KPZ wandering exponent for random walk in i.i.d. dynamic Beta random environment

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Joint work with Márton Balázs (Bristol)
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Coin tosses and random walk

Toss a coin: Heads with probability $p$, Tails with probability $1 - p$.

Tails=↑, Heads=→
Coin tosses and random walk

Toss a coin: Heads with probability $p$, Tails with probability $1 - p$.

$\text{Tails}=\uparrow, \text{ Heads}=\rightarrow$

Repeated tosses $\longleftrightarrow$ up-right path (Random Walk on $\mathbb{Z}^2$):

HHTHTTTTHHTHTT $\longleftrightarrow$
Classical results: LLN, CLT, LDP

\[ X_0 = 0, \quad X_n = \text{position on up-right path after } n \text{ tosses/steps.} \]

\[ X_n \cdot e_1 = \# H, \quad X_n \cdot e_2 = n - X_n \cdot e_1 = \# T \text{ (up to toss } n). \]
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**Law of Large Numbers (LLN): proportion of } H \rightarrow \text{ probability of } H\)

\( X_n/n \rightarrow \xi = pe_1 + (1 - p)e_2 \quad (\text{almost surely}). \)
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Law of Large Numbers (LLN): proportion of \(H\) → probability of \(H\)

\(X_n / n \to \xi = pe_1 + (1 - p)e_2\) (almost surely).

Central Limit Theorem (CLT): fluctuations around the mean are order \(\sqrt{n}\)

\[\frac{X_n - n\xi}{\sqrt{p(1 - p)n}} \to Z e_1 - Z e_2 \text{ (in distribution)}, \ Z \sim \text{Standard Normal.}\]
Classical results: LLN, CLT, LDP

$X_0 = 0, \quad X_n = \text{position on up-right path after } n \text{ tosses/steps.}$

$X_n \cdot e_1 = \#H, \quad X_n \cdot e_2 = n - X_n \cdot e_1 = \#T \text{ (up to toss } n).$

Law of Large Numbers (LLN): proportion of $H \rightarrow$ probability of $H$

$X_n/n \rightarrow \xi = pe_1 + (1 - p)e_2 \quad \text{(almost surely).}$

Central Limit Theorem (CLT): fluctuations around the mean are order $\sqrt{n}$

$$
\frac{X_n - n\xi}{\sqrt{p(1 - p)n}} \rightarrow Z e_1 - Z e_2 \text{ (in distribution), } Z \sim \text{Standard Normal.}
$$

Large Deviation Principle (LDP): $P(\text{proportion of } H \geq s > p) \approx e^{-nH(s)}$

$$
n^{-1} \log P\{X_n \cdot e_1 \geq sn\} \rightarrow H(s) = s \log \frac{s}{p} + (1 - s) \log \frac{1 - s}{1 - p}
$$

$H(s) = \text{entropy of coin } s \text{ relative to coin } p.$
**Conditioned random walk**

Large deviations also tell us that $X_{0,n} = (X_0, \ldots, X_n)$ conditioned on $X_n = n \mu = se_1 + (1 - s)e_2$

converges (in distribution) to a random walk with probability of Heads $= s$. 
Conditioned random walk

Large deviations also tell us that $X_{0,n} = (X_0, \ldots, X_n)$ conditioned on $X_n/n \approx \zeta = se_1 + (1 - s)e_2$

converges (in distribution) to a random walk with probability of Heads $= s$.

New random walk still has CLT fluctuations (of size $\sqrt{n}$).
Random walk in random environment

Take medium inhomogeneity into account.
Random walk in random environment

Take medium inhomogeneity into account.

Make \( p \) random and dependent on the number of heads and tails so far

\[ #H = i, \quad #T = j: \text{next } H \text{ has (random) probability } p_{i,j}. \]
Random walk in random environment

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Make $p$ random and dependent on the number of heads and tails so far

$\#H = i, \quad \#T = j$: next $H$ has (random) probability $p_{i,j}$.

$X_n$ is now a Markov chain with transitions

$P\{X_{n+1} = x + e_1 \mid X_n = x\} = p_x$

$P\{X_{n+1} = x + e_2 \mid X_n = x\} = 1 - p_x, \quad x \in \mathbb{Z}_+^2$. 

