Large deviations for certain inhomogeneous corner growth models

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(joint work with Elnur Emrah)
1 The (totally asymmetric) corner growth model
   1 Questions: general homogeneous shape theorem and large deviations
   2 Inhomogeneous exponential model
   3 Model with random parameters
2 Results and proof sketches
   1 The shape function: appearance of linear regions.
   2 Overview of quenched and annealed large deviation results.
   3 Sketch of quenched rate function computation.

Goal: To better understand what can happen in inhomogeneous models in the KPZ class at the level of large deviations through a solvable example.
Take $W(i, j) \geq 0, (i, j) \in \mathbb{N}^2$ i.i.d.
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$$G(m,n) = \max_{\text{up-right paths } \pi: (1,1) \to (m,n)} \sum_{(i,j) \in \pi} W(i,j)$$
Theorem (Martin, ’04)

Suppose the family \( \{W(i, j)\} \) are positive, i.i.d. random variables with

\[
\int_0^\infty \sqrt{P(W(1,1) > r)} \, dr < \infty.
\]

Then there exists a finite, concave, homogeneous function \( g : (0, \infty)^2 \to \mathbb{R}_+ \) such that

\[
\lim_{n \to \infty} n^{-1} G([ ns ], [ nt ]) = g(s, t).
\]
Theorem

Suppose the family \( \{W(i, j)\} \) are positive, i.i.d. random variables with \( E[e^{\lambda W(1,1)}] < \infty \) for some \( \lambda > 0 \) and that \( P(W(1, 1) > r) > 0 \) for all \( r > 0 \). Then there exists a finite, convex function \( J_{s,t}(r) : \mathbb{R} \rightarrow \mathbb{R}_+ \) such that

\[
\lim_{n \to \infty} -n^{-1} \log P(G(\lfloor ns \rfloor, \lfloor nt \rfloor) \geq nr) = J_{s,t}(r).
\]
Homogeneous left tail large deviations

Theorem

Suppose the family \{W(i,j)\} are non-negative, non-degenerate, i.i.d. random variables with \(E[e^{\lambda W(1,1)}] < \infty\) for some \(\lambda > 0\). Then there exist constants \(C > 0\) such that for \(r \in (0, g(s, t))\)

\[
P(\lfloor ns \rfloor, \lfloor nt \rfloor) \leq nr \leq C^{-1}e^{-Cn^2}.
\]

A lower bound with the same rate holds for all \(r \in (0, g(s, t))\) if \(P(W(1,1) \in [0, \epsilon)) > 0\) for all \(\epsilon > 0\).
Inhomogeneous exponential last passage percolation

Model introduced by Johansson '01.

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Inhomogeneous exponential last passage percolation

\[ W(i, j) \overset{\text{ind.}}{\sim} \text{Exp}(a_i + b_j) \]

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\[ a = (a_n)_{n \geq 1} \quad \text{and} \quad b = (b_n)_{n \geq 1} \]

\[ a_i, b_j \geq c > 0. \]
Inhomogeneous exponential last passage percolation

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- \( W(i, j) \overset{\text{ind.}}{\sim} \text{Exp}(a_i + b_j) \)
- \( \mathbf{a} = (a_n)_{n \geq 1} \) and \( \mathbf{b} = (b_n)_{n \geq 1} \)
- \( a_i, b_j \geq c > 0 \)
- Model introduced by Johansson '01.
Extend environment to include \((i, 0), (0, j), i, j \geq 0, \ W(0, 0) = 0\).
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Take \(z: -a_i < z < b_j, i, j \in \mathbb{N}\)

\[ W(i, 0) \sim \text{Exp}(a_i + z) \]
Stationary model

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\(W(i, 0) \sim \text{Exp}(a_i + z)\)

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Take \(z\): \(-a_i < z < b_j, \ i, j \in \mathbb{N}\)

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W(i, j) \sim \text{Exp}(a_i + b_j)
\]
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Passage time in this environment is denoted \(\hat{G}_z(i,j)\).
Stationary model

