbf{B} \cdot z = (z(t_0), z'(t_0))$$

Theorem (Picard-Lindelöf – linear case)

The linear operator:

$$(\mathbf{L}, \mathbf{B}_{t_0}) : \mathcal{C}^r(I) \to \mathcal{C}^0(I) \times \mathbb{R}^r,$$

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Some problems:

- How to find approximate solutions?
- How to bound the error of an approximate solution?

Previous Works

- Spectral methods:
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- A posteriori validation methods for differential equations:
  - Quasi-Newton fixed-point methods (Yamamoto, Lessard)
  - D-finite approach and iteration method (Benoit, Joldeș, Mezzarobba)

Reformulation with Integral Operator

Let $\varphi = f^{(r)} \in \mathcal{C}^0(I)$ with $f(t_0) = v_0 \ldots f^{(r-1)}(t_0) = v_{r-1}$.
Then for $i \in [0, r-1]$:

$$f^{(i)}(t) = \sum_{j=i}^{r-1} \frac{(t-t_0)^{j-i}}{(j-i)!} v_j + \int_{t_0}^{t} \int_{t_0}^{s_1} \cdots \int_{t_0}^{s_{r-1-i}} \varphi \mathrm{d}s_1 \ldots \mathrm{d}s_{r-i}.$$

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We get an integral reformulation:

$$\varphi + \mathbf{K} \cdot \varphi = \psi,$$

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- $\psi(t) = g(t) +$ (some function depending on the v_j's).

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- $\psi(t) = c - (z(t_0) + (t - t_0) z'(t_0)) \left(4 - \frac{3}{1 + e \cos t}\right).$

Chebyshev Polynomials

- The Chebyshev family of polynomials:

$$\begin{aligned} T_0(X) &= 1, \\ T_1(X) &= X, \\ T_{n+2}(X) &= 2XT_{n+1}(X) - T_n(X). \end{aligned}$$

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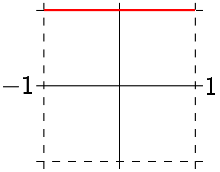

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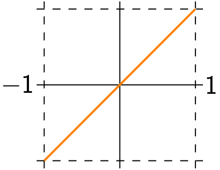

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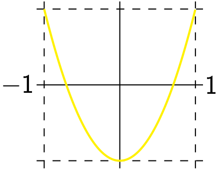

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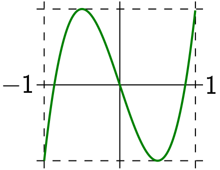

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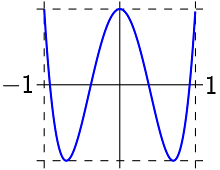

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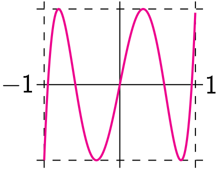

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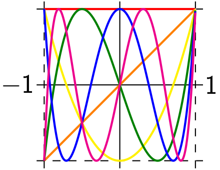

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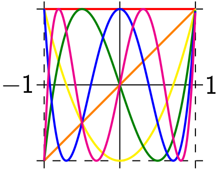

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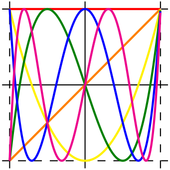

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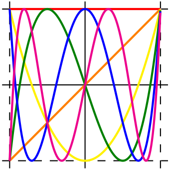

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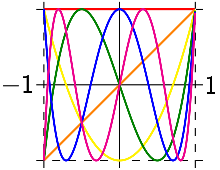

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- Integration:

$$\int T_n = \frac{1}{2}\left(\frac{T_{n+1}}{n+1} - \frac{T_{n-1}}{n-1}\right).$$

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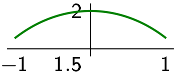
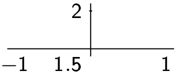
Theorem

If f is $\mathcal{C}^r$ $(r \geq 1)$, then $a_n = O(n^{-r})$.

- Approximation of $t \mapsto 4 - \dfrac{3}{1 + e\cos t}$ over $[-1, 1]$ ($e = 0.5$):





$\alpha(t)$

Approximating our Example

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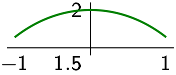

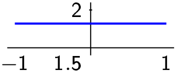

$\alpha(t)$ 1.82

$$|\alpha(t) - 1.82| \leq 0.2$$

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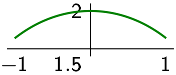

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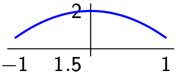

$1.82 - 0.18\, T_2(t)$

$$|\alpha(t) - (1.82 - 0.18\, T_2(t))| \leq 0.007$$

The Almost-Banded Structure of the Operator $\mathbf{K}$

$$\mathbf{K} \cdot \varphi(t) = \sum_{j=0}^{r-1} a_j(t) \int_{t_0}^{t} \frac{(t-s)^{r-1-j}}{(r-1-j)!} \varphi(s) \mathrm{d}s$$

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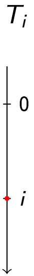

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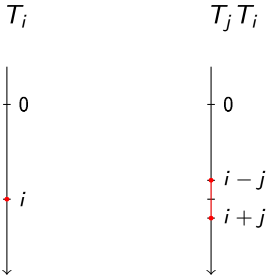

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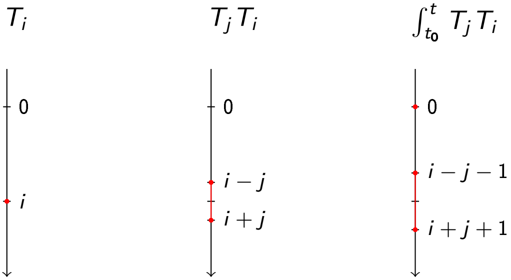

