How to replace the random walk representation in models which don't have one

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Gradient models: *Gaussian lattice free field:* $D \subset\subset \mathbb{Z}^d$

Hamiltonian:

$$H(\boldsymbol{\phi}) := \frac{1}{4d} \sum_{x,y \in \mathbb{Z}^d,\ |x-y|=1} \left(\phi_x - \phi_y\right)^2,\ \boldsymbol{\phi} = \{\phi_x\},\ \phi_x \in \mathbb{R},$$

$$\mu_D(d\boldsymbol{\phi}) = \frac{1}{Z_D} \exp\left[-H(\boldsymbol{\phi})\right] \prod_{x \in D} d\phi_x \prod_{x \notin D} \delta_0(d\phi_x).$$

Random walk representation of covariances: $X_n^D,\ n \in \mathbb{N}$, the standard symmetric random walk on $\mathbb{Z}^d$, killed when exiting D :

$$\mathrm{cov}_D\left(\phi_x, \phi_y\right) = E_x^{\mathrm{RW}}\left(\sum_n 1_y\left(X_n^D\right)\right)$$

Consequence: For $D = D_N := \{-N, \ldots, N\}^d$:

$$\mathrm{var}\,(\phi_0) \approx \begin{cases} \text{const} \times N & \text{for } d = 1 \\ \text{const} \times \log N & \text{for } d = 2 \\ O\,(1) & \text{for } d \geq 3 \end{cases}.$$

For $d \geq 3$, the limit field as $N \to \infty$ exists, with slowly decaying correlations

$$\mathrm{cov}_{\mathbb{Z}^d}\left(\phi_x, \phi_y\right) \sim \frac{\text{const}}{|x-y|^{d-2}}.$$

In contrast, the **massive** field with Hamiltonian

$$H\,(\phi) := \frac{1}{4d} \sum_{x,y \in \mathbb{Z}^d,\ |x-y|=1} \left(\phi_x - \phi_y\right)^2 + m \sum_x \phi_x^2,\ m > 0$$

has exponentially decaying correlations in all dimensions.

There are many generalizations and open problems:

- Replace $\left(\phi_x - \phi_y\right)^2$ by a more general function $V\left(\phi_x - \phi_y\right)$, V strongly convex: Helffer-Sjöstrand representation: RWRE. What if V is not convex? (Cotar-Deuschel 2009 using decimation, Adams-Kotecky-Müller 2016 using field theory).

- Gaussian fields without random walk representation: E.g. **membrane model**:

$$H(\phi) := \text{const} \times \sum_x (\Delta\phi_x)^2 ,$$

where

$$(\Delta\phi)_x := \frac{1}{2d} \sum_{y:y\sim x} \left(\phi_y - \phi_x\right).$$

For $d = 1$: Integrated random walk.

This model has "critical dimension" 4.

Effect of **local pinning** at the origin, so-called δ-pinning with pinning parameter $\varepsilon > 0$:

$$\mu_D^\varepsilon(d\phi) := \frac{1}{Z_D^\varepsilon} \exp\left[-H(\phi)\right] \prod_{x \in D} \left(d\phi_x + \varepsilon\delta_0(d\phi_x)\right) \prod_{x \in \partial D} \delta_0(\phi_x),$$

where

$$Z_D^\varepsilon := \int_{\mathbb{R}^D} \exp\left[-H_N(\phi)\right] \prod_x \left(d\phi_x + \varepsilon\delta_0(d\phi_x)\right) \prod_{x \in \partial D} \delta_0(\phi_x).$$

In the **gradient case: Strongly localizes** the free field, i.e. makes it "massive": If $D_N \uparrow \mathbb{Z}^d$, then

$$\lim_{N \to \infty} \mu_{D_N}^\varepsilon$$

exists for all d, and has **exponentially decaying correlations**, and a **positive density of zeros**. $d \geq 3$ by Brydges, Fröhlich, and Spencer 1982. $d = 2$: B-Brydges 2000, Deuschel-Velenik, Ioffe-Velenik 2000, B-Velenik 2001.