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X(i, j) = \hat{G}_z(i, j) - \hat{G}_z(i - 1, j)
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\[
X(i,j) = \hat{G}_z(i,j) - \hat{G}_z(i-1,j) \\
Y(i,j) = \hat{G}_z(i,j) - \hat{G}_z(i,j-1)
\]
Suppose that \((X, Y, W)\) are mutually independent exponential random variables with means \((a + z)^{-1}, (b - z)^{-1}, (a + b)^{-1}\) and define

\[
\check{X} = (X - Y)^+ + W, \quad \check{Y} = (Y - X)^+ + W, \quad \check{W} = X \wedge Y.
\]

Then

\[
(\check{X}, \check{Y}, \check{W}) \overset{d}{=} (X, Y, W).
\]

Proof: Compute the Laplace transform of \((\check{X}, \check{Y}, \check{W})\).
Stationary model - corner flipping

\[ \begin{align*}
W(i, j) & \sim \text{Exp}(a_i + b_j) \\
Y(i - 1, j) & \sim \text{Exp}(b_j - z) \\
X(i, j - 1) & \sim \text{Exp}(a_i + z)
\end{align*} \]
Stationary model - corner flipping

\[ Y(i-1, j) \sim \text{Exp}(b_j - z) \]

\[ X(i, j-1) \sim \text{Exp}(a_i + z) \]

\[ W(i, j) \sim \text{Exp}(a_i + b_j) \]

\[ X(i, j) = (X(i, j-1) - Y(i-1, j))^+ + W(i, j) \]

\[ Y(i, j) = (Y(i-1, j) - X(i, j-1))^+ + W(i, j) \]
Stationary model - corner flipping

\[ X(i, j) \sim \text{Exp}(a_i + z) \]

\[ Y(i, j) \sim \text{Exp}(b_j - z) \]

\[ \tilde{W}(i - 1, j - 1) \sim \text{Exp}(a_i + b_j) \]

\[ X(i, j) = (X(i, j - 1) - Y(i - 1, j))^+ + W(i, j) \]

\[ Y(i, j) = (Y(i - 1, j) - X(i, j - 1))^+ + W(i, j) \]

\[ \tilde{W}(i - 1, j - 1) = X(i, j - 1) \wedge Y(i - 1, j) \]
Stationary model - down right paths

By induction, mutual independence and distributions are preserved along down-right edge paths.

\[ X(i, 0) = W(i, 0) \sim \text{Exp}(a_i + z) \]
\[ Y(0, j) = W(0, j) \sim \text{Exp}(b_j - z) \]
Stationary model - down right paths

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\[ X(i, j) \sim \text{Exp}(a_i + z) \]
\[ Y(i, j) \sim \text{Exp}(b_j - z). \]
\[ W(i, j) \sim \text{ind.} \exp(a_i + b_j) \]

\[ a = (a_i), \ b = (b_j) \ \text{indep. i.i.d.}, \ E[a_1 + b_1] < \infty, \ a_i, b_j \geq c > 0 \ \text{(can be weaker)} \]
Random environment

\[ W(i,j) \overset{\text{ind.}}{\sim} \exp(a_i + b_j) \]

\[ \mathbf{a} = (a_i), \mathbf{b} = (b_j) \text{ indep. i.i.d.,} \]
\[ \mathbb{E}[a_1 + b_1] < \infty, a_i, b_j \geq c > 0 \text{ (can be weaker)} \]

\[ \mathsf{P}_{a,b}: \text{ conditioned on } (\mathbf{a}, \mathbf{b}), \]
\[ W(i,j) \overset{\text{ind.}}{\sim} \exp(a_i + b_j). \]
Random environment