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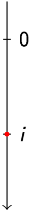
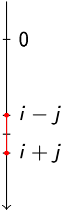
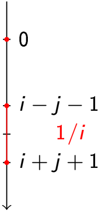
T_i



$T_j T_i$



$\int_{t_0}^{t} T_j T_i$



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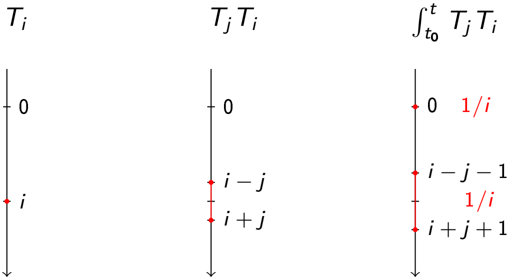

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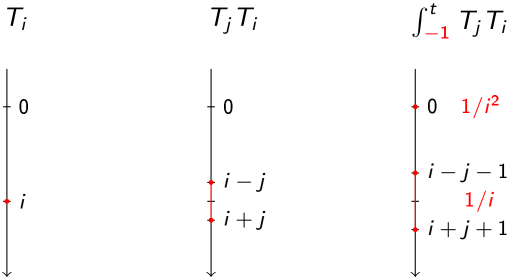

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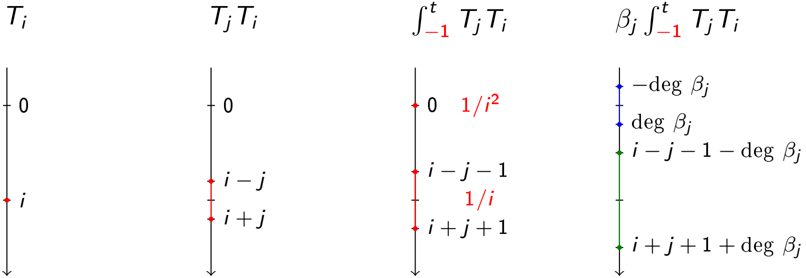

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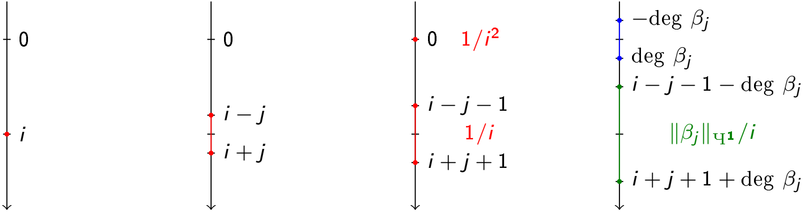

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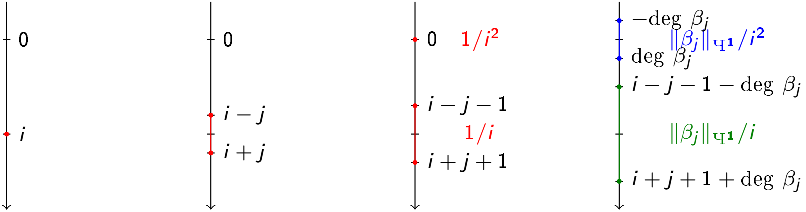

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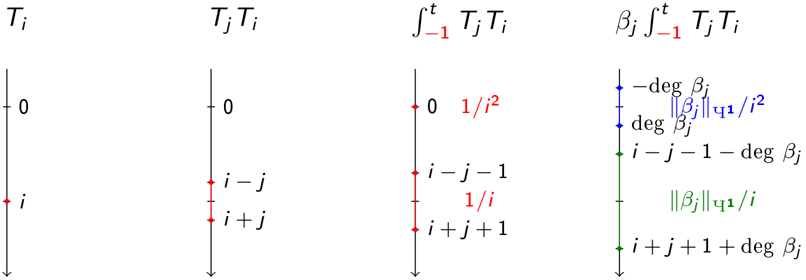

$$h = \max_{0\le j<r} \deg \beta_j, \qquad d = \max_{0\le j<r} (j+1+\deg \beta_j).$$

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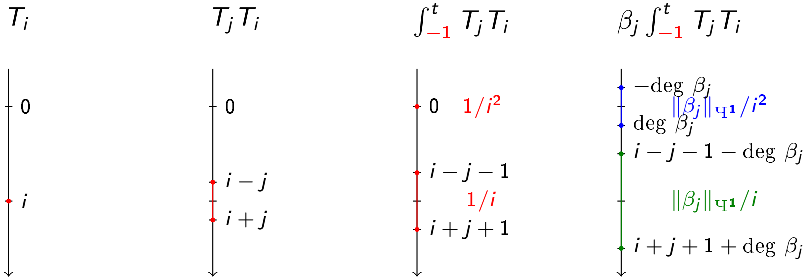
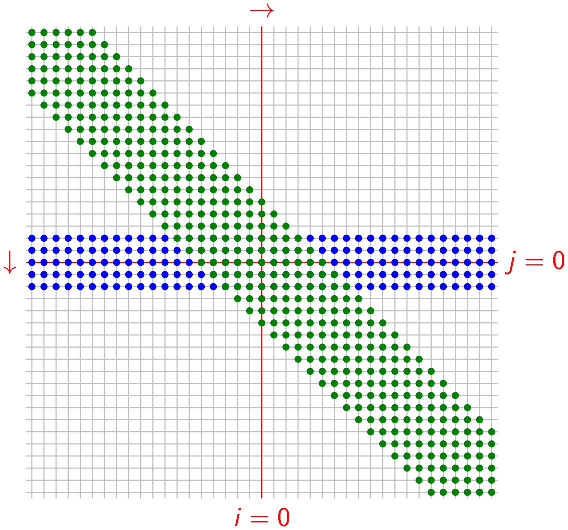


$$h = \max_{0\le j<r} \deg \beta_j, \qquad d = \max_{0\le j<r}(j+1+\deg \beta_j).$$