First step: **Positivity of the surface tension**. For all d and $\varepsilon > 0$

$$\xi^{\varepsilon} := \lim_{N\to\infty} \frac{1}{|D_N|} \log \frac{Z^{\varepsilon}_{D_N}}{Z_{D_N}} > 0, \ \forall \varepsilon > 0.$$

Second step: **Expansion**

$$\prod_{x\in D} (d\phi_x + \varepsilon\delta_0 (d\phi_x)) = \sum_{A\subset D} \varepsilon^{|A|} \prod_{x\in A} \delta_0 (d\phi_x) \prod_{x\in D\setminus A} d\phi_x,$$

leading to an expansion of the measure

$$\mu^{\varepsilon}_D = \sum_{A\subset D} \nu^{\varepsilon}_D (A)\, \mu_{D\setminus A}$$

where $\mu_{D\setminus A}$ is the free field on $D\setminus A$ and

$$\nu^{\varepsilon}_D (A) := \frac{Z^{\varepsilon=0}_{D\setminus A}}{Z^{\varepsilon}_D} \varepsilon^{|A|}.$$

Gradient model: Let $\mathcal{A}$ be the random subset with distribution ν_D^{ε}.

- $d \geq 3$: $\exists p(\varepsilon, d) > 0$ such that $\forall D,\ x \in D,\ C \subset D \setminus \{x\}$

$$\nu_D^{\varepsilon}(x \in \mathcal{A} | \mathcal{A} \setminus \{x\} = C) \geq p.$$

- Not true for $d = 2$, but $\forall B \subset D$

$$\nu_D^{\varepsilon}(\mathcal{A} \cap B = \emptyset) \leq (1 - p)^{|B|}.$$

Combined with a random walk representation:

$$\mathrm{cov}_{\mu_D^{\varepsilon}}\left(\phi_x, \phi_y\right) = \sum_{A \subset \mathbb{Z}^d} \nu^{\varepsilon}(A)\, E_x^{\mathrm{RW}}\left(\sum\nolimits_n 1_y\left(X_n^{A^c \cup D}\right)\right)$$

which decays exponentially in $|x - y|$.

A result in B.-Velenik 2001:

$$\mathsf{var}_{\mu_\varepsilon}(\phi_x) \leq \sum_n E_x^{\mathsf{RW}} \left((1 - p_\varepsilon)^{|R_n|} 1_x (X_n) \right).$$

Using refined LD properties for the range (Donsker-Varadhan, B., Sznitman, van den Berg-B-den Hollander), one gets the precise behavior, e.g. for $d = 2$, $\varepsilon \sim 0$.

The domination of ν by Bernoulli is rather delicate in $d = 2$.

Remark: The above expansion is particularly simple for the δ-pinning case. More general local attractions to 0 can be handled, too, for instance adding to the Hamiltonian a summand

$$-\sum_x \varepsilon I (|\phi_x| \leq a),$$

and leaving the reference measure Lebesgue.

Pinning for the **membrane model** (jointly with **Alessandra Cipriani and Noemi Kurt**): Gaussian model with Hamiltonian

$$H(\phi) := \sum_x (\Delta\phi)_x^2 = \left\langle \phi, \Delta^2 \phi \right\rangle,$$

where Δ is the discrete Laplace operator.

$$\mu_N := \frac{1}{Z_N} \exp\left[-H(\phi)\right] \prod_{x \in D_N} d\phi_x \prod_{x \notin D_N} \delta_0(\phi_x),$$

$$\mu_N^\varepsilon : \prod_{x \in D_N} d\phi_x \rightsquigarrow \prod_{x \in D_N} \left(d\phi_x + \varepsilon \delta_0(d\phi_x)\right).$$

In physics literature: Models for membranes: Stiffer than the gradient model. Leibler 1989, Lipowsky 1995, and others.