\[
\begin{array}{c|c|c|c}
\vdots & \vdots \\
W(i,j) & \overset{\text{ind.}}{\sim} & \text{Exp}(a_i + b_j) \\
\vdots & \vdots \\
\hline
b_j \\
\hline
\vdots & \vdots \\
\vdots & \vdots \\
\hline
b_1 \\
\hline
a_1 & \ldots & a_i & \ldots
\end{array}
\]

\[a = (a_i), \ b = (b_j) \ \text{indep. i.i.d.,} \ E[a_1 + b_1] < \infty, \ a_i, b_j \geq c > 0 \ (\text{can be weaker})\]

\[P_{a,b}: \text{conditioned on } (a, b), \]

\[W(i,j) \overset{\text{ind.}}{\sim} \text{Exp}(a_i + b_j).\]

\[\mathbb{P}: \text{average } P_{a,b} \text{ over } (a, b): \]

\[\mathbb{P}(\cdot) = E[P_{a,b}(\cdot)].\]
**Key points:**

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<td>$W(i, j) \sim_{\text{ind.}} \text{Exp}(a_i + b_j)$</td>
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<td>$b_1$</td>
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- $W(i, j)$ are independent, not identically distributed: if $i \neq i_1$ or $j \neq j_1$ usually.
- $W(i, j)$ are not independent, but identically distributed: $W(i_1, j_1)$. If $i \neq i_1$ or $j \neq j_1$, $\text{Cov}(W(i, j), W(i_1, j_1)) \approx 0$. 

**Random environment**

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Random environment

Key points:

\( P_{a,b} \): indep., not ident. dist.:
if \( i \neq i' \) or \( j \neq j' \)

\[ W(i, j) \sim \text{Exp}(a_i + b_j) \]

\( W(i, j) \neq W(i', j') \) (usually)
### Random environment

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\[
W(i, j) \sim \text{Exp}(a_i + b_j)
\]

**Key points:**

- $P_{a,b}$: indep., not ident. dist.:
  - if $i \neq i'$ or $j \neq j'$
  - $W(i, j) \nconverges \text{d}_W W(i', j')$ (usually)

$W(i, j) \perp W(i', j')$
Random environment

### Key points:

- **\( P_{a,b} \): indep., not ident. dist.:**
  - if \( i \neq i' \) or \( j \neq j' \)
  - \( W(i,j) \not\sim W(i',j') \) (usually)
  - \( W(i,j) \perp W(i',j') \)

- **\( P \): ident. dist., not indep.:**
  - \( W(i,j) \overset{d}{=} W(i',j') \).
Random environment

Key points:

\( \mathbb{P}_{a,b} \): indep., not ident. dist.:
if \( i \neq i' \) or \( j \neq j' \)

\[ W(i, j) \overset{d}{\neq} W(i', j') \] (usually)

\[ W(i, j) \perp W(i', j') \]

\( \mathbb{P} \): ident. dist., not indep.:

\[ W(i, j) \overset{d}{=} W(i', j') \]

If \( i = i' \) or \( j = j' \)

\[ \text{Cov}(W(i, j), W(i', j')) \neq 0. \]
**Lemma**

For $s, t > 0$, $\mathbb{P}^z$ almost surely and for almost all $(a, b)$ $P^z_{a, b}$ almost surely

$$g_z(s, t) := \lim_{n \to \infty} \frac{1}{n} \hat{G}_z([ns], [nt]) = s \mathbb{E} \left[ \frac{1}{a_1 + z} \right] + t \mathbb{E} \left[ \frac{1}{b_1 - z} \right].$$
Lemma

For \( s, t > 0, \mathbb{P}_z \) almost surely and for almost all \((a, b)\) \( \mathbb{P}_{a,b}^z \) almost surely

\[
g_z(s, t) := \lim_{n \to \infty} \frac{1}{n} \hat{G}_z([ns], [nt]) = s \mathbb{E} \left[ \frac{1}{a_1 + z} \right] + t \mathbb{E} \left[ \frac{1}{b_1 - z} \right].
\]

