Hyperbolicity of minimizers and random Hamilton-Jacobi equations

Qualitative methods in KPZ universality, CRIM, 2017

Joint works with R. Iturriaga and K. Khanin

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Stochastic Burgers equation on the circle

- E-Khanin-Mazel-Sinai (2000) studied the following equation:

$$u_t + u_x u = \nu u_{xx} + f^\omega(x,t), \quad x \in \mathbb{T}^1,\ \nu \geqslant 0,$$

where $f^\omega(x,t) = \sum_{k=1}^M F_k'(x)\dot{W}_k(t)$, $F_k(x)$ are C^3 functions on C^1.

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  2. Higher dimension: $x \in \mathbb{T}^d$.

Random Hamilton-Jacobi equations

- For $b \in \mathbb{R}^d$, consider

$$\varphi_t + \frac{1}{2}\|\nabla\varphi + b\|^2 = F^\omega(x,t), \quad x \in \mathbb{T}^d, \omega \in \Omega.$$

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- Focus on "kicked" case.

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- Dual semi-group:

$$\check{T}^\omega_{s,t} g(x) = \max_{\gamma(s)=x} \left\{ g(\gamma(t)) - \int_s^t L(\gamma, \dot\gamma, \tau) d\tau \right\}.$$

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- Theorem (A)

  Almost surely in ω*, there is unique* $\psi_\omega^-(x,n)$ *for* $-\infty < n \leqslant 0$ *such that*

$$\lim_{m \to -\infty} |||T^\omega_{m,n} g(x) - \psi^-_\omega(x,n)||| = 0,$$

  uniformly over all $g \in C(\mathbb{T}^d)$*.* $d = 1$ *(E-Khanin-Mazel-Sinai 2000),* $d \geqslant 1$ *(Iturriaga-Khanin 2002).*

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uniformly over all $g \in C(\mathbb{T}^d)$. $d = 1$ (E-Khanin-Mazel-Sinai 2000), $d \geqslant 1$ (Iturriaga-Khanin 2002).

  - Exists solution $\psi^+_\omega(x, n)$ for $0 \leqslant n < \infty$ for the dual semi-group.

Minimizers

- A curve $\gamma : I \to \mathbb{T}^d$ is called a minimizer if it minimizes $\int_a^b L(\gamma, \dot\gamma, t)dt$ on each $[a, b] \subset I$. Minimizers solve the Euler-Lagrange equation.

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- $-\psi^+(x, t)$ = action of infinite forward minimizer.

Regularity

- Specific form of kicks:

$$F_j^\omega = \sum_{i=1}^{M} F_i(x)\xi_j^i(\omega),$$

with $F_i \in C^3(\mathbb{T}^d)$, and $\xi_j(\omega) = (\xi_j^1, \cdots \xi_j^M)(\omega)$ is i.i.d. random vector with smooth densities.

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- Theorem (B)

Under suitable conditions for $\xi_j(\omega)$ and $F_1, \cdots, F_M$, there exists $x_0^{\omega} \in \mathbb{T}^d$, $r(\omega) > 0$ such that

$$\psi_{\omega}^{-}(x, 0) \text{ is } C^3 \text{ on } \|x - x_0^{\omega}\| < r(\omega).$$

$d = 1$ (E-Khanin-Mazel-Sinai), $d \geqslant 1$ (Khanin-Z)

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- Theorem (C)

  Under the same conditions, there exists (non-random) $\lambda > 0$ and $C(\omega) > 0$ such that

$$\| T^{\omega}_{m,0} g(x) - \psi^{-}_{\omega}(x, 0) \| \leqslant C(\omega) e^{-\lambda |m|}.$$

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  - For viscous equation

    $$\varphi_t + \frac{1}{2}\|\nabla\varphi(x)\|^2 = F^{\omega}(x,t) + \nu\Delta\varphi,$$

    the solution converges exponentially with exponent $\lambda(\nu)$. (Sinai)

The global minimizer

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- Proposition (EKMS, Iturriaga-Khanin)

  Almost surely, $Q^\omega(x,0)$ has a unique minimum.

  - Proof: convex analysis. Recall $F_j^\omega = \sum_{k=1}^M \xi_j^k(\omega) F_i(x)$. There is $\psi(x)$ such that

$$Q^\omega(x,0) = \sum_{k=1}^M \xi_0^k(\omega) F_i(x) + \psi(x).$$

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- Proposition

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  - The above two propositions implies Theorem B.