$x = (i, j)$
Random walk in random environment

Take medium inhomogeneity into account.

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\]

$HHTHTTT$ has probability

\[
p_{0,0} p_{1,0} (1 - p_{2,0}) p_{2,1} (1 - p_{3,1}) (1 - p_{3,2})
\]
Assume ω = \{p_x : x \in \mathbb{Z}^2\} are independent identically distributed (i.i.d.).
LLN, CLT, LDP

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If $\omega$ is averaged out, distribution of $X_{0,\infty}$ is the same as that of a random walk with probability of $H = \bar{p} = \mathbb{E}[p_0]$ and probability of $T = 1 - \bar{p}$.
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$\text{LLN: } X_n/n \rightarrow \xi = \bar{p}e_1 + (1 - \bar{p})e_2$. 
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**LLN:** $X_n/n \rightarrow \xi = \bar{p}e_1 + (1 - \bar{p})e_2$.

Almost every environment $\{p_x : x \in \mathbb{Z}_+^2\}$ and almost every path $X_{0,\infty}$.
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Almost every environment $\{p_x : x \in \mathbb{Z}^2\}$ and almost every path $X_{0,\infty}$.

Averaged CLT: if the environment is averaged out, then

$$\frac{X_n - n\xi}{\sqrt{\bar{p}(1 - \bar{p})n}} \rightarrow Z e_1 - Z e_2 \text{ (in distribution), } Z \text{ Standard Normal.}$$
Also, Quenched CLT (R-A, Seppäläinen ’05): for almost every environment \( \{ p_x : x \in \mathbb{Z}_+^2 \} \)

\[
\frac{X_n - n\xi}{\sqrt{\bar{p}(1 - \bar{p})n}} \to Z_{e_1} - Z_{e_2} \text{ (in distribution), } Z \text{ Standard Normal.}
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Also, Quenched CLT (R-A, Seppäläinen '05): for almost every environment \( \{p_x : x \in \mathbb{Z}^2_+\} \)

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Note: once environment is fixed, \( X_n \) is no longer a random walk with i.i.d. increments.
LLN, CLT, LDP

Averaged LDP: when environment is averaged out and $s > \bar{p}$

$$-n^{-1} \log P\{X_n \cdot e_1 \geq sn\} \to H_a(s) = s \log \frac{s}{\bar{p}} + (1 - s) \log \frac{1 - s}{1 - \bar{p}}.$$

Also, Quenched LDP (R-A, Seppäläinen, Yilmaz '13):

for almost every environment $p = \{p_x : x \in \mathbb{Z}^2\}$

$$-n^{-1} \log P\{X_n \cdot e_1 \geq sn\} \to H_q(s) = \frac{1}{n} \sum_{i=1}^{n} \log \left( 1 - p_{X_i} \right).$$

$H_q$ is deterministic but in general does not have an explicit expression

unless $s = \bar{p}$, in which case both $p = 0$.
**LLN, CLT, LDP**

**Averaged LDP**: when environment is averaged out and $s > \bar{p}$

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Also, **Quenched LDP** (R-A, Seppäläinen, Yilmaz ’13):

for almost every environment $\omega = \{ p_x : x \in \mathbb{Z}_+^2 \}$

$$- n^{-1} \log P^\omega \{ X_n \cdot e_1 \geq s n \} \to H_q(s).$$
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**Averaged LDP:** when environment is averaged out and $s > \bar{p}$

\[-n^{-1} \log P\{X_n \cdot e_1 \geq sn\} \to H_a(s) = s \log \frac{s}{\bar{p}} + (1 - s) \log \frac{1 - s}{1 - \bar{p}}.\]

Also, **Quenched LDP** (R-A, Seppäläinen, Yilmaz '13):

for almost every environment $\omega = \{p_x : x \in \mathbb{Z}_+^2\}$

\[-n^{-1} \log P^\omega\{X_n \cdot e_1 \geq sn\} \to H_q(s).\]

$H_q$ is deterministic but in general does not have an explicit expression (though some variational formulas are available).