The difficulty: *Much* less is known or true (for instance correlation inequalities, random walk representation).

The critical dimension is 4 : For $d \geq 5$, the field (without pinning) exists on $\mathbb{Z}^d$ with decay of correlations of order $|x-y|^{4-d}$. For $d=4$, the variance of ϕ_0 is of order $\log N$. $\rightsquigarrow$ in the class of logarithmically correlated models.

Remark: Interesting for $d=1$: Sinai, Caravenna-Deuschel, Dembo-Gao and others.

Question: Does pinning localize the field in a strong sense?

Positivity of the surface tension for $d \geq 2$ was proved by Sakagawa just recently. (earlier result $d \geq 4$).

Bernoulli domination: Trivial for $d \geq 5$, but not known for $d \leq 4$. My conjecture: False for $d = 2, 3$, but true in the above form for $d = 4$.

Theorem Let $d \geq 5$. For all $\varepsilon > 0$, there exist $\eta(d,\varepsilon), C(d,\varepsilon) > 0$ such that

$$\left|\mathrm{cov}_N^{\varepsilon}\left(\phi_x, \phi_y\right)\right| \leq C(\varepsilon, d) \exp\left[-\eta(d,\varepsilon)|x-y|\right].$$

(We don't know of $\lim_{N\to\infty} \mu_N^{\varepsilon}$ exists).

Main problem: Random walk representations seem to be of no use.

Method of proof: Expansion as above

$$\mathrm{cov}_N^{\varepsilon}\left(\phi_x, \phi_y\right) = \sum_{A \subset D_N} \nu^{\varepsilon}(A) G_A(x,y)$$

where here $y \mapsto G_A(x,y)$ for $x \in D_N \backslash A$ satisfies

$$\begin{aligned} G_A(x,y) &= 0,\ y \in A \cup D_N^{\mathsf{c}}, \\ \Delta_y^2 G_A(x,y) &= \delta_{x,y}. \end{aligned}$$

Question: If A is "sufficiently dense" does this imply the exponential decay?

Question about PDEs (in continuous space): Let

$$\Omega := \mathbb{R}^d \backslash \bigcup_{x \in \mathbb{Z}^d} B_r(x), \; r < 1/2,$$

where $B_r(x)$ is the closed ball of radius r around x. Consider in Ω the equation $\Delta^2 u = f$ with f of compact support, and Dirichlet boundary conditions for u and ∇u. How does one prove exponential decay of u?

Remark: The delicacy comes from the boundary conditions. If $\Delta u = 0$ on the boundary, then easy.

Vladimir Mazya: Crucial is an equivalence of norms: Fur u's satisfying the boundary conditions

$$\|u\|_{H^2(\Omega)}^2 := \|u\|_{2,\Omega}^2 + \|\nabla u\|_{2,\Omega}^2 + \left\|\nabla^2 u\right\|_{2,\Omega}^2 \sim \left\|\nabla^2 u\right\|_{L_2(\Omega)}^2$$

which uses the high density of the "trapping" regions $B_r(x)$ and the boundary conditions.

$C_n := B_{\text{const}\times n}(0)$ and smooth interpolating $\eta_n = 1$ on C_{n+1}^{c} and $= 0$ on C_n. For u with $\Delta^2 u = 0$ in $\Omega \cap C_{n+1}^{\mathsf{c}}$

$$
\begin{aligned}
\|u\|^2_{H^2\left(C_{n+1}^{\mathsf{c}}\right)} &= \|\eta_n u\|^2_{H^2\left(C_{n+1}^{\mathsf{c}}\right)} \leq \|\eta_n u\|^2_{H^2(\Omega)} \leq \text{const} \times \left\|\nabla^2 (\eta_n u)\right\|^2_{L_2(\Omega)} \\
&\leq \text{const} \times \left\|\nabla^2 (\eta_n u)\right\|^2_{L_2(C_{n+1}\setminus C_n)} \\
&\leq \text{const} \times \|u\|_{H^2(C_{n+1}\setminus C_n)} \\
&\leq \text{const} \times \left(\|u\|^2_{H^2(C_n^{\mathsf{c}})} - \|u\|^2_{H^2\left(C_{n+1}^{\mathsf{c}}\right)} \right)
\end{aligned}
$$

which proves the exponential decay. This argument works for any Ω which is “porous” enough.