Proof:

\[
\hat{G}_z(n, n) = \sum_{1 \leq i \leq n} \left[ \hat{G}_z(i, 0) - \hat{G}_z(i - 1, 0) \right] + \sum_{1 \leq j \leq n} \left[ \hat{G}_z(n, j) - \hat{G}_z(n, j - 1) \right]
\]

\[
= \sum_{1 \leq i \leq n} X(i, 0) + \sum_{1 \leq j \leq n} Y(n, j).
\]
Lemma

For $s, t > 0$, $\mathbb{P}^z$ almost surely and for almost all $(a, b)$ $\mathbb{P}^z_{a,b}$ almost surely

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Proof:

$$\hat{G}_z(n, n) = \sum_{1 \leq i \leq n} \left[ \hat{G}_z(i, 0) - \hat{G}_z(i - 1, 0) \right] + \sum_{1 \leq j \leq n} \left[ \hat{G}_z(n, j) - \hat{G}_z(n, j - 1) \right]$$

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Proof:

$$\hat{G}_z(n, n) = \sum_{1 \leq i \leq n} \left( \hat{G}_z(i, 0) - \hat{G}_z(i - 1, 0) \right) + \sum_{1 \leq j \leq n} \left( \hat{G}_z(n, j) - \hat{G}_z(n, j - 1) \right)$$

$$= \sum_{1 \leq i \leq n} X(i, 0) + \sum_{1 \leq j \leq n} Y(n, j).$$

These sums are marginally i.i.d. under $\mathbb{P}^z$ (not mutually indep. under $\mathbb{P}_{a,b}^z$ or $\mathbb{P}^z$).
Notation: $\alpha = \text{essinf}\{a_1\}, \beta = \text{essinf}\{b_1\}$. 

Theorem (Emrah '15)
For $s, t \geq 0$, $P$ almost surely and for almost all $p, a, b, q$:

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{u=1}^{u=n} \min \left\{ \bar{\alpha} + \bar{\beta} \right\} = sE_a + tE_b.
$$

Remark: These results also hold if $a$ and $b$ are both separately ergodic, rather than a pair of independent i.i.d. sequences. The formulas only depend on marginal distributions of $a_1$ and $b_1$ separately.
Notation: $\alpha = \text{essinf}\{a_1\}$, $\beta = \text{essinf}\{b_1\}$.

**Theorem (Emrah '15)**

For $s, t > 0$, $\mathbb{P}$ almost surely and for almost all $(a, b)$ $\mathbb{P}_{a,b}$ almost surely

$$g(s, t) := \lim_{n \to \infty} \frac{1}{n} G(\lfloor ns \rfloor, \lfloor nt \rfloor) = \min_{-\alpha \leq z \leq \beta} \left\{ s \mathbb{E} \left[ \frac{1}{a_1 + z} \right] + t \mathbb{E} \left[ \frac{1}{b_1 - z} \right] \right\}$$
Notation: $\alpha = \operatorname{essinf}\{a_1\}$, $\beta = \operatorname{essinf}\{b_1\}$.

**Theorem (Emrah '15)**

For $s, t > 0$, $\mathbb{P}$ almost surely and for almost all $(a, b)$ $\mathbb{P}_{a,b}$ almost surely

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Remark:

- These results also hold if $a$ and $b$ are both separately ergodic, rather than a pair of independent i.i.d. sequences. The formulas only depend on marginal distributions of $a_1$ and $b_1$ separately.
Asymptotic shape of the cluster

Key properties of the shape function $g_p$:

- $g$ is strictly concave in $S$,
- linear in $S_1$ and $S_2$.

$S_1$, $S_2 \nsubseteq H$ if $E_{r_p} \alpha - \beta > 0$ up to $S_1$ and $E_{r_p} \beta - \alpha > 0$ up to $S_2$. 

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Asymptotic shape of the cluster

Key properties of the shape function $g(s, t)$:

- $g$ is strictly concave in $S$,
- linear in $S_1$ and $S_2$.

