Martingale solutions to the KPZ equation

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Qualitative Methods in KPZ Universality
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Joint work with Patricia Gonçalves, Massimiliano Gubinelli, Marielle Simon

Aim: weak KPZ universality

Want convergence of weakly asymmetric (1+1)-dimensional growth models to KPZ equation

$$\partial_t h = \Delta h + (\partial_x h)^2 + \xi.$$

Examples of models:

- Exclusion type dynamics (variants of WASEP);
- zero range processes;
- interacting Brownian motions (Ginzburg-Landau $\nabla\varphi$ model):

$$\mathrm{d}x^j = \big(\big(\frac{1}{2}+\varepsilon\big)V'(r^{j+1}) - \big(\frac{1}{2}-\varepsilon\big)V'(r^j)\big)\,\mathrm{d}t + \mathrm{d}w^j; \quad r^j = x^j - x^{j-1};$$

- Hairer-Quastel model:

$$\partial_t v = \Delta v + \varepsilon F(\partial_x v) + \eta.$$

Different descriptions of KPZ

$$\partial_t h = \Delta h + |\partial_x h|^2 + \xi.$$

$h(t, \cdot)$ has Brownian regularity, so $|\partial_x h|^2 = ?$

- Cole-Hopf transformation: Bertini-Giacomin (1997) set $h := \log w$

$$\partial_t w = \Delta w + w\xi$$

  (Itô SPDE, $w > 0$ by Mueller (1991)). Equation for e^h but not for h.

- Hairer (2013), Friz-Hairer (2014), Gubinelli-P. (2017), Kupiainen-Marcozzi (2016): rough paths / regularity structures / paracontrolled distributions / renormalization group approach control solution h as continuous functional of "polynomials" of ξ.

- Martingale approach: Gonçalves-Jara (2014), Gubinelli-Jara (2013), Gubinelli-P. (2015), based on stationarity and time-reversal.

For convergence we need analogous description of approximating system!

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Martingale problem for the KPZ equation

$$\partial_t h = \Delta h + |\partial_x h|^2 + \xi$$

- Try to implement martingale problem: for $\varphi \in \mathcal{S}$

$$M(\varphi) = h(\varphi) - h_0(\varphi) - \int_0^\cdot h_s(\Delta\varphi)\,\mathrm{d}s - \int_0^\cdot |\partial_x h_s|^2(\varphi)\,\mathrm{d}s$$

  should be continuous martingale with $\langle M(\varphi)\rangle_t = \|\varphi\|_{L^2}^2 t$.
- h supported on non-differentiable functions, so $\int_0^\cdot |\partial_x h_s|^2(\varphi)\,\mathrm{d}s = ?$
- Possible solution: restrict to measures supported on modelled / paracontrolled distributions ("smooth in new topology").
- For convergence: difficult to verify that limit points satisfy this.
- Softer condition than modelled / paracontrolled which still gives well-posedness?

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From martingale problem to energy solutions

Consider $u = \partial_x h$, solution to Burgers equation

$$\partial_t u = \Delta u + \partial_x u^2 + \partial_x \xi;$$

has invariant probability distribution; recover KPZ easily.

- $\partial_x u^2$ ill-defined.
- Naively: for $\varphi \in \mathcal{S}$

$$M(\varphi) = u(\varphi) - u_0(\varphi) - \int_0^{\cdot} u_s(\Delta\varphi)\mathrm{d}s + \lim_n \int_0^{\cdot} (\rho_n * u_s)^2(\partial_x \varphi)\mathrm{d}s$$

cont. martingale, $\langle M(\varphi)\rangle_t = 2\|\partial_x\varphi\|_{L^2}^2 t$. No chance for uniqueness.

- Gonçalves-Jara (2014): energy solution if additionally

$$\mathbb{E}\Big[\Big|\int_s^t ((\rho_n * u_r)^2(\partial_x\varphi) - (\rho_m * u_r)^2(\partial_x\varphi))\mathrm{d}r\Big|^2\Big] \lesssim \frac{|t-s|}{n \wedge m}.$$

$\Rightarrow u(\varphi) - M(\varphi)$ has zero quadratic varation (Dirichlet process), get Itô formula (e.g. Russo-Vallois).

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Funaki-Quastel strategy

Energy solution: martingale solution to

$$\partial_t u = \Delta u + \partial_x u^2 + \partial_x \xi;$$

with $u(\varphi) - M(\varphi)$ of zero quadratic variation.