$H_q(s) > H_a(s)$ unless $s = \bar{p}$, in which case both $= 0$. 
**Solvable model**

Explicit computations are possible when $p_x \sim \text{Beta}(\alpha, \beta)$, $\alpha, \beta > 0$.

**Example:** $\text{Beta}(1, 1) = \text{Uniform}(0, 1)$. 
**Solvable model**

Explicit computations are possible when \( p_x \sim \text{Beta}(\alpha, \beta), \alpha, \beta > 0 \).

**Example:** \( \text{Beta}(1, 1) = \text{Uniform}(0, 1) \).

**LLN velocity:** \( \bar{p} = \frac{\alpha}{\alpha + \beta} \) and \( \xi = \frac{\alpha e_1 + \beta e_2}{\alpha + \beta} \).
Solvable model

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LLN velocity: $\bar{p} = \frac{\alpha}{\alpha + \beta}$ and $\xi = \frac{\alpha e_1 + \beta e_2}{\alpha + \beta}$.

Can also compute the quenched rate $H_q(s)$ explicitly (later).
**KPZ fluctuation exponent**

Barraquand and Corwin ’15 observed a connection to KPZ:

**Theorem.** For the Beta(α, β) case

\[
\log P^\omega \{ X_n \cdot e_1 \geq s n \} + n H_q(s) \xrightarrow{\sigma(s)n^{1/3}} \text{GUE (in distribution)}
\]

(\(\sigma(s)\) is known explicitly in terms of polygamma functions \(\psi_1\) and \(\psi_2\)).

Proved by Barraquand and Corwin ’15 for \(s\) far enough from \(\bar{p}\) then by Thiery and Le Doussal ’16 for all \(s \neq \bar{p}\).
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$$\log P^\omega \{ X_n \cdot e_1 \geq sn \} + nH_q(s) \quad \xrightarrow{\sigma(s)n^{1/3}} \quad \text{GUE \ (in distribution)}$$

($\sigma(s)$ is known explicitly in terms of polygamma functions $\psi_1$ and $\psi_2$).

Proved by Barraquand and Corwin ’15 for $s$ far enough from $\bar{p}$ then by Thiery and Le Doussal ’16 for all $s \neq \bar{p}$.

**Question:** Does the path have the KPZ wandering exponent of $2/3$?
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**Question:** Does the path have the KPZ wandering exponent of 2/3?

But how could it? We know the CLT holds, both quenched and averaged!

What is going on?!
Conditioned RWRE

What happens if we condition on $X_n \approx n\zeta$ for $\zeta \neq \xi$? (i.e. on an atypical fraction of $H$)

Subtlety: order of conditioning and averaging

Annealed: average environment first (and get a classical random walk) then condition. New process is another (classical) random walk. Nothing new.

Quenched: fix a typical environment and then condition. What is the resulting process? (Not a classical random walk)

Averaged: average out the environment in the above. What is the resulting process? (Again, not a classical random walk)
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**Busemann function**

**Theorem** (Balázs, R-A, Seppäläinen ’16). For almost every choice of the environment \( \omega = \{ p_x : x \in \mathbb{Z}_+ \} \), limit

\[
B^\zeta(x, y) = \lim_{n \to \infty} \left[ \log P^\omega(X_n \approx n\zeta \mid X_0 = x) - \log P^\omega(X_n \approx n\zeta \mid X_0 = y) \right]
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exists and \( H_q(s) = -s \mathbb{E}[B^\zeta(0, e_1)] - (1 - s) \mathbb{E}[B^\zeta(0, e_2)] \)

where \( \zeta = se_1 + (1 - s)e_2 \).
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$$B_ω(x, y) = \lim_{n→∞} \left[ \log P_ω(X_n ≈ nζ | X_0 = x) − \log P_ω(X_n ≈ nζ | X_0 = y) \right]$$

exists and $H_q(s) = −sE[B_ω(0, e_1)] − (1 − s)E[B_ω(0, e_2)]$

where $ζ = se_1 + (1 − s)e_2$.