$S_1$, $S_2$ are not equal if

$$E \left( r_{p_1} - \alpha q - r_{p_2} \right) \neq 0$$

and

$$E \left( r_{b_1} - \beta q - r_{b_2} \right) \neq 0$$

$S_1$, $S_2$ are equal if

$$E \left( r_{p_1} - \alpha q - r_{p_2} \right) = 0$$

and

$$E \left( r_{b_1} - \beta q - r_{b_2} \right) = 0$$

where $r_{p_i}$ and $r_{b_i}$ are functions of $p_i$ and $b_i$, respectively.
Asymptotic shape of the cluster

Key properties of the shape function $g(s, t)$:

- $g$ is strictly concave in $S$, linear in $S_1$ and $S_2$. 

$$S_1 \quad S_2 \quad S$$

0

$t$

$s$
Asymptotic shape of the cluster

Key properties of the shape function $g(s, t)$:

- $g$ is strictly concave in $S$, linear in $S_1$ and $S_2$.
- $S_1, S_2 \neq \emptyset$ iff
  
  - $E[(a_1 - \alpha)^{-2}] < \infty \quad (S_1)$
  - $E[(b_1 - \beta)^{-2}] < \infty \quad (S_2)$
Quenched right tail rate function

**Theorem**

For almost all \((a, b)\), for any \(s, t > 0\) and \(r \geq g(s, t)\)

\[
J_{s,t}(r) = \lim_{n \to \infty} -n^{-1} \log P_{a,b}(n^{-1}G([ns], [nt]) \geq r)
\]

\[
= \sup_{\lambda \in (0, \alpha + \beta), z \in (-\alpha, \beta - \lambda)} \left\{ r\lambda + sE \log \left( \frac{a_1 + z}{a_1 + z + \lambda} \right) - tE \log \left( \frac{b_1 - z}{b_1 - z - \lambda} \right) \right\}
\]
Quenched right tail rate function

**Theorem**

*For almost all* $(a, b)$, *for any* $s, t > 0$ *and* $r \geq g(s, t)$

$$J_{s,t}(r) = \lim_{n \to \infty} -n^{-1} \log P_{a,b}(n^{-1}G([ns], [nt]) \geq r)$$

$$= \sup_{\lambda \in (0, \alpha + \beta)} \left\{ r\lambda + sE \log \left( \frac{a_1 + z}{a_1 + z + \lambda} \right) - tE \log \left( \frac{b_1 - z}{b_1 - z - \lambda} \right) \right\}$$

**Remarks:**

- The same theorem also holds if $(a, b)$ is totally ergodic rather than a pair of independent i.i.d. sequences. Again, these formulas only depend on marginal distributions of $a_1$ and $b_1$ separately.
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**Theorem**

For almost all \((a, b)\), for any \(s, t > 0\) and \(r \geq g(s, t)\)

\[
J_{s,t}(r) = \lim_{n \to \infty} -n^{-1} \log P_{a,b} \left( n^{-1} G(\lfloor ns \rfloor, \lfloor nt \rfloor) \geq r \right)
\]

\[
= \sup_{\lambda \in (0, \alpha + \beta)} \left\{ r\lambda + s \mathbb{E} \log \left( \frac{a_1 + z}{a_1 + z + \lambda} \right) - t \mathbb{E} \log \left( \frac{b_1 - z}{b_1 - z - \lambda} \right) \right\}
\]

Remarks:

- The same theorem also holds if \((a, b)\) is totally ergodic rather than a pair of independent i.i.d. sequences. Again, these formulas only depend on marginal distributions of \(a_1\) and \(b_1\) separately.

- Rate \(n\) LDP for \(n^{-1} G(\lfloor ns \rfloor, \lfloor nt \rfloor)\) under \(P_{a,b}\) with rate function

\[
I_{s,t}(r) = J_{s,t}(r)1_{\{r \geq g(s,t)\}} + \infty 1_{\{r < g(s,t)\}}.
\]
Expected fluctuations

Quenched fluctuations are TW$_{GUE}$ in $S$, but not in $S_1, S_2$.