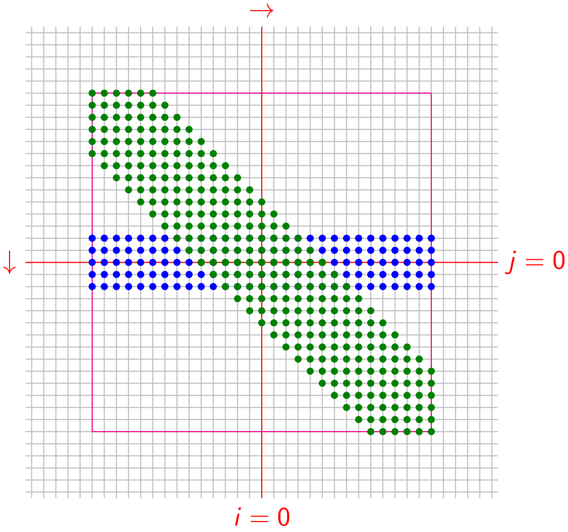
$\Rightarrow$ $\mathbf{K}$ is an (h, d)-**almost banded** operator.

The Almost-Banded Structure of the Operator $\mathbf{K}$



The infinite-dimensional operator $\mathbf{K}$.

The Almost-Banded Structure of the Operator $\mathbf{K}$



The final-dimensional truncation $\mathbf{K}^{[N]}$.

The Almost-Banded Structure of the Operator $\mathbf{K}$ - Example

$$\mathbf{K} \cdot \varphi = t\left(4 - \frac{3}{1 + e\cos t}\right)\int_{t_0}^{t} \varphi(s)\mathrm{d}s + \left(-4 + \frac{3}{1 + e\cos t}\right)\int_{t_0}^{t} s\varphi(s)\mathrm{d}s$$

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$$\mathbf{K} \cdot \varphi \approx t(1.82 - 0.18\, T_2(t)) \int_{t_0}^{t} \varphi(s) \mathrm{d}s + (-1.82 + 0.18\, T_2(t)) \int_{t_0}^{t} s\varphi(s) \mathrm{d}s$$

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1.342500	-0.606667	-0.326250	0.364000	-0.155417	0.086667	-0.056875	0.040444	-0.030333	0.023636	-0.018958	0.015556	-0.013000	0.011030	-0.009479	0.008235	-0.007222	0.006386	-0.005687	0.005098	-0.004596
1.730000	-0.652500	-0.576667	0.438125	-0.115333	0.070208	-0.049429	0.036042	-0.027460	0.021625	-0.017475	0.014417	-0.012098	0.010298	-0.008872	0.007723	-0.006784	0.006007	-0.005356	0.004806	-0.004336
0.320000	0.060000	-0.271458	-0.036000	0.093833	-0.008571	0.004500	-0.004000	0.003000	-0.002338	0.001875	-0.001538	0.001286	-0.001091	0.000937	-0.000814	0.000714	-0.000632	0.000562	-0.000504	0.000455
-0.090000	0.109583	0.030000	-0.137375	0.006000	0.036042	0.002571	-0.002625	0.001429	-0.001125	0.000909	-0.000750	0.000629	-0.000536	0.000462	-0.000402	0.000353	-0.000313	0.000279	-0.000250	0.000226
-0.022500	0	0.052917	0	-0.065167	0	0.024036	0	-0.000536	0	0	0	0	0	0	0	0	0	0	0	0
0	-0.003750	0	0.028375	0	-0.040327	0	0.016104	0	-0.000402	0	0	0	0	0	0	0	0	0	0	0
0	0	-0.001875	0	0.018167	0	-0.027527	0	0.011548	0	-0.000312	0	0	0	0	0	0	0	0	0	0
0	0	0	-0.001125	0	0.012708	0	-0.020021	0	0.008688	0	-0.000250	0	0	0	0	0	0	0	0	0
0	0	0	0	-0.000750	0	0.009411	0	-0.015230	0	0.006774	0	-0.000205	0	0	0	0	0	0	0	0
0	0	0	0	0	-0.000536	0	0.007257	0	-0.011981	0	0.005431	0	-0.000170	0	0	0	0	0	0	0
0	0	0	0	0	0	-0.000402	0	0.005770	0	-0.009675	0	0.004451	0	-0.000144	0	0	0	0	0	0
0	0	0	0	0	0	0	-0.000312	0	0.004699	0	-0.007978	0	0.003715	0	-0.000124	0	0	0	0	0
0	0	0	0	0	0	0	0	-0.000250	0	0.003902	0	-0.006692	0	0.003147	0	-0.000107	0	0	0	0
0	0	0	0	0	0	0	0	0	-0.000205	0	0.003292	0	-0.005694	0	0.002701	0	-0.000094	0	0	0
0	0	0	0	0	0	0	0	0	0	-0.000170	0	0.002815	0	-0.004905	0	0.002343	0	-0.000083	0	0
0	0	0	0	0	0	0	0	0	0	0	-0.000144	0	0.002435	0	-0.004269	0	0.002052	0	-0.000074	0
0	0	0	0	0	0	0	0	0	0	0	0	-0.000124	0	0.002127	0	-0.003749	0	0.001812	0	-0.000066
0	0	0	0	0	0	0	0	0	0	0	0	0	-0.000107	0	0.001874	0	-0.003319	0	0.001612	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.000094	0	0.001663	0	-0.002959	0	0.001443
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.000083	0	0.001487	0	-0.002655	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.000074	0	0.001337	0	-0.002395