$e^{-B_ω(0, x)}$ is a harmonic function:

$$e^{-B_ω(0, x)} = p_x e^{-B_ω(0, x+e_1)} + (1 − p_x) e^{-B_ω(0, x+e_2)}.$$
**Busemann function**

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$$B^\zeta(x, y) = \lim_{n \to \infty} \left[ \log P^\omega(X_n \approx n\zeta \mid X_0 = x) - \log P^\omega(X_n \approx n\zeta \mid X_0 = y) \right]$$

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$e^{-B^\zeta(0, x)}$ is a harmonic function:

$$e^{-B^\zeta(0, x)} = p_x e^{-B^\zeta(0, x+e_1)} + (1 - p_x) e^{-B^\zeta(0, x+e_2)}.$$

This comes from the Markov property

$$P^\omega(X_n \approx n\zeta \mid X_0 = x) = p_x P^\omega(X_n \approx n\zeta \mid X_0 = x+e_1) + (1-p_x) P^\omega(X_n \approx n\zeta \mid X_0 = x+e_2)$$

(then divide by $P^\omega(X_n \approx n\zeta \mid X_0 = 0)$ and take $n \to \infty$).
**Quenched conditioned RWRE**

Define $\pi^\zeta$ as a Doob transform of $p$ by the harmonic function $e^{-B^\zeta(0,x)}$:

$$
\pi^\zeta_{x,x+e_1} = p_x \frac{e^{-B^\zeta(0,x+e_1)}}{e^{-B^\zeta(0,x)}} \quad \text{and} \quad \pi^\zeta_{x,x+e_2} = (1 - p_x) \frac{e^{-B^\zeta(0,x+e_2)}}{e^{-B^\zeta(0,x)}}.
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(They do add up to 1.)
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(They do add up to 1.)

Theorem (Balázs, R-A, Seppäläinen '16). For almost every choice of the environment $\omega = \{p_x : x \in \mathbb{Z}_+\}$, the quenched distribution of $X_{0,m}$, conditional on $X_n \approx n\zeta$, converges as $n \to \infty$ to that of a Markov chain with transitions $\pi^\zeta$.

Note: $\zeta = \xi$ gives $B^\xi \equiv 0$ and $\pi^\xi \equiv p$. 
Quenched conditioned RWRE

Define $\pi^\zeta$ as a Doob transform of $p$ by the harmonic function $e^{-B^\zeta(0,x)}$:

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(They do add up to 1.)

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**Note:** $\zeta = \xi$ gives $B^\xi \equiv 0$ and $\pi^\xi \equiv p$.

So, if $\zeta \neq \xi$, the new process is another random walk in a stationary but very correlated random environment.
Distribution of $\pi^\zeta$

In the solvable $Beta(\alpha, \beta)$ case we can identify $\pi^\zeta$ explicitly.
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Fix a parameter $\lambda > 0$ (depending on $\zeta$).
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Fix a parameter $\lambda > 0$ (depending on $\zeta$).

Let $\{U_{k \epsilon_1} : k \geq 0\}$ be i.i.d. Beta($\alpha + \lambda, \beta$).

Let $\{V_{k \epsilon_2}^{-1} : k \geq 0\}$ be i.i.d. Beta($\lambda, \alpha$).

Let $\{\tilde{p}_x : x \in \mathbb{N}^2\}$ be i.i.d. Beta($\alpha, \beta$).

All three families are mutually independent.
Distribution of $\pi^\zeta$

For the rest of the edges of $\mathbb{Z}_2^+$ define $U$s and $V$s via induction

$$U' = \frac{\tilde{p} V + (1 - \tilde{p}) U}{V}, \quad V' = \frac{\tilde{p} V + (1 - \tilde{p}) U}{U}.$$ 

And define $\pi^\zeta_{x,x+e_1} = \frac{V_x - 1}{V_x - U_x} \in (0, 1)$ and $\pi^\zeta_{x,x+e_2} = 1 - \pi^\zeta_{x,x+e_1}$. 

