An Introduction to hyperbolic sigma models and Edge Reinforced Random Walk

Thomas Spencer
IAS, Princeton NJ

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Linearly Edge Reinforced Random Walk

History dependent walk $W_n \in \mathbb{Z}^d$, $n \in \mathbb{Z}^+$:

Walk takes nearest neighbor steps and favors edges $j, k, \in \mathbb{Z}^d$, $|j - k| = 1$, it has visited in the past.

Introduced by P. Diaconis in 1986 while wandering the streets of Paris. He liked to return to streets he had visited in the past.

Why Linearly Edge Reinforced?

Related to Polya's Urn.

Partially exchangeable process - generalization of de Finetti

Not Markovian but is a superposition of Markov Processes

Equivalent to a random walk in a random environment - must average over environment.

Definition of Reinforced Random Walk

Let $C_{jk}(n)$ = number of times the walk has crossed edge jk up to time n and let $\beta > 0$.

$$Prob\{W_{n+1} = k | W_n = j\} = \frac{1 + C_{jk}(n)/\beta}{\mathcal{N}_\beta}, \quad |j - k| = 1.$$

$\mathcal{N}$ is the normalization: $\mathcal{N}_\beta = \sum_{k'} (1 + C_{jk'}(n)/\beta), \quad |j - k'| = 1$

$0 < \beta \ll 1$, **strong** reinforcement, (high temperature)
$\beta \gg 1$, **weak** reinforcement, (low temperature)

Long time behavior of W_n , n, large?

Is ERRW recurrent? Localized?

Localization:

$$Prob_{\beta}\{|W_n - W_0| \geq R\} \leq Ce^{-R/\ell}\ , \quad \ell(\beta) = \text{localization length}$$

Is ERRW Transient? Diffusive?

Is there a Phase Transition as we vary the reinforcement β ?

P. Diaconis and D. Coppersmith (1986):

ERRW $\approx$ random walk in a random environment.

Environment: The rate at which an edge j, j' is crossed $w_{j,j'} > 0$ are correlated random variables. (conductances)

Explicit Joint Distribution of $w_{j,j'} > 0$ appeared in an unpublished paper Diaconis and Coppersmith.

Complicated statistical mechanics model -
Can explicitly compute the partition function!

The generator for the RW is a weighted Laplacian L

$$v^t \cdot L\, v = \sum_{|j-j'|=1} w_{j,j'} (v_j - v_{j'})^2$$

However, L is NOT uniformly elliptic.

Important: The distribution of $w_{j,j'}$ depends on the **starting point** of the Walk.

Some results

R. Pemantle analyzed ERRW on the Regular tree. Showed that it has sharp transition from recurrent to transient.

Merkl and Rolles studied one dimensional strips of Width W and show the ERRW is localized with $\ell \approx \beta W$

In 2D Merkl and Rolles prove the conductance has a power law decay away from the origin - via a deformation argument.

Sabot and Zeng: In 2D ERRW is recurrent for all β.

Conjecture: In 2D the walk is exponentially localized for all β.

Model for Quantum Particle on Z^d scattered by Impurities

Anderson model of electron on a lattice scattered by a random potential.

$$H = -\Delta + \lambda v_j \ , \ \ \ell_2(Z^d)$$

where v_j iid and λ is the strength of disorder.

To Understand: Eigenstates, Time evolution.

Localization means that the eigenfunctions decay about some center of the lattice: Insulator

"Equivalent" **dual model** in statistical mechanics with Hyperbolic symmetry - Wegner 1980, Replicas

1982 Efetov: $U(1,1|2)$ SUSY - Rigorous equivalence but **very** complicated.

In 1991 Martin Zirnbauer defined a simplified version of Efetov's dual model: SUSY Hyperbolic sigma model $H^{2|2}$,

Two for dimension of hyperbolic space and 2 for the dimension space of Grassmann variables.

In 1D Zirnbauer proved localization - exponential decay of correlations for all β.

In any dimension spin correlations of $H^{2|2}$ can be expressed as a random walk in a random environment.

Theorem (DSZ '10): $H^{2|2}$ has Phase transition for $D \geq 3$:

What is the relation between $H^{2|2}$ and ERRW?

In a **surprising work**, Sabot and Tarres ('12) found a precise connection of $H^{2|2}$ with **ERRW**.

This made it possible to prove a phase transition for ERRW and for VRJP in 3D. (Disertori Merkl, Rolles, Sabot, Tarres.)

There is also excellent related work by Angel, Crawford and Kozma.

What are lattice Sigma Models?

Spins with values in Symmetric space $\mathcal{S}$

$$\textbf{Spin}: \quad S(j) \in \mathcal{S}, \quad j \in Z^d \cap \Lambda = \; box$$

$$S_j = \pm 1 \quad Z_2 = \mathcal{S} \quad Ising\ \ Model$$

$$S_j = (\cos\theta_j, \sin\theta_j) \in U(1) = \mathcal{S} \quad XY\ model$$

$$S_j \in \ \ Unit\ \ Sphere = \mathcal{S} \quad Classical\ \ Heisenberg$$

$$S_j \in \ \ H^2 = \mathcal{S} \quad Hyperbolic\ sigma\ model$$

$$S_j \in H^{(2|2)} = \mathcal{S} \quad SUSY\ Hyperbolic\ sigma\ model$$

Energy $E_\Lambda(S)$ of a Spin Configuration

$$E_\Lambda(S, \epsilon) = \sum_{j \sim j' \in \Lambda} |S_j - S_{j'}|^2 + \epsilon \sum_j S_j^{(1)}$$

Here $j \sim j'$ are adjacent vertices in the box $Z^d \cap \Lambda$

$\epsilon \geq 0$ external magnetic field

The $|\cdot|$ is distance in the symmetric space.