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$$
\begin{aligned}
\widetilde{\varphi} = & -0.6\,T_0 - 1.19\,T_1 + 0.62\,T_2 + 0.17\,T_3 - 0.05\,T_4 - 0.01\,T_5 \\
& + 2.1 \cdot 10^{-3}\,T_6 + 3.2 \cdot 10^{-3}\,T_7 - 5.8 \cdot 10^{-5}\,T_8 - 7.6 \cdot 10^{-6}\,T_9 + 1.2 \cdot 10^{-6}\,T_{10} \\
& + 1.4 \cdot 10^{-7}\,T_{11} - 1.9 \cdot 10^{-8}\,T_{12} - 2.0 \cdot 10^{-9}\,T_{13} + 2.6 \cdot 10^{-10}\,T_{14} + 2.5 \cdot 10^{-11}\,T_{15} \\
& - 3.0 \cdot 10^{-12}\,T_{16} - 2.6 \cdot 10^{-13}\,T_{17} + 3.0 \cdot 10^{-14}\,T_{18} + 2.5 \cdot 10^{-15}\,T_{19} - 2.6 \cdot 10^{-16}\,T_{20}
\end{aligned}
$$

General Ideas for Validation of Linear Problems

- Recall: For the integral equation of unknown φ

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we want to validate an approximate solution $\widetilde{\varphi}$:

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But problems:

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$$\varphi + \mathbf{K} \cdot \varphi = \psi \Leftrightarrow \mathbf{T} \cdot \varphi = \varphi,$$

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- If $\|\mathrm{D}\mathbf{T}\|_{\text{Ч}^1} = \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\text{Ч}^1} = k < 1$, $\mathbf{T}$ is contractive and we get a tight enclosure of the approximation error:

$$\frac{\|\mathbf{T} \cdot \widetilde{\varphi} - \widetilde{\varphi}\|_{\text{Ч}^1}}{1 + k} \leq \|\widetilde{\varphi} - \varphi^*\|_{\text{Ч}^1} \leq \frac{\|\mathbf{T} \cdot \widetilde{\varphi} - \widetilde{\varphi}\|_{\text{Ч}^1}}{1 - k}.$$

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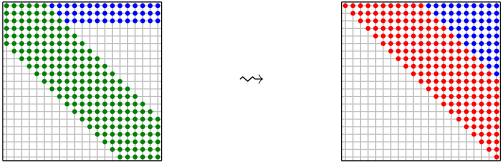
- Computing the (dense) inverse, using Olver and Townsend's algorithm: $O(n^2(h+d))$.



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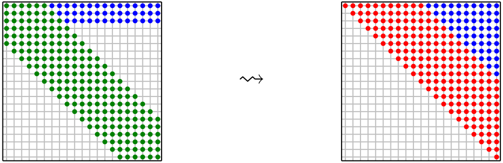
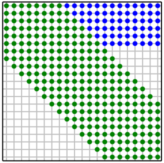


- Computing an (h', d') almost-banded approximate inverse: $O(n(h'+d')(h+d))$.