**Lemma:** $$(U_0, V_0; \pi^\zeta)$$ has the same distribution as $$(U, V; \tilde{p})$$.

**Corollary:** $f \pi^\zeta_{x,x+y}; x+y+e_1: y \in \mathbb{Z}_2^+$ has the same distribution as for $x = 0$. 

**Distribution of $\pi^\zeta$**

For the rest of the edges of $\mathbb{Z}_+^2$ define $U_s$ and $V_s$ via induction

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And define $\pi^\zeta_{x, x+e_1} = \frac{V_x - 1}{V_x - U_x} \in (0, 1)$ and $\pi^\zeta_{x, x+e_2} = 1 - \pi^\zeta_{x, x+e_1}$. 

**Lemma:** $(U_0', V_0'; \tilde{p}, \pi^\zeta)$ has the same distribution as $(U_0, V_0; \tilde{p}, \pi^\zeta)$. 

**Corollary:** $f(\pi^\zeta_{x, x+y}; x, x+y+e_1; \pi^\zeta_{x+e_2})$ has the same distribution as for $x = 0$. 

![Diagram](Image)
Distribution of $\pi^\zeta$

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**Distribution of** $\pi^\zeta$

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Distribution of $\pi^\zeta$

For the rest of the edges of $\mathbb{Z}_+^2$ define $U$s and $V$s via induction

\[ U' = \frac{\tilde{\rho} V + (1 - \tilde{\rho}) U}{V}, \quad V' = \frac{\tilde{\rho} V + (1 - \tilde{\rho}) U}{U}. \]

And define $\pi^\zeta_{x,x+e_1} = \frac{V_x - 1}{V_x - U_x} \in (0, 1)$ and $\pi^\zeta_{x,x+e_2} = 1 - \pi^\zeta_{x,x+e_1}$. 

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$$U' = \frac{\tilde{p}V + (1 - \tilde{p})U}{V}, \quad V' = \frac{\tilde{p}V + (1 - \tilde{p})U}{U}.$$ 

And define $\pi^\zeta_{x,x+e_1} = \frac{V_x - 1}{V_x - U_x} \in (0, 1)$ and $\pi^\zeta_{x,x+e_2} = 1 - \pi^\zeta_{x,x+e_1}$.

Lemma: $(U_0, V_0; \zeta)$ has the same distribution as $(U, V; \tilde{p})$.

Corollary: $f_{\zeta}(x; x+y; x+y+e_1)$ has the same distribution as for $x = 0$. 

- $\tilde{p}$
- $U'$
- $V'$
- $\pi^\zeta$
- $1 - \pi^\zeta$
Distribution of $\pi^\zeta$

For the rest of the edges of $\Z_+^2$ define $U$s and $V$s via induction

$$U' = \frac{\tilde{p} V + (1 - \tilde{p}) U}{V}, \quad V' = \frac{\tilde{p} V + (1 - \tilde{p}) U}{U}.$$  

And define $\pi^\zeta_{x, x+e_1} = \frac{V_x - 1}{V_x - U_x} \in (0, 1)$ and $\pi^\zeta_{x, x+e_2} = 1 - \pi^\zeta_{x, x+e_1}$. 

Lemma: $(U'_0, V'_0, \pi^\zeta)$ has the same distribution as $(U, V, \tilde{p})$.

Corollary: $f_{\pi^\zeta_{x, x+y}}(x+y) \sim_{x+2} \Z_+^2$ and $g$ has the same distribution as $x = 0$. 

\[ \text{Diagram:} \] 

- $V$ and $U$ with $\tilde{p}$ indicating the probability of transition and $1 - \pi^\zeta$ indicating the probability of staying in the same state. 
- $U'$ and $V'$ showing the transition from current state to next state. 

Distribution of $\pi^\zeta$

For the rest of the edges of $\mathbb{Z}_2^2$ define $U$s and $V$s via induction

$$U' = \frac{\p V + (1 - \p)U}{V}, \quad V' = \frac{\p V + (1 - \p)U}{U}.$$ 

And define $\pi^\zeta_{x,x+e_1} = \frac{V_x - 1}{V_x - U_x} \in (0, 1)$ and $\pi^\zeta_{x,x+e_2} = 1 - \pi^\zeta_{x,x+e_1}$.

Lemma: $(U', V', \pi^\zeta)$ has the same distribution as $(U, V, \p)$. 