Q1: Can we “see” different scaling exponents in the rate functions?

Q2: What happens when $p, t, q, l_1, l_2$?
Expected fluctuations

- Quenched fluctuations are TW_{GUE} in $S$, but not in $S_1, S_2$.
- Q1: Can we “see” different scaling exponents in the rate functions?
Expected fluctuations

Quenched fluctuations are \( TW_{GUE} \) in \( S \), but not in \( S_1, S_2 \).

Q1: Can we “see” different scaling exponents in the rate functions?

Q2: What happens when \( (s, t) \in l_1, l_2 \)?
Proposition

For any \( s, t > 0 \), let \( \zeta \in [-\alpha, \beta] \) solve (uniquely) \( g_\zeta(s, t) = g(s, t) \). As \( \epsilon \downarrow 0 \), there are explicit values \( C_1, C_2, C_3 \) depending on \( s, t, \zeta \) such that

\[
J_{s,t}(g(s, t) + \epsilon) = \begin{cases}
C_1(s, t) \epsilon^2 + o(\epsilon^2) & (s, t) \in S_1 \\
\frac{1}{2} C_2(s, t, \alpha) \epsilon^{3/2} + o(\epsilon^{3/2}) & (s, t) \in l_1 \\
C_2(s, t, \zeta) \epsilon^{3/2} + o(\epsilon^{3/2}) & (s, t) \in S \\
\frac{1}{2} C_2(s, t, \beta) \epsilon^{3/2} + o(\epsilon^{3/2}) & (s, t) \in l_2 \\
C_3(s, t) \epsilon^2 + o(\epsilon^2) & (s, t) \in S_2
\end{cases}
\]
Scaling and the quenched rate functions

Proposition

For any $s, t > 0$, let $\zeta \in [-\alpha, \beta]$ solve (uniquely) $g_\zeta(s, t) = g(s, t)$. As $\epsilon \downarrow 0$, there are explicit values $C_1, C_2, C_3$ depending on $s, t, \zeta$ such that

$$J_{s,t}(g(s, t) + \epsilon) = \begin{cases} 
C_1(s, t) \epsilon^2 + o(\epsilon^2) & (s, t) \in S_1 \\
\frac{1}{2}C_2(s, t, \alpha) \epsilon^{3/2} + o(\epsilon^{3/2}) & (s, t) \in l_1 \\
C_2(s, t, \zeta) \epsilon^{3/2} + o(\epsilon^{3/2}) & (s, t) \in S \\
\frac{1}{2}C_2(s, t, \beta) \epsilon^{3/2} + o(\epsilon^{3/2}) & (s, t) \in l_2 \\
C_3(s, t) \epsilon^2 + o(\epsilon^2) & (s, t) \in S_2 
\end{cases}$$
Heuristically consistent with $TW_{GUE}$ fluct in $S$. 
Heuristically consistent with $TW_{GUE}$ fluct in $S$. Take $(s, t) \in S$ and set

$$C = s \mathbb{E} \left[ \frac{1}{(a + \zeta)^3} \right] + t \mathbb{E} \left[ \frac{1}{(b - \zeta)^3} \right] = \frac{1}{2} \partial_z^2 g_z(s, t) \bigg|_{z=\zeta}$$

In the notation of the previous result, $C_2(s, t, \zeta) = 4/3 C^{-1/2}$. 
Heuristically consistent with $TW_{GUE}$ fluct in $S$. Take $(s, t) \in S$ and set

$$C = s E \left[ \frac{1}{(a + \xi)^3} \right] + t E \left[ \frac{1}{(b - \xi)^3} \right] = \frac{1}{2} \partial_z^2 g_z(s, t) \bigg|_{z=\xi}$$