- Existence trivial;
- no way to compare two energy solutions, uniqueness?
- Inspired by Funaki-Quastel (2015): $u^\varepsilon = \delta_\varepsilon * u$. Itô: $w^\varepsilon = e^{\partial_x^{-1} u^\varepsilon}$ solves

$$\mathrm{d}w_t^\varepsilon = \Delta w^\varepsilon \mathrm{d}t + w_t^\varepsilon(\mathrm{d}\partial_x^{-1} M_t^\varepsilon + (\partial_x^{-1}(\delta_\varepsilon * (\partial_x u_t^2)) - [(u_t^\varepsilon)^2 - \|\delta_\varepsilon\|_{L^2}^2])\mathrm{d}t).$$

Lemma (Uniqueness criterion)

If for $c \in \mathbb{R}$

$$\int_0^\cdot w_t^\varepsilon(\partial_x^{-1}(\delta_\varepsilon * (\partial_x u_t^2)) - [(u_t^\varepsilon)^2 - \|\delta_\varepsilon\|_{L^2}^2])\mathrm{d}t \longrightarrow \int_0^\cdot w_t c\,\mathrm{d}t,$$

then $u = \partial_x \log w$ for unique solution $\partial_t w = \Delta w + w\xi + cw$.

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How to check uniqueness criterion?

- Uniqueness criterion needs control of additive functional

$$\int_0^\cdot w_t^\varepsilon(\partial_x^{-1}(\delta_\varepsilon * (\partial_x u_t^2)) - [(u_t^\varepsilon)^2 - \|\delta_\varepsilon\|_{L^2}^2])\mathrm{d}t$$
$$= \int_0^\cdot e^{\partial_x^{-1} u_t * \delta_\varepsilon}(\partial_x^{-1}(\delta_\varepsilon * (\partial_x u_t^2)) - [(u_t * \delta_\varepsilon)^2 - \|\delta_\varepsilon\|_{L^2}^2])\mathrm{d}t.$$

- How to control additive functionals of energy solutions? First consider Markov processes.

Intermezzo: Martingale trick

X Markov, generator $\mathcal{L}$, $X_0 \sim \mu$ stationary.

- symmetric $\mathcal{L}_S = (\mathcal{L} + \mathcal{L}^*)/2$, antisymmetric $\mathcal{L}_A = (\mathcal{L} - \mathcal{L}^*)/2$.
- $\hat{X}_t = X_{T-t}$ Markov w/ generator $\mathcal{L}^* = \mathcal{L}_S - \mathcal{L}_A$.
- Dynkin's formula (Itô):

$$F(X_T) - F(X_0) - \int_0^T \mathcal{L}F(X_s)\mathrm{d}s = M_T^F,$$

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Martingale trick for Burgers equation

$$\partial_t u = \Delta u + \partial_x u^2 + \partial_x \xi.$$

Formally:

- Space white noise invariant.
- symmetric $\mathcal{L}_S = (\mathcal{L} + \mathcal{L}^*)/2$, antisymmetric $\mathcal{L}_A = (\mathcal{L} - \mathcal{L}^*)/2$.
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$\Rightarrow$ martingale trick should bound $\mathbb{E}\left[\left|\int_0^t \mathcal{L}_S F(u_s)\mathrm{d}s\right|^p\right]$ via white noise and OU-generator $\mathcal{L}_S$.

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Gubinelli-Jara solution

Gubinelli-Jara (2013): stationary energy solution

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is FB-solution if $\hat{u}_t = u_{T-t}$ energy solution of

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Gubinelli-Jara (2013): existence of FB-solutions, martingale trick works.

- $\Rightarrow$ control $\int_0^t F(u_s) \mathrm{d}s$ if we solve ∞-dim Poisson eq.

$$\mathcal{L}_S G = F \qquad \text{in } L^2(\text{white noise}).$$

- L^2(white noise) Gaussian Hilbert space $\Rightarrow$ chaos expansion:

$$F = \sum_n I_n(f_n) \text{ for } f_n \in L^2(\mathbb{R}^n)$$

$$\mathcal{L}_S I_n(g_n) = I_n(\Delta g_n) \qquad \Rightarrow \qquad \text{solve } \Delta g_n = f_n.$$

Reduces problem to finite-dim PDE!

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Bounds for Burgers nonlinearity

u FB-solution to

$$\partial_t u = \Delta u + \partial_x u^2 + \partial_x \xi.$$

- Martingale trick + chaos decomposition: bound $\mathbb{E}\Big[\Big| \int_0^t F(u_s)\mathrm{d}s\Big|^p\Big]$;
- $F(u) = \partial_x u^2 \Rightarrow$ sharp bounds on Burgers nonlinearity;
- get $\int_0^{\cdot} (\partial_x u_s^2)(\varphi)\mathrm{d}s \in C^{3/4-}$ for $\varphi \in \mathcal{S}$ – but not better!
- Consequence: $u \mapsto u(\varphi)$ not in domain of generator.
- Open problem: describe Burgers generator.

