Hyperbolicity of minimizers and random Hamilton-Jacobi equations
Qualitative methods in KPZ universality, CRIM, 2017
Joint works with R. Iturriaga and K. Khanin

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E-Khanin-Mazel-Sinai (2000) studied the following equation:

\[ u_t + u_x u = \nu u_{xx} + f^\omega(x, t), \quad x \in \mathbb{T}^1, \nu \geq 0, \]

where \( f^\omega(x, t) = \sum_{k=1}^M F'_k(x) \dot{W}_k(t), \) \( F_k(x) \) are \( C^3 \) functions on \( C^1 \).
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For the invisible case $\nu = 0$, they proved the one force one solution principle, and much more.
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where \( f^\omega(x, t) = \sum_{k=1}^{M} F'_k(x) \dot{W}_k(t), F_k(x) \) are \( C^3 \) functions on \( C^1 \).

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**Generalizations:**

1. **Non-compact case:** \( x \in \mathbb{R} \). See Bakhtin-Cator-Khanin (2014), Bakhtin (2016), Bakhtin-Li (2017) for one force one solution principle.
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  1. Non-compact case: \( x \in \mathbb{R} \). See Bakhtin-Cator-Khanin (2014), Bakhtin (2016), Bakhtin-Li (2017) for one force one solution principle.
  2. Higher dimension: \( x \in \mathbb{T}^d \).
Random Hamilton-Jacobi equations

- For $b \in \mathbb{R}^d$, consider

$$\varphi_t + \frac{1}{2}\|\nabla \varphi + b\|^2 = F^\omega(x, t), \quad x \in \mathbb{T}^d, \omega \in \Omega.$$  

Recovers the Burgers equation via $u = \nabla \varphi + b$.  

The choice of the random potential:

1. (White noise) $F^\omega p x, t q = \sum_{k} f_{k} p x q \mathbb{M}^\omega k^9 w_{k} p t q$.

2. (Periodically kicked) $F^\omega p x, t q$ are i.i.d. randomly chosen functions.

Focus on "kicked" case.
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- The choice of the random potential:

  1. (White noise)

$$F^\omega(x, t) = \sum_{k} F_k(x, t) \delta_{k}, \quad \delta_k = \delta_{k, t}.$$ 

  2. (Periodically kicked)

$$F^\omega(x, t) = \sum_{j} F_j(x, t) \delta_{j,k}.$$ 

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$$F^\omega(x, t) = \sum_{j \in \mathbb{Z}} F_j^\omega(x) \delta(t - j).$$
Random Hamilton-Jacobi equations

- For \( b \in \mathbb{R}^d \), consider

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\phi_t + \frac{1}{2} \| \nabla \phi + b \|^2 = F^\omega(x, t), \quad x \in \mathbb{T}^d, \omega \in \Omega.
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- Focus on “kicked” case.
The Lax-Oleinik semi-group

- Lagrangian $L^\omega(x, v, t) = \frac{1}{2}v^2 - b \cdot v + F^\omega(x, t)$, $b \in \mathbb{R}^d$. 
The Lax-Oleinik semi-group

- Lagrangian \( L^\omega(x, v, t) = \frac{1}{2}v^2 - b \cdot v + F^\omega(x, t), b \in \mathbb{R}^d \).
- Define (Lax-Oleinik semi-group)

\[
T^\omega_{s,t}g(x) = \min_{\gamma(t) = x} \left\{ g(\gamma(s)) + \int_{s}^{t} L(\gamma, \dot{\gamma}, \tau) d\tau \right\}.
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- Then \( T^\omega_{s,t}g(x) \) solves

\[
\begin{align*}
\varphi_t + \frac{1}{2} \| \nabla \varphi + b \|^2 &= F^\omega(x, t) \\
\varphi(x, 0) &= g(x).
\end{align*}
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- Define (Lax-Oleinik semi-group)

$$T^\omega_{s,t}g(x) = \min_{\gamma(t)=x} \left\{ g(\gamma(s)) + \int_s^t L(\gamma, \dot{\gamma}, \tau) \, d\tau \right\}.$$ 