Approximate Inverse for our Example

0.161442	0.292063	0.309506	-0.202732	0.032680	-0.025359	0.022468	-0.016198	0.012178	-0.009592	0.007748	-0.006386	0.005355	-0.004555	0.003922	-0.003413	0.002997	-0.002652	0.002364	-0.002123	0.001915
-0.951737	1.182179	0.521477	-0.177420	-0.063659	0.010819	0.004034	-0.002709	0.002226	-0.002039	0.001797	-0.001567	0.001369	-0.001201	0.001059	-0.000939	0.000836	-0.000749	0.000675	-0.000611	0.000555
0.013779	-0.233695	1.197801	0.162217	-0.147560	0.020291	-0.013304	0.012961	-0.009866	0.007675	-0.006197	0.005110	-0.004285	0.003645	-0.003138	0.002731	-0.002398	0.002122	-0.001892	0.001699	-0.001532
0.137573	-0.112050	-0.075379	1.156558	0.009202	-0.048363	-0.000583	0.002026	-0.000322	0.000261	-0.000260	0.000227	-0.000198	0.000174	-0.000153	0.000136	-0.000121	0.000108	-0.000098	0.000088	-0.000080
0.003107	0.020279	-0.060441	-0.014077	1.079375	-0.001761	-0.025165	-0.001125	0.001745	-0.000666	0.000523	-0.000443	0.000372	-0.000316	0.000272	-0.000237	0.000208	-0.000184	0.000164	-0.000147	0.000133
-0.007791	0.007936	0.004269	-0.034920	-0.000521	1.043723	0.000033	-0.017202	0.000018	0.000559	0.000015	-0.000020	0.000011	-0.000010	0.000009	-0.000008	0.000007	-0.000006	0.000006	-0.000005	0.000005
-0.000031	-0.000830	0.003439	0.000576	-0.020460	0.000072	1.028868	0.000046	-0.012114	0.000027	0.000385	0.000018	-0.000020	0.000013	-0.000011	0.000010	-0.000009	0.000008	-0.000007	0.000006	-0.000005
0.000259	-0.000232	-0.000142	0.001781	0.000017	-0.013596	-0.000001	1.020722	-0.000001	-0.008981	-0.000000	0.000307	-0.000000	-0.000002	-0.000000	0.000000	-0.000000	0.000000	-0.000000	0.000000	-0.000000
0.000003	0.000023	-0.000079	-0.000016	0.001018	-0.000002	-0.009855	-0.000001	1.015624	-0.000001	-0.006950	-0.000001	0.000241	-0.000000	-0.000001	-0.000000	0.000000	-0.000000	0.000000	-0.000000	0.000000
-0.000006	0.000006	0.000003	-0.000032	-0.000000	0.000666	0.000000	-0.007508	0.000000	1.012219	0.000000	-0.005544	0.000000	0.000194	0.000000	-0.000001	0.000000	0.000000	0.000000	-0.000000	0.000000
-0.000000	-0.000000	0.000002	0.000000	-0.000014	0.000000	0.000475	0.000000	-0.005923	0.000000	1.009828	0.000000	-0.004527	0.000000	0.000161	0.000000	-0.000001	0.000000	0.000000	0.000000	-0.000000
0.000000	-0.000000	-0.000000	0.000001	0.000000	-0.000007	-0.000000	0.000357	-0.000000	-0.004798	-0.000000	1.008081	-0.000000	-0.003767	-0.000000	0.000135	-0.000000	-0.000001	-0.000000	0.000000	-0.000000
0.000000	0.000000	-0.000000	-0.000000	0.000000	-0.000000	-0.000004	-0.000000	0.000279	-0.000000	-0.003969	-0.000000	1.006764	-0.000000	-0.003185	-0.000000	0.000116	-0.000000	-0.000000	-0.000000	0.000000
-0.000000	0.000000	0.000000	-0.000000	-0.000000	0.000000	0.000000	-0.000003	0.000000	0.000224	0.000000	-0.003339	0.000000	1.005746	0.000000	-0.002729	0.000000	0.000100	0.000000	-0.000000	0.000000
-0.000000	-0.000000	0.000000	0.000000	-0.000000	0.000000	0.000000	0.000000	-0.000002	0.000000	0.000184	0.000000	-0.002849	0.000000	1.004943	0.000000	-0.002364	0.000000	0.000088	0.000000	-0.000000
0.000000	-0.000000	-0.000000	0.000000	0.000000	-0.000000	-0.000000	0.000000	-0.000000	-0.000001	-0.000000	0.000154	-0.000000	-0.002460	-0.000000	1.004298	-0.000000	-0.002068	-0.000000	0.000077	-0.000000
0.000000	0.000000	-0.000000	-0.000000	0.000000	-0.000000	-0.000000	-0.000000	0.000000	-0.000000	-0.000001	-0.000000	0.000131	-0.000000	-0.002146	-0.000000	1.003772	-0.000000	-0.001825	-0.000000	0.000069
-0.000000	0.000000	0.000000	-0.000000	-0.000000	0.000000	0.000000	-0.000000	0.000000	0.000000	0.000000	-0.000001	0.000000	0.000113	0.000000	-0.001888	0.000000	1.003337	0.000000	-0.001622	0.000000
-0.000000	-0.000000	0.000000	0.000000	-0.000000	0.000000	0.000000	0.000000	-0.000000	0.000000	0.000000	0.000000	-0.000000	0.000000	0.000098	0.000000	-0.001675	0.000000	1.002973	0.000000	-0.001451
0.000000	-0.000000	-0.000000	0.000000	0.000000	-0.000000	-0.000000	0.000000	-0.000000	-0.000000	-0.000000	0.000000	-0.000000	-0.000000	-0.000000	0.000086	-0.000000	-0.001496	-0.000000	1.002664	-0.000000
0.000000	0.000000	-0.000000	-0.000000	0.000000	-0.000000	-0.000000	-0.000000	0.000000	-0.000000	-0.000000	-0.000000	0.000000	-0.000000	-0.000000	-0.000000	0.000076	-0.000000	-0.001344	-0.000000	1.002403

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-0.951737	1.182179	0.521477	-0.177420	-0.063659	0.010819	0.004034	-0.002709	0.002226	-0.002039	0.001797	-0.001567	0.001369	-0.001201	0.001059	-0.000939	0.000836	-0.000749	0.000675	-0.000611	0.000555
0.013779	-0.233695	1.197801	0.162217	-0.147560	0.020291	-0.013304	0.012961	-0.009866	0.007675	-0.006197	0.005110	-0.004285	0.003645	-0.003138	0.002731	-0.002398	0.002122	-0.001892	0.001699	-0.001532
0.137573	-0.112050	-0.075379	1.156558	0.009202	-0.048363	-0.000583	0.002026	0	0	0	0	0	0	0	0	0	0	0	0	0
0.003107	0.020279	-0.060441	-0.014077	1.079375	-0.001761	-0.025165	-0.001125	0.001745	-0.000666	0.000523	0	0	0	0	0	0	0	0	0	0
-0.007791	0.007936	0.004269	-0.034920	0	1.043723	0	-0.017202	0	0.000559	0	0	0	0	0	0	0	0	0	0	0
0	0	0.003439	0	-0.020460	0	1.028868	0	-0.012114	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0.001781	0	-0.013596	0	1.020722	0	-0.008981	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0.001018	0	-0.009855	0	1.015624	0	-0.006950	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0.000666	0	-0.007508	0	1.012219	0	-0.005544	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	-0.005923	0	1.009828	0	-0.004527	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	-0.004798	0	1.008081	0	-0.003767	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	-0.003969	0	1.006764	0	-0.003185	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	-0.003339	0	1.005746	0	-0.002729	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	-0.002849	0	1.004943	0	-0.002364	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	-0.002460	0	1.004298	0	-0.002068	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.002146	0	1.003772	0	-0.001825	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.001888	0	1.003337	0	-0.001622	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.001675	0	1.002973	0	-0.001451
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.001496	0	1.002664	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.001344	0	1.002403

Computing the Operator Norm (1/2)

- Decomposition of the operator norm:

$$\|\mathbf{I}-\mathbf{A}(\mathbf{I}+\mathbf{K})\|_{\mathcal{H}^1} \leq \|\mathbf{I}-\mathbf{A}(\mathbf{I}+\mathbf{K}^{[N]})\|_{\mathcal{H}^1} + \|\mathbf{A}(\mathbf{K}-\mathbf{K}^{[N]})\|_{\mathcal{H}^1}.$$

Computing the Operator Norm (1/2)