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Uniqueness of FB-solutions I

- Uniqueness criterion: need control of

$$\int_0^\cdot e^{\partial_x^{-1} u_t * \delta_\varepsilon} (\partial_x^{-1}(\delta_\varepsilon * (\partial_x u_t^2)) - [(u_t * \delta_\varepsilon)^2 - \|\delta_\varepsilon\|_{L^2}^2]) \mathrm{d}t.$$

- Problem: $(\partial_x^{-1}(\delta_\varepsilon * (\partial_x u_t^2)) - [(u_t * \delta_\varepsilon)^2 - \|\delta_\varepsilon\|_{L^2}^2])$ in second chaos, but $e^{\partial_x^{-1} u_t * \delta_\varepsilon} (\partial_x^{-1}(\delta_\varepsilon * (\partial_x u_t^2)) - [(u_t * \delta_\varepsilon)^2 - \|\delta_\varepsilon\|_{L^2}^2])$ not! Tedious to solve Poisson equation. What to do?

Intermezzo: Martingale trick w/o Poisson equation

- Martingale trick bounds $\mathbb{E}\Big[\Big|\int_0^t \mathcal{L}_S F(u_s)\mathrm{d}s\Big|^p\Big]$.
- What if we cannot solve Poisson equation $\mathcal{L}_S F = G$? Use Kipnis-Varadhan lemma!
- Idea: solve $(\mathcal{L}_S - \lambda)F_\lambda = G$ instead. $\lambda > 0$ enforces spectral gap,

$$F_\lambda = \int_0^\infty e^{t(\mathcal{L}_S - \lambda)} G \mathrm{d}t.$$

- Apply martingale trick to $\mathbb{E}\Big[\Big|\int_0^t \mathcal{L}_S F_\lambda(u_s)\mathrm{d}s\Big|^2\Big]$, send $\lambda \to 0$.
- Duality: $\mathbb{E}\Big[\Big|\int_0^t G(u_s)\mathrm{d}s\Big|^2\Big] \lesssim T\|G\|_{-1}^2$ for

$$\|G\|_{-1}^2 = \sup_H \{2\mathbb{E}[G(u_0)H(u_0)] + \mathbb{E}[H(u_0)\mathcal{L}_S H(u_0)]\}.$$

- Note: $\mathbb{E}[G(u_0)] \neq 0$, then $\|G\|_{-1}^2 = \infty$.

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Uniqueness of FB-solutions II

Aim: control $\int_0^\cdot e^{\partial_x^{-1} u_t * \delta_\varepsilon}(\partial_x^{-1}(\delta_\varepsilon * (\partial_x u_t^2)) - [(u_t * \delta_\varepsilon)^2 - \|\delta_\varepsilon\|_{L^2}^2])\mathrm{d}t$.

- Kipnis-Varadhan:

$$\mathbb{E}\Big[\Big|\int_0^t F^\varepsilon(u_t)\mathrm{d}t\Big|^2\Big] \lesssim T\|F^\varepsilon\|_{-1}^2.$$

- But $\mathbb{E}[F^\varepsilon(u_0)] \neq 0$, so $\|F^\varepsilon\|_{-1}^2 = \infty$! Solution: consider

$$G^\varepsilon(u_t) = F^\varepsilon(u_t) - \frac{1}{12} w_t^\varepsilon.$$

Tedious computation: $\lim_\varepsilon \|G^\varepsilon\|_{-1}^2 = 0$.

- So for $w_t = e^{\partial_x^{-1} u_t} = \lim_{\varepsilon \to 0} w_t^\varepsilon$:

$$\mathrm{d}w_t = \Delta w \mathrm{d}t + w_t \mathrm{d}\partial_x^{-1} M_t + \frac{1}{12} w_t \mathrm{d}t.$$

Theorem (Gubinelli-P. '15)

FB-solution to Burgers equation is unique and $u = \partial_x \log w$.
(First probabilistic solution for a "truely singular" SPDE).

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Aim: control $\int_0^\cdot e^{\partial_x^{-1} u_t * \delta_\varepsilon}(\partial_x^{-1}(\delta_\varepsilon * (\partial_x u_t^2)) - [(u_t * \delta_\varepsilon)^2 - \|\delta_\varepsilon\|_{L^2}^2])\mathrm{d}t$.