- Then $T^\omega_{s,t}g(x)$ solves

$$\begin{cases}
\varphi_t + \frac{1}{2} \| \nabla \varphi + b \|^2 = F^\omega(x, t) \\
\varphi(x, 0) = g(x).
\end{cases}$$

- Dual semi-group:

$$\tilde{T}^\omega_{s,t}g(x) = \max_{\gamma(s)=x} \left\{ g(\gamma(t)) - \int_s^t L(\gamma, \dot{\gamma}, \tau) \, d\tau \right\}.$$
One force, one solution

- Define

\[ \| \varphi \| = \frac{1}{2} (\max \varphi - \min \varphi). \]
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- Theorem (A)

> Almost surely in \( \omega \), there is unique \( \psi^{-}(x, n) \) for \( -\infty < n \leq 0 \) such that

\[ \lim_{m \to -\infty} \left\| T_{m,n} g(x) - \psi^{-}(x, n) \right\| = 0, \]

uniformly over all \( g \in C(\mathbb{T}^d) \). \( d = 1 \) (E-Khanin-Mazel-Sinai 2000), \( d \geq 1 \) (Iturriaga-Khanin 2002).
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  - Exists solution \(\psi^+_\omega(x, n)\) for \(0 \leq n < \infty\) for the dual semi-group.
Minimizers

- A curve $\gamma : I \rightarrow \mathbb{T}^d$ is called a minimizer if it minimizes $\int_a^b L(\gamma, \dot{\gamma}, t) dt$ on each $[a, b] \subset I$. Minimizers solve the Euler-Lagrange equation.

Ke Zhang  Random Hamilton-Jacobi
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- At each point $x$, there exits a minimizer $\gamma_x^- : (-\infty, 0] \rightarrow \mathbb{T}^d$ and $\gamma_x^+ : [0, \infty) \rightarrow \mathbb{T}^d$ with $\gamma_x^\pm(0) = x$. Called the infinite backward minimizers.
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- One force one solution $\simeq$ Convergence of backward minimizers to each other.
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- One force one solution \( \simeq \) Convergence of backward minimizers to each other.

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- \( -\psi^+ (x, t) = \) action of infinite forward minimizer.
Regularity

- Specific form of kicks:

\[ F_j^\omega = \sum_{i=1}^{M} F_i(x) \xi_j^i(\omega), \]

with \( F_i \in C^3(\mathbb{T}^d) \), and \( \xi_j(\omega) = (\xi_j^1, \cdots \xi_j^M)(\omega) \) is i.i.d. random vector with smooth densities.
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- Theorem (B)

Under suitable conditions for \( \xi_j(\omega) \) and \( F_1, \cdots, F_M \), there exists \( x_0^\omega \in \mathbb{T}^d, r(\omega) > 0 \) such that

\[ \psi_{\omega}^-(x, 0) \text{ is } C^3 \text{ on } \|x - x_0^\omega\| < r(\omega). \]

\( d = 1 \) (E-Khanin-Mazel-Sinai), \( d \geq 1 \) (Khanin-Z)
Theorem (C)

Under the same conditions, there exists (non-random) $\lambda > 0$ and $C > 0$ such that

$\| \psi \|_p < C \| \phi \|_p e^{-\lambda m}$.

For viscous equation $\psi \in C^0(\mathbb{T}, \mathbb{R})$, $\psi \in C^1(\mathbb{T}, \mathbb{R})$,

$\phi_t \sim \frac{1}{2} \nabla \phi \cdot F + \nu \Delta \phi$,

the solution converges exponentially with exponent $\lambda \nu > 0$. (Sinai)
Exponential convergence

- Theorem (C)

Under the same conditions, there exists (non-random) $\lambda > 0$ and $C(\omega) > 0$ such that

$$\| T^\omega_{m,0} g(x) - \psi^-_\omega(x, 0) \| \leq C(\omega) e^{-\lambda|m|}.$$ 

$d = 1$ (EKMS, Boritchev), $d \geq 1$ (Iturriaga-Khanin-Z)
Exponential convergence

- Theorem (C)