- Decomposition of the operator norm:

$$\|\mathbf{I}-\mathbf{A}(\mathbf{I}+\mathbf{K})\|_{\text{Ч}^1} \leq \underbrace{\|\mathbf{I}-\mathbf{A}(\mathbf{I}+\mathbf{K}^{[N]})\|_{\text{Ч}^1}}_{\text{Approximation error}} + \|\mathbf{A}(\mathbf{K}-\mathbf{K}^{[N]})\|_{\text{Ч}^1}.$$

Computing the Operator Norm (1/2)

- Decomposition of the operator norm:

$$\|\mathbf{I}-\mathbf{A}(\mathbf{I}+\mathbf{K})\|_{\text{ч}^1} \leq \underbrace{\|\mathbf{I}-\mathbf{A}(\mathbf{I}+\mathbf{K}^{[N]})\|_{\text{ч}^1}}_{\text{Approximation error}} + \underbrace{\|\mathbf{A}(\mathbf{K}-\mathbf{K}^{[N]})\|_{\text{ч}^1}}_{\text{Truncation error}}.$$

Computing the Operator Norm (1/2)

- Decomposition of the operator norm:

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\text{Ч}^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\text{Ч}^1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\text{Ч}^1}.$$

Computing the Operator Norm (1/2)

- Decomposition of the operator norm:

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\text{Ч}^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\text{Ч}^1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\text{Ч}^1}.$$

- Addition and Multiplication are trivially handled.

Computing the Operator Norm (1/2)

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- Computing Ч^1-norm = maximum of 1-norms of the columns.

Computing the Operator Norm (1/2)

- Decomposition of the operator norm:

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\text{Ч}^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\text{Ч}^1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\text{Ч}^1}.$$

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- With (h', d')-almost-banded A: $O(n(h' + d')(h + d))$.

Computing the Operator Norm (1/2)

- Decomposition of the operator norm:

$$\|\mathbf{I}-\mathbf{A}(\mathbf{I}+\mathbf{K})\|_{Ч^1} \leq \|\mathbf{I}-\mathbf{A}(\mathbf{I}+\mathbf{K}^{[N]})\|_{Ч^1} + \|\mathbf{A}(\mathbf{K}-\mathbf{K}^{[N]})\|_{Ч^1}.$$

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Example

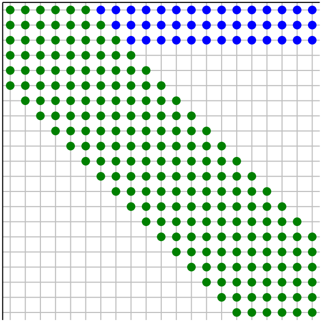
In our case, the approximation error is:

$$\|\mathbf{I}-\mathbf{A}\left(\mathbf{I}+\mathbf{K}^{[N]}\right)\|_{Ч^1} \leq 1.5 \cdot 10^{-3}$$

Computing the Operator Norm (2/2)

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\mathcal{Y}^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\mathcal{Y}^1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\mathcal{Y}^1}.$$

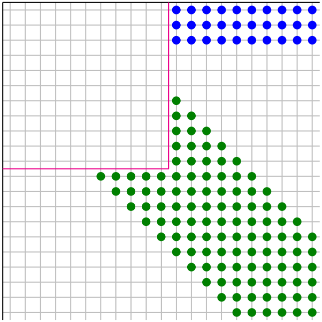
$$\|\mathbf{I} - \mathbf{A}(\mathbf{I}+\mathbf{K})\|_{\ell^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I}+\mathbf{K}^{[N]})\|_{\ell^1} + \|\mathbf{A}(\mathbf{K}-\mathbf{K}^{[N]})\|_{\ell^1}.$$



$\mathbf{K}$

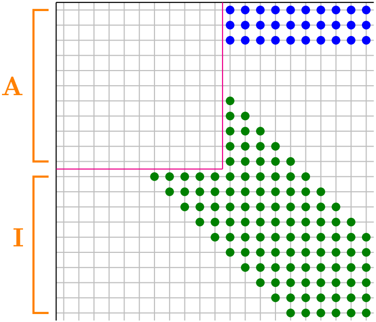
Computing the Operator Norm (2/2)

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\ell^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\ell^1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\ell^1}.$$



$\mathbf{K} - \mathbf{K}^{[N]}$

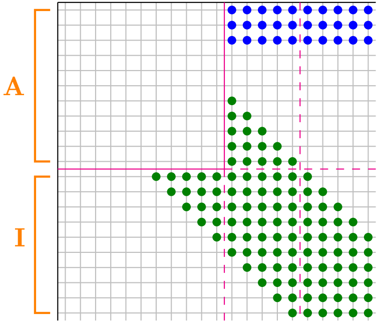
$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\ell^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\ell^1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\ell^1}.$$



$\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})$

Computing the Operator Norm (2/2)

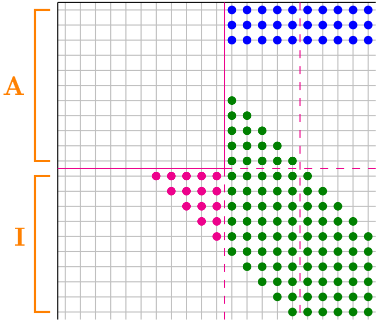
$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\ell^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\ell^1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\ell^1}.$$



$\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})$

Computing the Operator Norm (2/2)

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\text{ч}^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\text{ч}^1} + \color{red}{\|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\text{ч}^1}}.$$

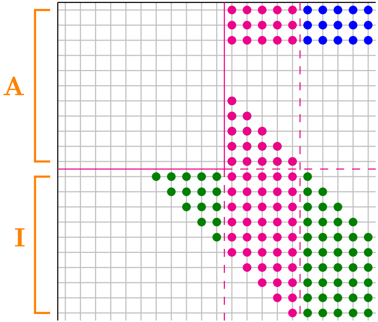


$$\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})$$

- Direct computation.