[Diagram showing the distribution and relationships between $U$, $V$, $U'$, and $V'$ with arrows and labels for $\p$, $1 - \pi^\zeta$, and $\pi^\zeta$.]
**Distribution of $\pi^\zeta$**

For the rest of the edges of $\mathbb{Z}_2^+$ define $U$s and $V$s via induction

$$U' = \frac{\hat{p}V + (1 - \hat{p})U}{V}, \quad V' = \frac{\hat{p}V + (1 - \hat{p})U}{U}.$$ 

And define $\pi^\zeta_{x,x+e_1} = \frac{V_x - 1}{V_x - U_x} \in (0, 1)$ and $\pi^\zeta_{x,x+e_2} = 1 - \pi^\zeta_{x,x+e_1}$.

**Lemma:** $(U', V', \pi^\zeta)$ has the same distribution as $(U, V, \hat{p})$.

**Corollary:** $\{\pi^\zeta_{x+y,x+y+e_1} : y \in \mathbb{Z}_2^+\}$ has the same distribution as for $x = 0$. 
Bijection between velocity $\zeta$ and boundary parameter $\lambda$

$\lambda \in [0, \infty]$ is in one-to-one correspondence with $\zeta$ via

$$
\zeta_1 = \frac{\psi_1(\lambda) - \psi_1(\alpha + \lambda)}{\psi_1(\lambda) - \psi_1(\alpha + \beta + \lambda)} \in \left[ \frac{\alpha}{\alpha + \beta}, 1 \right], \quad \zeta_2 = 1 - \zeta_1
$$

with $\lambda = 0 \iff \zeta = e_1$ and $\lambda = \infty \iff \zeta = \xi = \left( \frac{\alpha}{\alpha + \beta}, \frac{\beta}{\alpha + \beta} \right)$.

$\psi_1$ is the trigamma function: $\psi_1 = (\log \Gamma)''$.

For rest of velocities, $\zeta_1 \in \left[ 0, \frac{\alpha}{\alpha + \beta} \right]$, switch role of Us and Vs.
Formula for quenched rate

\[(B^\zeta(0, e_1), B^\zeta(0, e_2)) \sim (\log U_0, \log V_0) \text{ with parameter } \lambda(\zeta).\]
Formula for quenched rate

\[(B^\zeta(0, e_1), B^\zeta(0, e_2)) \sim (\log U_0, \log V_0) \text{ with parameter } \lambda(\zeta).\]

\[H_q(s) = -s\mathbb{E}[B^\zeta(0, e_1)] - (1 - s)\mathbb{E}[B^\zeta(0, e_2)] \quad (\zeta = se_1 + (1 - s)e_2)\]
**Formula for quenched rate**

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\[= -s \mathbb{E}[\log U] - (1 - s) \mathbb{E}[\log V] \]
**Formula for quenched rate**

\((B^\zeta(0, e_1), B^\zeta(0, e_2)) \sim (\log U_0, \log V_0)\) with parameter \(\lambda(\zeta)\).

\[
H_q(s) = -s \mathbb{E}[B^\zeta(0, e_1)] - (1 - s) \mathbb{E}[B^\zeta(0, e_2)] \quad (\zeta = se_1 + (1 - s)e_2)
\]

\[
= -s \mathbb{E}[\log U] - (1 - s) \mathbb{E}[\log V]
\]

\[
= s \psi_0(\alpha + \beta + \lambda(\zeta)) + (1 - s)\psi_0(\lambda(\zeta)) - \psi_0(\alpha + \lambda(\zeta))
\]

for \(s \in [\frac{\alpha}{\alpha+\beta}, 1]\).

(For \(s \in [0, \frac{\alpha}{\alpha+\beta})\) switch the role of the axes.)

(Barraquand and Corwin '15 got this formula first, by a more direct computation.)
KPZ behavior of averaged conditioned RWRE

Because of the correlations in the environment, averaging the new RWRE does not give a classical RW (nor a Markov chain).
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Furthermore, the aforementioned quenched CLT does not apply (as it was proved for i.i.d. environment, later improved to allow some mixing).