Under the same conditions, there exists (non-random) \( \lambda > 0 \) and \( C(\omega) > 0 \) such that

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\| T_{m,0}^{\omega} g(x) - \psi_{\omega}^{-}(x, 0) \| \leq C(\omega) e^{-\lambda|m|}.
\]

\( d = 1 \) (EKMS, Boritchev), \( d \geq 1 \) (Iturriaga-Khanin-Z)

- For viscous equation

\[
\varphi_t + \frac{1}{2} \| \nabla \varphi(x) \|^2 = F^\omega(x, t) + \nu \Delta \varphi,
\]

the solution converges exponentially with exponent \( \lambda(\nu) \).

(Sinai)
The global minimizer

- Recall: $\psi^-(x, t)$ is the action of infinite backward minimizer, $-\psi^+(x, t)$ action of infinite forward minimizer.
The global minimizer

- Recall: $\psi^-(x, t)$ is the action of infinite backward minimizer, $-\psi^+(x, t)$ action of infinite forward minimizer.
- The minimum of

\[ Q_\omega(x, t) := \psi_\omega^-(x, t) - \psi_\omega^+(x, t) \]

corresponds global minimizers.
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- Proposition (EKMS, Iturriaga-Khanin)
  
  *Almost surely, \( Q^\omega (x, 0) \) has a unique minimum.*
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- Proposition (EKMS, Iturriaga-Khanin)

  *Almost surely, $Q^\omega(x, 0)$ has a unique minimum.*

  - Proof: convex analysis. Recall $F^\omega_j = \sum_{k=1}^M \xi^k_j(\omega) F_i(x)$. There is $\psi(x)$ such that

    $$Q^\omega(x, 0) = \sum_{k=1}^M \xi^k_0(\omega) F_i(x) + \psi(x).$$
Hyperbolicity of the global minimizer
Hyperbolicity of the global minimizer

- **Proposition**

  Orbit of \((x_0^\omega, v_0^\omega)\) is *nonuniformly hyperbolic* under the dynamics of Euler-Lagrange flow. Namely, the Lyapunov exponent of the orbit \((x_n^\omega, v_n^\omega)\) satisfies

  \[
  \lambda_1 \leq \cdots \leq \lambda_d < 0 < \lambda_{d+1} \leq \cdots \leq \lambda_{2d}.
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The graph \(\{(x, \nabla \psi^- (x, 0))\}\) contains the local unstable manifold of \((x_0^\omega, v_0^\omega)\).
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Proposition

The graph \(\{(x, \nabla \psi^-_{\omega}(x, 0))\}\) contains the local unstable manifold of \((x_0^\omega, v_0^\omega)\).

- The above two propositions implies Theorem B.
Green bundles

- An orbit \((x_n, \nu_n)\) is disconjugate if

\[
D\Phi_{m,n}^\omega(x_m, \nu_m) \left( \{0\} \times \mathbb{R}^d \right) \cap \left( \{0\} \times \mathbb{R}^d \right).
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\]

- Then the following limits exist:

\[
G^u(x_n, v_n) = \lim_{m \to -\infty} D\Phi_{m,n}^{\omega}(x_m, v_m) \left( \{0\} \times \mathbb{R}^d \right),
\]

\[
G^s(x_n, v_n) = \lim_{k \to \infty} (D\Phi_{n,k}^{\omega})^{-1}(x_n, v_n) \left( \{0\} \times \mathbb{R}^d \right),
\]

If \(G^u(x_n, v_n)\) & \(G^s(x_n, v_n)\), then \((x_n, v_n)\) is hyperbolic.
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- An orbit \((x_n, v_n)\) is disconjugate if
  \[D\Phi^\omega_{m,n}(x_m, v_m) \left(\{0\} \times \mathbb{R}^d\right) \cap \left(\{0\} \times \mathbb{R}^d\right).