Computing the Operator Norm (2/2)

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I}+\mathbf{K})\|_{\ell^1} \le \|\mathbf{I} - \mathbf{A}(\mathbf{I}+\mathbf{K}^{[N]})\|_{\ell^1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\ell^1}.$$

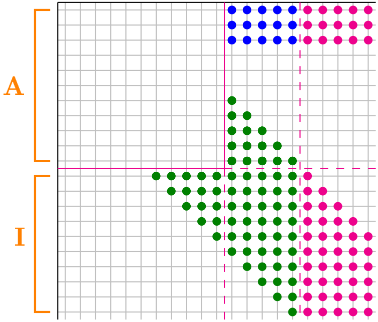


$\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})$

- Direct computation
- Apply $\mathbf{A}$ and direct computation.

Computing the Operator Norm (2/2)

$$\|\mathbf{I}-\mathbf{A}(\mathbf{I}+\mathbf{K})\|_{\ell^1} \leq \|\mathbf{I}-\mathbf{A}(\mathbf{I}+\mathbf{K}^{[N]})\|_{\ell^1} + \|\mathbf{A}(\mathbf{K}-\mathbf{K}^{[N]})\|_{\ell^1}.$$

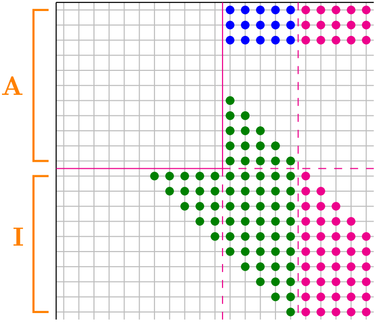


$\mathbf{A}(\mathbf{K}-\mathbf{K}^{[N]})$

- Direct computation
- Apply $\mathbf{A}$ and direct computation.
- Bound the remaining *infinite* number of columns:

Computing the Operator Norm (2/2)

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I}+\mathbf{K})\|_{\ell^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I}+\mathbf{K}^{[N]})\|_{\ell^1} + \|\mathbf{A}(\mathbf{K}-\mathbf{K}^{[N]})\|_{\ell^1}.$$

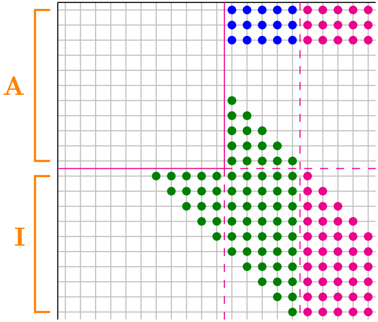


$\mathbf{A}(\mathbf{K}-\mathbf{K}^{[N]})$

- Direct computation
- Apply $\mathbf{A}$ and direct computation.
- Bound the remaining *infinite* number of columns:
  - Using the bounds in $1/i$ and $1/i^2$: possibly big overestimations.

Computing the Operator Norm (2/2)

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\ell^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\ell^1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\ell^1}.$$

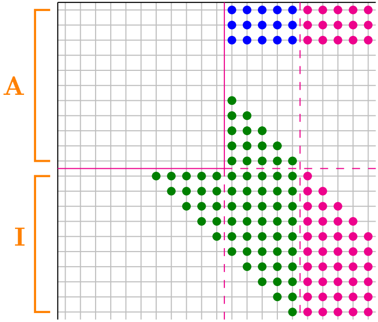


$\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})$

- Direct computation
- Apply $\mathbf{A}$ and direct computation.
- Bound the remaining *infinite* number of columns:
  - Using the bounds in $1/i$ and $1/i^2$: possibly big overestimations.
  - Using a first order difference method: differences in $1/i^2$ and $1/i^4$.

Computing the Operator Norm (2/2)

$$\|\mathbf{I}-\mathbf{A}(\mathbf{I}+\mathbf{K})\|_{\ell^1} \leq \|\mathbf{I}-\mathbf{A}(\mathbf{I}+\mathbf{K}^{[N]})\|_{\ell^1} + \|\mathbf{A}(\mathbf{K}-\mathbf{K}^{[N]})\|_{\ell^1}.$$



$\mathbf{A}(\mathbf{K}-\mathbf{K}^{[N]})$

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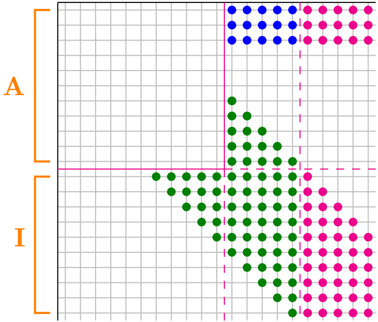
Truncation error of the example

Computing the Operator Norm (2/2)

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I}+\mathbf{K})\|_{\ell^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I}+\mathbf{K}^{[N]})\|_{\ell^1} + \|\mathbf{A}(\mathbf{K}-\mathbf{K}^{[N]})\|_{\ell^1}.$$



$\mathbf{A}(\mathbf{K}-\mathbf{K}^{[N]})$

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- Apply $\mathbf{A}$ and direct computation.
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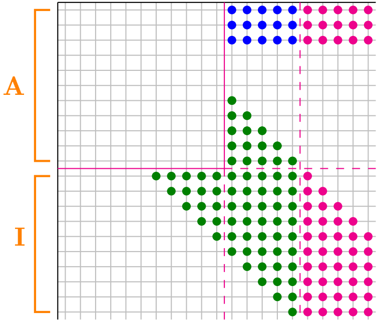
Truncation error of the example

$1.3 \cdot 10^{-3}$

Computing the Operator Norm (2/2)

$$\|\mathbf{I}-\mathbf{A}(\mathbf{I}+\mathbf{K})\|_{\ell^1} \le \|\mathbf{I}-\mathbf{A}(\mathbf{I}+\mathbf{K}^{[N]})\|_{\ell^1} + \|\mathbf{A}(\mathbf{K}-\mathbf{K}^{[N]})\|_{\ell^1}.$$