- Kipnis-Varadhan:

$$\mathbb{E}\Big[\Big|\int_0^t F^\varepsilon(u_t)\mathrm{d}t\Big|^2\Big] \lesssim T\|F^\varepsilon\|_{-1}^2.$$

- But $\mathbb{E}[F^\varepsilon(u_0)] \neq 0$, so $\|F^\varepsilon\|_{-1}^2 = \infty$! Solution: consider

$$G^\varepsilon(u_t) = F^\varepsilon(u_t) - \frac{1}{12} w_t^\varepsilon.$$

  Tedious computation: $\lim_\varepsilon \|G^\varepsilon\|_{-1}^2 = 0$.

- So for $w_t = e^{\partial_x^{-1} u_t} = \lim_{\varepsilon \to 0} w_t^\varepsilon$:

$$\mathrm{d}w_t = \Delta w \mathrm{d}t + w_t \mathrm{d}\partial_x^{-1} M_t + \frac{1}{12} w_t \mathrm{d}t.$$

Theorem (Gubinelli-P. '15)

FB-solution to Burgers equation is unique and $u = \partial_x \log w$.
(First probabilistic solution for a "truely singular" SPDE).

Simple extensions

- Works on torus $\mathbb{R}/\mathbb{Z}$ and on $\mathbb{R}$.
- Uniqueness criterion works without stationarity or forward-backward structure.
- $\Rightarrow$ energy solutions with $\mathrm{law}(u) \ll \mathrm{law}(u_{FB})$ are unique Gubinelli-P. (2017).
- $\Rightarrow$ bounded entropy perturbations of stationary weakly asymmetric systems converge to Burgers Gonçalves-Jara-Sethuraman (2015).
- Extension from Burgers to KPZ is easy:

$$\partial_t h = \Delta h + (\partial_x h)^2 + \xi = \Delta h + u^2 + \xi.$$

  u unique $\Rightarrow$ u^2 unique $\Rightarrow$ h unique.
- $h_t = h_t^{CH} + \frac{1}{12}t$ for Cole-Hopf solution h^{CH}. Already observed by Funaki-Quastel (2015).

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How to apply this

For convergence to FB-solutions we need:

- Martingale characterization (easy);
- energy condition = zero quadratic variation nonlinearity (easy);
- forward-backward decomposition (needs similar structure in approximating model; satisfied for all examples from above).

Example: WASEP with open boundaries

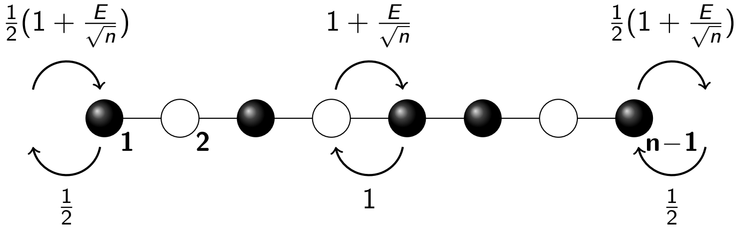

Figure: Jump rates. Leftmost and rightmost rates are the entrance/exit rates. Compare also Corwin-Shen (2016).

- Product Bernoulli measure w/ density 1/2 invariant;
- $\mathcal{L}_S$: generator of dynamics with $E = 0$;
- LLN: $\eta(0, n\cdot) \longrightarrow \frac{1}{2}$;
- CLT: $n^{1/2}(\eta(0, n\cdot) - 1/2) \longrightarrow$ white noise;
- Set $u^n(t, x) := n^{1/2}(u(n^2 t, nx) - 1/2)$.

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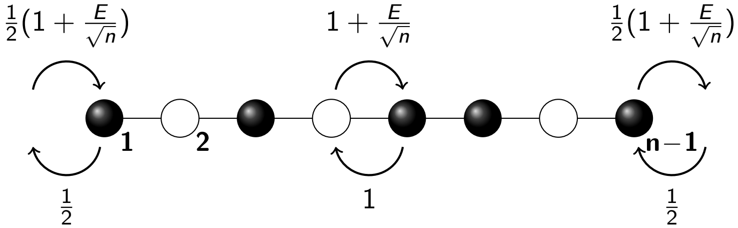

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Scaling limit

- u^n solves

$$\begin{aligned} \mathrm{d}u^n &= S^n \mathrm{d}t + A^n \mathrm{d}t + \mathrm{d}M^n \\ &= \Delta^{(n)}_{\mathrm{Dir}} u^n \mathrm{d}t + A^n \mathrm{d}t + \mathrm{d}M^n; \end{aligned}$$

- time-reversed process satisfies same equation with $-A^n$.