- Then the following limits exist:
  \[\mathcal{G}^u(x_n, v_n) = \lim_{m \to -\infty} D\Phi^\omega_{m,n}(x_m, v_m) \left(\{0\} \times \mathbb{R}^d\right),\]
  \[\mathcal{G}^s(x_n, v_n) = \lim_{k \to \infty} (D\Phi^\omega_{n,k})^{-1}(x_n, v_n) \left(\{0\} \times \mathbb{R}^d\right),\]
- If \(\mathcal{G}^u(x_n, v_n) \cap \mathcal{G}^s(x_n, v_n)\), then \((x_n, v_n)\) is hyperbolic.
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- If \(\mathcal{G}^u(x_n, v_n) \cap \mathcal{G}^s(x_n, v_n)\), then \((x_n, v_n)\) is hyperbolic.

- Uses the works of M.-C. Arnaud. (also Green, Bialy-McKay, Contreras-Iturriaga)
(Non-degeneracy of minimum) There exists $a(\omega) > 0$ and $R > 0$ such that

$$Q_\omega^\infty(x, 0) - Q_\omega^\infty(x_0^\omega, 0) \geq a(\omega)\|x - x_0^\omega\|^2, \quad \|x - x_0^\omega\| < R.$$ 

Proof (convex analysis)
Non-degeneracy of minimum implies hyperbolicity

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\[ Q_\omega^\infty(x, 0) - Q_\omega^\infty(x_0^\omega, 0) \geq a(\omega)\|x - x_0^\omega\|^2, \quad \|x - x_0^\omega\| < R. \]

Proof (convex analysis)

- (M.-C. Arnaud) the above implies $\mathcal{G}^u \pitchfork \mathcal{G}^s$, and hence implies hyperbolicity of $(x_n^\omega, v_n^\omega)$. 
Unstable manifolds

- Orbit of $(y_0, \eta_0) := (y_0, \nabla \psi_\omega(y_0, 0))$, denoted $\{(y_n, \eta_n)\}$, is a minimizer on $(-\infty, 0]$. 
Unstable manifolds

- Orbit of \((y_0, \eta_0) := (y_0, \nabla \psi_\omega(y_0, 0))\), denoted \(\{(y_n, \eta_n)\}\), is a minimizer on \((-\infty, 0]\).
- Want to show \((y_0, \eta_0)\) is on the unstable manifold of \((x_0^\omega, v_0^\omega)\).
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- Near a hyperbolic orbit, backward non-expanding orbit must be exponentially stable.
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- Near a hyperbolic orbit, backward non-expanding orbit must be exponentially stable.
- To prove \((y_n, \eta_n)\) is stable, note that \(Q^\omega_\omega(\cdot, n)\) is a Lyapunov function in the sense

\[
0 \leq Q^\omega_\omega(y_i, i) - Q^\omega_\omega(x_i^\omega, i) \leq Q^\omega_\omega(y_j, j) - Q^\omega_\omega(x_j^\omega, j)
\]

if \(i < j\).
Unstable manifolds

- Orbit of \((y_0, \eta_0) \coloneqq (y_0, \nabla \psi_{\omega}^{-}(y_0, 0))\), denoted \(\{(y_n, \eta_n)\}\), is a minimizer on \((-\infty, 0]\).
- Want to show \((y_0, \eta_0)\) is on the unstable manifold of \((x^\omega_0, v^\omega_0)\).
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\]

if \(i < j\).
- \(Q^\infty_\omega\) is non-degenerate in the sense that

\[
a(\omega)\|x - x^\omega_0\|^2 \leq Q^\infty_\omega (x, i) - Q^\infty_\omega (x^\omega_i, i) \leq K(\omega)\|x - x^\omega_0\|^2.
\]
Unstable manifolds