The Partition function $Z_\Lambda(\beta, \epsilon)$

$$Z_\Lambda(\beta, \epsilon) \equiv \int e^{-\beta E_\Lambda(S;\epsilon)} \prod_{j \in \Lambda} d\mu(S_j)$$

$\beta \propto (Temp)^{-1}$. For $x \in Z^d$, the Spin-Spin correlation:

$$< S_0 \cdot S_x >_\Lambda (\beta, \epsilon) \equiv$$

$$\mathrm{Z}_\Lambda(\beta, \epsilon)^{-1} \int S_0 \cdot S_x \, e^{-\beta E_\Lambda(S;\epsilon)} \prod_{j \in \Lambda} d\mu(S_j)$$

Low Temperature favors minimum energy configuration of spins - 0 modes, the saddle manifold dominates .

This is why Wigner Dyson (mean field) statistics should also hold for 3D random band-They have the same saddle manifold governed by symmetry. Fluctuations about saddle manifold should be small in 3D.

Hyperbolic sigma model in 3 Dimensions

The hyperbolic spins indexed by $j \in Z^3$ may be expressed as

$$S_j = (x_j, y_j, z_j), \quad z_j^2 - x_j^2 - y_j^2 = 1, \quad z_j > 0$$

$$Z_\Lambda(\beta, \epsilon) = \int e^{-\beta \sum_{j\sim j'} S_j \cdot S_{j'} - \epsilon \sum_{j \in \Lambda} z_j} \, d\mu(S_j), \ \epsilon > 0$$

Horospherical coordinates: $s_j, t_j \in \mathbb{R}$

$$z = \cosh t + s^2 e^t/2, \quad x = \sinh t - s^2 e^t/2, \quad y = se^t.$$

$$E(S) = \beta \sum_{j \sim j'} \{cosh(t_j - t_{j'}) + \frac{1}{2}(s_j - s_{j'})^2 e^{(t_j + t_{j'})}\} + \epsilon \sum_{j \in \Lambda} z_j$$

H^2 Random Environment

Generator:

$$[s\,;D_{\beta,\epsilon}(t)\,s]_\Lambda = \beta \sum\nolimits_{(j'\sim j)} \mathrm{e}^{t_j+t_{j'}}(s_j - s_{j'})^2 + \epsilon \sum\nolimits_{k\in\Lambda} \mathrm{e}^{t_k} s_k^2$$

$D_{\beta,\epsilon}(t) > 0$ is a finite difference elliptic operator

Note: $D_{\beta,\epsilon} = -\beta\Delta + \epsilon$ when $t_j \equiv 0$,

The **local conductance** $= e^{t_j+t_{j'}}$

$\epsilon > 0$, denotes small killing term.

D is **not uniformly elliptic**, since $t_j \in \mathbb{R}$ are unbounded

Effective Energy E'(t)

Integrate out s variables.

$$E'_\Lambda(\{t_j\}) = \beta \sum_{j \sim j'} cosh(t_j - t_{j'}) + 1/2 \log \det D_{\beta,\epsilon}(t)$$

Spin-Spin correlation, $x \in Z^d$

$$< y_0 \, y_x > = < s_0 e^{t_0} s_x e^{t_x} > (\beta, \epsilon) = < e^{t_0 + t_x} D_{\beta,\epsilon}^{-1}(0, x) >' (\beta, \epsilon)$$

$$Z(\beta, \epsilon) = \int e^{-E'_\Lambda(t)} \prod e^{t_j} dt_j$$

Formal Symmetry

$$E'(\{t_j\}) = E'(\{t_j + a\}) \quad for \;\; \epsilon = 0$$

Theorem (Sp-Zirnbauer) In 3 Dimensions, for all $\beta > 0, \ \epsilon > 0$, the hyperbolic model only has diffusive behavior as $\epsilon \downarrow 0$

$$< s_0 e^{t_0} s_x e^{t_x} > (\beta, \epsilon) \propto (-\Delta + \epsilon)^{-1}(0, x) \approx 1/|x|$$

No disordered phase.
Sketch of Proof:

A) $E'_\Lambda(\{t_j\})$ is a **convex** in t, Hessian $\geq -\beta\Delta > 0$.

B) Uniform bounds on moments of local conductance

$$< e^{p(t_j + t_{j'})} >, \quad p = \pm 1, \pm 2, ...$$

$\rightarrow$ Diffusive behavior of RW in this environment.

The SUSY Hyperbolic Model - $H^{(2|2)}$

$$E_{SUSY}(\{t_j\}) = \beta \sum_{j\sim j'} cosh(t_j - t_{j'}) - 1/2 \log \det D_{\beta,\epsilon}(t)$$

where D is finite difference elliptic:

$$[s\,;D_{\beta,\epsilon}(t)\,s]_\Lambda = \beta \sum\nolimits_{(j'\sim j)} \mathrm{e}^{t_j+t_{j'}}(s_i - s_j)^2 + \epsilon \sum\nolimits_{k\in\Lambda} \mathrm{e}^{t_k} s_k^2$$

Spin-Spin correlation:

$$< s_0 e^{t_0} s_x e^{t_x} > (\beta,\epsilon) =< e^{t_0+t_x} D_{\beta,\epsilon}(t)^{-1}(0,x) >_{SUSY} (\beta,\epsilon)$$

E_{SUSY} is **NOT convex** and a **phase transition occurs**.