**Theorem:** For $\zeta \neq \xi$, $\exists C, c$: $\forall n \in \mathbb{N}$ and $b$ large,

$$\mathbb{E} P^{\pi^\zeta} \{|X_n - n\zeta| \geq bn^{2/3}\} \leq Cb^{-3}$$

and

$$\mathbb{E} P^{\pi^\zeta} [|X_n - n\zeta|] \geq cn^{2/3}.$$
KPZ behavior of averaged conditioned RWRE

Because of the correlations in the environment, averaging the new RWRE does not give a classical RW (nor a Markov chain).

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**Theorem:** For $\zeta \neq \xi$, $\exists C, c: \forall n \in \mathbb{N}$ and $b$ large,

\[
\mathbb{E} P^{\pi_{\zeta}} \{ |X_n - n\zeta| \geq bn^{2/3} \} \leq Cb^{-3}
\]

and

\[
\mathbb{E} P^{\pi_{\zeta}} [|X_n - n\zeta|] \geq cn^{2/3}.
\]

Complements the aforementioned results saying KPZ fluctuations exponent (for $\log P^\omega (X_n \approx n\zeta)$) is $1/3$. 
KPZ behavior in some other RWREs

There is an earlier result with a KPZ wandering exponent for a RWRE...
KPZ behavior in some other RWREs

There is an earlier result with a KPZ wandering exponent for a RWRE:

The limit of Seppäläinen’s inverse gamma polymer, pinned to go in a given direction.

This is again a random walk in a very correlated random environment and its path has fluctuation exponent 2/3 (Georgiou, R-A, Seppäläinen, Yilmaz ’15).
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But it is different from the one described in this talk, even though Beta random variables appear in its description too!
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But it is different from the one described in this talk, even though Beta random variables appear in its description too!

In both models, solvability comes from the Beta-Gamma algebra. Namely:

If $A$ is $\text{Gamma}(a + b, c)$ and $B$ is an independent $\text{Beta}(a, b)$, then $AB$ and $A(1 − B)$ are independent $\text{Gamma}(a, c)$ and $\text{Gamma}(b, c)$. 
Existence of Busemann limit: coupling

For $\lambda > 0$ recall the system of edge variables $U$ and $V$.

Denote them by $U^\lambda$ and $V^\lambda$. 
Existence of Busemann limit: coupling

For $\lambda \geq 0$ recall the system of edge variables $U$ and $V$. Denote them by $U^\lambda$ and $V^\lambda$.

Can couple all of them (through uniform random variables) so that $U^\lambda$ is increasing in $\lambda$, $V^\lambda$ is decreasing in $\lambda$, and the two are continuous.
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Define $p^\lambda_x = \frac{U^\lambda_x (V^\lambda_x - 1)}{V^\lambda_x - U^\lambda_x} \in (0, 1)$.

**Theorem:** $\{p^\lambda_x : x \in \mathbb{Z}_+^2\}$ are i.i.d. Beta($\alpha, \beta$) random variables (regardless of $\lambda$!).
**Existence of Busemann limit: coupling**

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Denote them by $U^\lambda$ and $V^\lambda$.

Can couple all of them (through uniform random variables) so that $U^\lambda$ is increasing in $\lambda$, $V^\lambda$ is decreasing in $\lambda$, and the two are continuous.

Define $p_x^\lambda = \frac{U_x^\lambda(V_x^\lambda - 1)}{V_x^\lambda - U_x^\lambda} \in (0, 1)$.

**Theorem:** $\{p_x^\lambda : x \in \mathbb{Z}_+^2\}$ are i.i.d. Beta($\alpha, \beta$) random variables (regardless of $\lambda$!).