Green bundles

- An orbit (x_n, v_n) is disconjugate if

$$D\Phi^{\omega}_{m,n}(x_m, v_m)\left(\{0\} \times \mathbb{R}^d\right) \pitchfork \left(\{0\} \times \mathbb{R}^d\right).$$

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- Then the following limits exist:

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- If $\mathcal{G}^u(x_n, v_n) \pitchfork \mathcal{G}^s(x_n, v_n)$, then (x_n, v_n) is hyperbolic.
- Uses the works of M.-C. Arnaud. (also Green, Bialy-McKay, Contreras-Iturriaga)

Non-degeneracy of minimum implies hyperbolicity

- (Non-degeneracy of minimum) There exists $a(\omega) > 0$ and $R > 0$ such that

$$Q_\omega^\infty(x, 0) - Q_\omega^\infty(x_0^\omega, 0) \geqslant a(\omega)\|x - x_0^\omega\|^2, \quad \|x - x_0^\omega\| < R.$$

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- (M.-C. Arnaud) the above implies $\mathcal{G}^u \pitchfork \mathcal{G}^s$, and hence implies hyperbolicity of (x_n^ω, v_n^ω).

Unstable manifolds

- Orbit of $(y_0, \eta_0) := (y_0, \nabla\psi_\omega^-(y_0, 0))$, denoted $\{(y_n, \eta_n)\}$, is a minimizer on $(-\infty, 0]$.

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- Orbit of $(y_0, \eta_0) := (y_0, \nabla\psi^-_\omega(y_0, 0))$, denoted $\{(y_n, \eta_n)\}$, is a minimizer on $(-\infty, 0]$.
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- To prove (y_n, η_n) is stable, note that $Q^\infty_\omega(\cdot, n)$ is a Lyapunov function in the sense

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- Q_ω^∞ is non-degenerate in the sense that

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- Use the above observations to prove Theorem B.

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$$\|(y_n,\eta_n) - (x_n^\omega, v_n^\omega)\| \leqslant C(\omega)e^{-\lambda N}.$$

- $Q_\omega^\infty(\cdot,0)$ is not a Lyapunov function for finite minimizers.

Exponential convergence

- To prove Theorem C, denote $\psi_N^-(x,0) = T_{-N,0}g(x)$, and $(y_0, \eta_0) = (y_0, \nabla\psi_N^-(x,0))$.
- Want to show for some $-N \leqslant n < 0$,

$$\|(y_n, \eta_n) - (x_n^\omega, v_n^\omega)\| \leqslant C(\omega)e^{-\lambda N}.$$

- $Q_\omega^\infty(\cdot, 0)$ is not a Lyapunov function for finite minimizers.
- But if $\|\psi_N^-(x,0) - \psi_\omega^-(x,0)\| = O(\delta)$, then $Q_\omega^\infty(\cdot, 0)$ is an $\delta-$approximate Lyapunov function in the sense that

$$Q_\omega^\infty(y_i, i) - Q_\omega^\infty(x_i^\omega, i) \leqslant Q_\omega^\infty(y_j, j) - Q_\omega^\infty(x_j^\omega, j) + \delta.$$

Upgrade argument

1. Use a priori convergence (Theorem A), to get $\|\psi_N^-(x,0) - \psi_\omega^-(x,0)\| = O(\delta)$.

This proves Theorem C.

Upgrade argument

1. Use a priori convergence (Theorem A), to get $\|\psi_N^-(x,0) - \psi_\omega^-(x,0)\| = O(\delta)$.
2. Use the fact that $Q_\omega^\infty(\cdot,0)$ is an $\delta-$approximate Lyapunov function, and hyperbolic theory to prove for some n

$$\|(y_n,\eta_n) - (x_n^\omega, v_n^\omega)\| \leqslant \delta^{100}.$$

This proves Theorem C.

Upgrade argument

1. Use a priori convergence (Theorem A), to get $\|\|\psi_N^-(x,0) - \psi_\omega^-(x,0)\|\| = O(\delta)$.
2. Use the fact that $Q_\omega^\infty(\cdot,0)$ is an $\delta-$approximate Lyapunov function, and hyperbolic theory to prove for some n

$$\|(y_n, \eta_n) - (x_n^\omega, v_n^\omega)\| \leqslant \delta^{100}.$$

3. Use step 2 to get $\|\|\psi_N^-(x,0) - \psi_\omega^-(x,0)\|\| = O(\delta^{100})$ and repeat from step 1.

This proves Theorem C.

Thank you!