- Orbit of \((y_0, \eta_0) := (y_0, \nabla \psi_{\omega}(y_0, 0))\), denoted \(\{(y_n, \eta_n)\}\), is a minimizer on \((-\infty, 0]\).
- Want to show \((y_0, \eta_0)\) is on the unstable manifold of \((x_0^\omega, v_0^\omega)\).
- Near a hyperbolic orbit, backward non-expanding orbit must be exponentially stable.
- To prove \((y_n, \eta_n)\) is stable, note that \(Q^\infty(\cdot, n)\) is a Lyapunov function in the sense

\[
0 \leq Q^\infty(y_i, i) - Q^\infty(x_i^\omega, i) \leq Q^\infty(y_j, j) - Q^\infty(x_j^\omega, j)
\]

if \(i < j\).
- \(Q^\infty\) is non-degenerate in the sense that

\[
a(\omega)\|x - x_0^\omega\|^2 \leq Q^\infty(x, i) - Q^\infty(x_i^\omega, i) \leq K(\omega)\|x - x_0^\omega\|^2.
\]
- Use the above observations to prove Theorem B.
To prove Theorem C, denote $\psi_N^-(x, 0) = T_{-N,0} g(x)$, and $(y_0, \eta_0) = (y_0, \nabla \psi_N^-(x, 0))$. 
Exponential convergence

- To prove Theorem C, denote $\psi_{-N}^{-}(x, 0) = T_{-N, 0} g(x)$, and $(y_0, \eta_0) = (y_0, \nabla \psi_{-N}^{-}(x, 0))$.
- Want to show for some $-N \leq n < 0$,

$$
\| (y_n, \eta_n) - (x_n^\omega, v_n^\omega) \| \leq C(\omega) e^{-\lambda N}.
$$
To prove Theorem C, denote $\psi_N^-(x, 0) = T_{-N,0}g(x)$, and $(y_0, \eta_0) = (y_0, \nabla \psi_N^-(x, 0))$.

Want to show for some $-N \leq n < 0$,

$$\|(y_n, \eta_n) - (x_n^\omega, v_n^\omega)\| \leq C(\omega)e^{-\lambda N}.$$ 

$Q_\omega^\infty(\cdot, 0)$ is not a Lyapunov function for finite minimizers.
Exponential convergence

- To prove Theorem C, denote $\psi_N^-(x, 0) = T_{-N, 0}g(x)$, and $(y_0, \eta_0) = (y_0, \nabla \psi_N^-(x, 0))$.
- Want to show for some $-N \leq n < 0$,

$$\| (y_n, \eta_n) - (x_n^\omega, v_n^\omega) \| \leq C(\omega) e^{-\lambda N}.$$  

- $Q_0^\infty(\cdot, 0)$ is not a Lyapunov function for finite minimizers.
- But if $\| \psi_N^-(x, 0) - \psi_0^\omega(x, 0) \| = O(\delta)$, then $Q_0^\infty(\cdot, 0)$ is an $\delta$—approximate Lyapunov function in the sense that

$$Q_0^\infty(y_i, i) - Q_0^\infty(x_i^\omega, i) \leq Q_0^\infty(y_j, j) - Q_0^\infty(x_j^\omega, j) + \delta.$$
Upgrade argument

1. Use a priori convergence (Theorem A), to get
\[ \| \psi_N(x, 0) - \psi_\omega(x, 0) \| = O(\delta). \]

This proves Theorem C.
Upgrade argument

1. Use a priori convergence (Theorem A), to get
\[ \| \psi_N^{-1}(x, 0) - \psi_\omega^{-1}(x, 0) \| = O(\delta). \]

2. Use the fact that \( Q_\omega^\infty(\cdot, 0) \) is an \( \delta \)–approximate Lyapunov function, and hyperbolic theory to prove for some \( n \)
\[ \| (y_n, \eta_n) - (x_n^\omega, v_n^\omega) \| \leq \delta^{100}. \]

This proves Theorem C.
Upgrade argument

1. Use a priori convergence (Theorem A), to get
\[ \| \psi_N^-(x, 0) - \psi_\omega^-(x, 0) \| = O(\delta). \]

2. Use the fact that \( Q_\omega^\infty(\cdot, 0) \) is an \( \delta \)-approximate Lyapunov function, and hyperbolic theory to prove for some \( n \)
\[
\| (y_n, \eta_n) - (x_n^\omega, v_n^\omega) \| \leq \delta^{100}.
\]

3. Use step 2 to get \( \| \psi_N^-(x, 0) - \psi_\omega^-(x, 0) \| = O(\delta^{100}) \) and repeat from step 1.

This proves Theorem C.
Thank you!