So we can use these as transitions for the Beta RWRE.
Existence of Busemann limit: cocycle

Given $\zeta$, let $\lambda = \lambda(\zeta)$ and define

$$B^\zeta(x, x + e_1) = \log U^\lambda_x \quad \text{and} \quad B^\zeta(x, x + e_2) = \log V^\lambda_x.$$
Existence of Busemann limit: cocycle

Given $\zeta$, let $\lambda = \lambda(\zeta)$ and define

$$B^\zeta(x, x + e_1) = \log U_x^\lambda \quad \text{and} \quad B^\zeta(x, x + e_2) = \log V_x^\lambda.$$  

The inductive definition of the $U$s and $V$s ensures the cocycle property:

$$B^\zeta(x, x + e_1) + B^\zeta(x + e_1, x + e_1 + e_2) = B^\zeta(x, x + e_2) + B^\zeta(x + e_2, x + e_1 + e_2).$$
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Given $\zeta$, let $\lambda = \lambda(\zeta)$ and define

$$B^\zeta(x, x + e_1) = \log U_{x}^{\lambda} \quad \text{and} \quad B^\zeta(x, x + e_2) = \log V_{x}^{\lambda}.$$  

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Can then define $B^\zeta(0, x)$ for all $x \in \mathbb{Z}_+^2$ by adding over edge-values along any up-right path from 0 to x.
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Can then define $B^\zeta(0, x)$ for all $x \in \mathbb{Z}_+^2$ by adding over edge-values along any up-right path from 0 to $x$.

Then define $B^\zeta(x, y) = B^\zeta(0, y) - B^\zeta(0, x)$ and we have the cocycle property: $B^\zeta(x, y) + B^\zeta(y, z) = B^\zeta(x, z)$. 
Existence of Busemann limit: dual polymer

Take $\zeta$, $\lambda = \lambda(\zeta)$, $x \in \mathbb{N}^2$ and consider the rectangle with corners 0 and $x$. 

![Rectangle diagram]

Note how path $x_0; n$ accumulates a product of $p$'s and $(1 \cdot p)$'s, until it hits the north-east boundary. I.e. $B(0; x)$ is almost the same as $\log P! (X_n = x_i; X_0 = 0)$.

Remark: This connects the RWRE to a polymer with boundary conditions, which leads to the KPZ wandering exponent.
Existence of Busemann limit: dual polymer

Take $\zeta$, $\lambda = \lambda(\zeta)$, $x \in \mathbb{N}^2$ and consider the rectangle with corners 0 and $x$.

Define edge weights $\sigma_{u, u+e_1} = p_u^\lambda$ and $\sigma_{u, u+e_2} = 1 - p_u^\lambda$ inside.
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**Lemma:** \[ \sum_{x_0=0, x_n=x} \prod_{i=0}^{n-1} \sigma_{x_i,x_{i+1}} = e^{B_\zeta(0,x)}. \]
Existence of Busemann limit: dual polymer

Take \( \zeta, \lambda = \lambda(\zeta), x \in \mathbb{N}^2 \) and consider the rectangle with corners 0 and x.

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**Lemma:** \[
\sum_{x_0=0,x_n=x} \prod_{i=0}^{n-1} \sigma_{x_i,x_{i+1}} = e^{B(\zeta)(0,x)}.
\]

Note how path \( x_0,n \) accumulates a product of \( p \)'s and \( (1 - p) \)'s, until it hits the north-east boundary.

I.e. \( B(\zeta)(0,x) \) is almost the same as \( \log P_\omega(X_n = x |, X_0 = 0) \).
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**Remark:** This connects the RWRE to a polymer with boundary conditions, which leads to the KPZ wandering exponent.
Existence of Busemann limit: comparison lemma

By a monotonicity of $B^\zeta(0, x)$ in the edge weights $\sigma$ the above gives:

**Lemma:** With probability one, for $n$ large and $\eta' \cdot e_1 < \zeta \cdot e_1 < \eta \cdot e_1$

$$B^n(0, e_1) \leq \log P^\omega(X_n \approx n\zeta \mid X_0 = 0) - \log P^\omega(X_n \approx n\zeta \mid X_1 = e_1) \leq B^{n'}(0, e_1).$$

\[\eta' \quad \zeta \quad \eta\]
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Now take $n \to \infty$ then $\eta$ and $\eta' \to \zeta$ to get that

$$\lim\{\log P^\omega(X_n \approx n\zeta | X_0 = 0) - \log P^\omega(X_n \approx n\zeta | X_1 = e_1)\}$$

exists (almost surely) and equals $B^\zeta(0, e_1)$. 

\[\text{□}\]
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□ Thank You!