$\mathbf{A}(\mathbf{K}-\mathbf{K}^{[N]})$

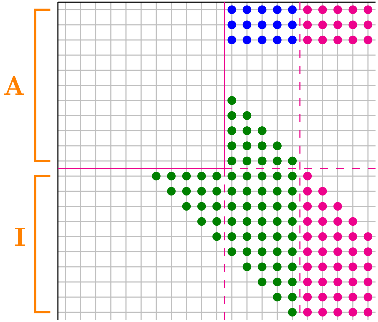
- Direct computation
- Apply $\mathbf{A}$ and direct computation.
- Bound the remaining *infinite* number of columns:
  - Using the bounds in $1/i$ and $1/i^2$: possibly big overestimations.
  - Using a first order difference method: differences in $1/i^2$ and $1/i^4$.

Truncation error of the example

$1.3 \cdot 10^{-3}$ $\quad$ $5.2 \cdot 10^{-3}$

Computing the Operator Norm (2/2)

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\ell^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\ell^1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\ell^1}.$$



$\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})$

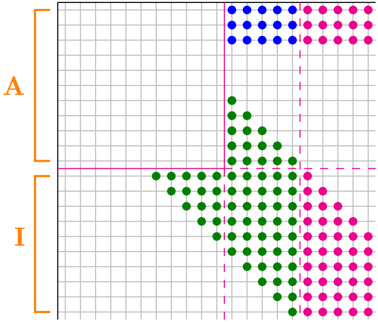
- Direct computation
- Apply $\mathbf{A}$ and direct computation.
- Bound the remaining *infinite* number of columns:
  - Using the bounds in $1/i$ and $1/i^2$: possibly big overestimations.
  - Using a first order difference method: differences in $1/i^2$ and $1/i^4$.

Truncation error of the example

$1.3 \cdot 10^{-3}$	$5.2 \cdot 10^{-3}$	$9.4 \cdot 10^{-3} + 2.7 \cdot 10^{-3}$

Computing the Operator Norm (2/2)

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\ell^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\ell^1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\ell^1}.$$



$\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})$

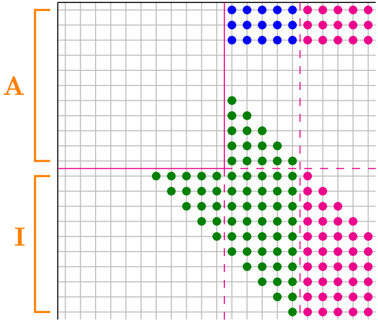
- Direct computation
- Apply $\mathbf{A}$ and direct computation.
- Bound the remaining *infinite* number of columns:
  - Using the bounds in $1/i$ and $1/i^2$: possibly big overestimations.
  - Using a first order difference method: differences in $1/i^2$ and $1/i^4$.

Truncation error of the example

$1.3 \cdot 10^{-3}$ $\quad 5.2 \cdot 10^{-3}$ $\quad 1.21 \cdot 10^{-2}$

Computing the Operator Norm (2/2)

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\ell^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\ell^1} + \color{red}{\|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\ell^1}}.$$



$\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})$

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- Apply $\mathbf{A}$ and direct computation.
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Truncation error of the example

$1.3 \cdot 10^{-3}$ $\quad 5.2 \cdot 10^{-3}$ $\quad 1.21 \cdot 10^{-2}$ $\quad \Rightarrow$ $\quad \mathbf{1.21 \cdot 10^{-2}}$

Bring our Example to the End

- $k \leq \mathbf{1.5 \cdot 10^{-3}} + \mathbf{1.21 \cdot 10^{-2}}$.

Bring our Example to the End

- $k \leq 1.36 \cdot 10^{-2}$.

Bring our Example to the End

- $\boldsymbol{k \leq 1.36 \cdot 10^{-2}}$.
- $\|\mathbf{T} \cdot \widetilde{\varphi} - \widetilde{\varphi}\|_{\text{Ч}^1} = \|\mathbf{A}(\widetilde{\varphi} + \mathbf{K} \cdot \widetilde{\varphi} - \psi)\|_{\text{Ч}^1} = 6.48 \cdot 10^{-16}$.

Bring our Example to the End

- $k \leq \mathbf{1.36 \cdot 10^{-2}}$.
- $\|\mathbf{T} \cdot \widetilde{\varphi} - \widetilde{\varphi}\|_{\mathbb{Y}^1} = \|\mathbf{A}(\widetilde{\varphi} + \mathbf{K} \cdot \widetilde{\varphi} - \psi)\|_{\mathbb{Y}^1} = 6.48 \cdot 10^{-16}$.
- Hence:

$$\frac{6.48 \cdot 10^{-16}}{1 + k} \leq \|\widetilde{\varphi} - \varphi^*\|_{\mathbb{Y}^1} \leq \frac{6.48 \cdot 10^{-16}}{1 - k}$$

Bring our Example to the End

- $k \leq \mathbf{1.36 \cdot 10^{-2}}$.
- $\|\mathbf{T} \cdot \widetilde{\varphi} - \widetilde{\varphi}\|_{\text{Ч}^1} = \|\mathbf{A}(\widetilde{\varphi} + \mathbf{K} \cdot \widetilde{\varphi} - \psi)\|_{\text{Ч}^1} = 6.48 \cdot 10^{-16}$.
- Hence:

$$6.39 \cdot 10^{-16} \qquad \leq \qquad \|\widetilde{\varphi} - \varphi^*\|_{\text{Ч}^1} \qquad \leq \qquad 6.57 \cdot 10^{-16}$$

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  - Other orthogonal families of polynomials.
- A COQ implementation to certify this algorithm.