Use approach of Gonçalves-Jara (2014):

- Show tightness for S^n, A^n and M^n; deduce tightness of u^n;
- second order Boltzmann-Gibbs principle:

$$\int_0^T \left(A^n(t) - E\partial_x(u^n(t)^2)\right)\mathrm{d}t \longrightarrow 0;$$

- straightforward: $\partial_t M^n \longrightarrow \partial_x \xi$;
- $\Rightarrow$ any limit point is FB-solution to Burgers eq. with Dirichlet b.c.

  Gonçalves-P.-Simon (in preparation)

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Burgers equation with Dirichlet boundary conditions

$$\partial_t u = \Delta u + \partial_x u^2 + \partial_x \xi,$$

$u\colon \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ with $u(0) = u(1) = 0$. u is FB-solution if:

- for $\varphi \in C^2[0,1]$ with $\varphi(0) = \varphi(1) = 0$:

  $$M(\varphi) = u(\varphi) - u_0(\varphi) - \int_0^\cdot u_s(\Delta\varphi)\mathrm{d}s + \lim_n \int_0^\cdot (\rho_n * u_s)^2(\partial_x\varphi)\mathrm{d}s$$

  cont. martingale, $\langle M(\varphi)\rangle_t = 2\|\partial_x\varphi\|_{L^2}^2 t$.
- $u(\varphi) - M(\varphi)$ has zero quadratic varation;
- u satisfies FB-condition: $u_t \sim$ white noise for all t, time reversed process solves $\partial_t \hat{u} = \Delta\hat{u} - \partial_x \hat{u}^2 + \partial_x \hat{\xi}$.

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Uniqueness for Dirichlet boundary conditions I

$$\partial_t u = \Delta u + \partial_x u^2 + \partial_x \xi,$$

$u\colon \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ with $u(0) = u(1) = 0$.

- Regularize e.g. w/ heat kernel: $u_t^\varepsilon(x) = u_t(p_\varepsilon(x,\cdot))$;
- $w_t^\varepsilon = e^{\partial_x^{-1} u_t^\varepsilon}$, get

$$\mathrm{d}w_t^\varepsilon = \Delta w^\varepsilon \mathrm{d}t + w_t^\varepsilon(\mathrm{d}\partial_x^{-1} M_t^\varepsilon + R_t^\varepsilon \mathrm{d}t + c\mathrm{d}t) + w_t^\varepsilon(\delta_0^\varepsilon + \delta_1^\varepsilon)\mathrm{d}t$$

$\partial_x w^\varepsilon(0) = w^\varepsilon(0)\partial_x\partial_x^{-1}u^\varepsilon(0) = w^\varepsilon(0)u^\varepsilon(0) = 0$ and $\partial_x w^\varepsilon(1) = 0$ (von Neumann bc)

- As before $R^\varepsilon \to 0$, $c = 1/12$.

So far so boring. But:

- $w_t^\varepsilon(\delta_0^\varepsilon + \delta_1^\varepsilon) \longrightarrow w_t(\delta_0 + \delta_1)$, very singular drift;
- kill this by test function φ w/ $\partial_x\varphi(0) = -\varphi(0)$, $\partial_x\varphi(1) = \varphi(1)$;
- get Robins bc for $w = \lim^\varepsilon w^\varepsilon$! Compare also Gerencser-Hairer (2017).

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Uniqueness for Dirichlet boundary conditions II

Theorem (Gonçalves-Simon-P. (in preparation))

$\exists$ unique FB-solution u to Burgers with Dirichlet bc, and a unique FB-solution h to

$$\partial_t h = \Delta h + (\partial_x h)^2 + \xi$$

with von Neumann bc $\partial_x h_t(0) = \partial_x h_t(1) = 0$.

$$h_t = h_t^{CH} + \frac{1}{12} t$$

for Cole-Hopf solution h^{CH} to

$$\partial_t h = \Delta h + (\partial_x h)^2 + \xi$$

with von Neumann bc $\partial_x h_t(0) = -1; \partial_x h_t(1) = 1$.

Conclusion

- Energy solutions formulate KPZ/Burgers equation as a martingale problem.
- At stationarity: uniqueness via Cole-Hopf and martingale trick.
- Extends to very simple non-stationary regimes, but general case still open.
- Powerful tool for proving convergence to Burgers equation, general recipe due to Gonçalves-Jara (2014).
- Extension to boundary conditions more interesting than expected.
- All rests on Cole-Hopf transform, probabilistic understanding of other singular SPDEs out of reach...

Thank you