In our case: Everything is on the lattice, and “porosity” is defined via ν^{ε}, the law of the random trap configuration $\mathcal{A}$.

First step: One defines a **random** weighted Sobolev norm: $E \subset \mathbb{Z}^d$

$$\|u\|^2_{H^2(E),\mathcal{A}} := \sum_{x\in E} \frac{u(x)^2}{1 + d(x, \mathcal{A}^o)^{2d+3}} + \sum_{x\in E} \frac{|\nabla u(x)|^2}{1 + d(x, \mathcal{A}^o)^{d+2}} + \sum_{x\in E} \left|\nabla^2 u(x)\right|,$$

where $\mathcal{A}^o$ is the set of lattice inner points of $\mathcal{A}$. Then

$$\|u\|^2_{H^2(E),\mathcal{A}} \leq \text{const}(d) \times \sum_{x\in E} \left|\nabla^2 u(x)\right|.$$

In the Mazja-argument: important that the partial summations inside $C_{n+1} \backslash C_n$ leads via derivatives of η_n to coefficients which are small when $d(x, \mathcal{A}^o)$ is large. This leads to *random* $C_n := C_{n,\mathcal{A}}$ which are growing faster in regions where $\mathcal{A}^o$ is thin.

More precisely: Define

$$\delta_{\mathcal{A}}(x,y) := \min_{\gamma : x \to y} \sum_{k=1}^{|\gamma|} \frac{1}{1 + d\left(\gamma_k, \mathcal{A}^0\right)^{2d+3}},$$

and put

$$C_n := \{x : \delta_{\mathcal{A}}(0,x) \leq 10n\}.$$

Up to now, nothing really depends on d.

For $d \geq 5$ it is easy to prove that ν^{ε} dominates a Bernoulli measure in a strong sense:

$$\nu^{\varepsilon}(x \in \mathcal{A} | \mathcal{A} \setminus \{x\} = B) \geq p^{-}(\varepsilon) > 0.$$

and then C_n and η_n can be chosen such that $\mathbf{diam}(C_n) \leq \mathbf{const} \times n$ with overwhelming ν^{ε}-probability.

Remark: Although not a problem for the theorem, the necessity to use $\mathcal{A}^0$ is awkward: At an isolated point, we don't know what the boundary conditions are.

Open problems:

- What about $d = 4$ or even $d = 2, 3$? Positivity of surface tension was proved by Sakagawa. The strong Bernoulli domination is not true. We don't know if the weaker version

$$\nu^{\varepsilon}(\mathcal{A} \cap B = \emptyset) \leq (1 - p(\varepsilon))^{|B|}, \ \forall B$$

  holds. An adaptation of the Mazja-type argument would probably be possible.

- The only place where the Bernoulli domination is used is to prove the linear increase of $\textbf{diam}\,(C_n)$ which may follow from much weaker notions.

- Properties of non Gaussian cases? For instance

$$H(\phi) := \sum_x V(\Delta \phi_x), \; V \text{ convex.}$$

- Wetting transitions? Here one conditions on $\{\phi_x \geq 0, \; \forall x\}$, but still has a pinning parameter ε. Does there exists $\varepsilon_{\text{cr}}(d) > 0$, such that for $\varepsilon < \varepsilon_{\text{cr}}$ is not pinned (i.e. entropy repulsion wins), and for $\varepsilon > \varepsilon_{\text{cr}}$, pinning wins.

- Many interesting problems other than pinning: E.g. scaling limits of level sets in the critical dimension 4?