In the notation of the previous result, $C_2(s, t, \xi) = 4/3 C^{-1/2}$. For $n$ large and large enough $r$ (but not $O(n^{3/3})$), we might expect

$$P_{a, b}(G([ns], [nt]) - ng(s, t) \geq n^{1/3} C^{1/3} r)$$
Heuristically consistent with $TW_{GUE}$ fluct in $S$. Take $(s, t) \in S$ and set

$$C = s E \left[ \frac{1}{(a + \zeta)^3} \right] + t E \left[ \frac{1}{(b - \zeta)^3} \right] = \frac{1}{2} \partial_z^2 g_z(s, t) \bigg|_{z=\zeta}$$

In the notation of the previous result, $C_2(s, t, \zeta) = 4/3 C^{-1/2}$. For $n$ large and large enough $r$ (but not $O(n^{2/3})$), we might expect

$$P_{a, b}(G(\lfloor ns \rfloor, \lfloor nt \rfloor) - ng(s, t) \geq n^{1/3} C^{1/3} r) \approx e^{-n J_{s, t}(g(s,t) + C^{1/3} n^{-2/3} r)}$$
Scaling and the quenched rate functions

Heuristically consistent with $TW_{GUE}$ fluct in $S$. Take $(s, t) \in S$ and set

$$C = s E \left[ \frac{1}{(a + \zeta)^3} \right] + t E \left[ \frac{1}{(b - \zeta)^3} \right] = \frac{1}{2} \partial_z^2 g_z(s, t)|_{z=\zeta}$$

In the notation of the previous result, $C_2(s, t, \zeta) = 4/3 C^{-1/2}$. For $n$ large and large enough $r$ (but not $O(n^{2/3})$), we might expect

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$$\approx e^{-\frac{4}{3} C^{-\frac{1}{2}} (C^{\frac{1}{3}} n^{-\frac{2}{3}} r)^{\frac{3}{2}}} n = e^{-\frac{4}{3} r^{\frac{3}{2}}}$$

which agrees with the leading order $TW_{GUE}$ right tail.
Annealed large deviations

**Theorem**

For $s, t > 0$ and $r \geq g(s, t)$,

$$
J_{s,t}(r) = \lim_{n \to \infty} -n^{-1} \log P \left( n^{-1} G([ns], [nt]) \geq r \right)
$$

$$
= \sup_{\lambda \in (0, \alpha + \beta)} \sup_{z \in (-\alpha, \beta - \lambda)} \left\{ r \lambda - s \log E \left[ \frac{a_1 + z + \lambda}{a_1 + z} \right] - t \log E \left[ \frac{b_1 - z}{b_1 - z - \lambda} \right] \right\}
$$

Remark:

We do not have the rate $n$ left tail rate function in this case, but we can show existence of $P_{p0}$, $g_p$, $t_q$, with $\lim \sup _{n \to \infty} n^{-1} \log P (n^{-1} G([ns], [nt]) \geq r)$. (i.e. there are rate $n$ annealed left tail large deviations)
Annealed large deviations

**Theorem**

For $s, t > 0$ and $r \geq g(s, t)$,

$$
\mathcal{J}_{s,t}(r) = \lim_{n \to \infty} -n^{-1} \log \mathbb{P} \left( n^{-1} G (\lfloor ns \rfloor, \lfloor nt \rfloor) \geq r \right)
$$

$$
= \sup_{\lambda \in (0, \alpha + \beta)} \left\{ r \lambda - s \log E \left[ \frac{a_1 + z + \lambda}{a_1 + z} \right] - t \log E \left[ \frac{b_1 - z}{b_1 - z - \lambda} \right] \right\}
$$

Remark:

- We do not have the rate $n$ left tail rate function in this case, but we can show existence of $r \in (0, g(s, t))$ with

$$
\limsup - \frac{1}{n} \mathbb{P} \left( n^{-1} G (\lfloor ns \rfloor, \lfloor nt \rfloor) \leq r \right) < \infty.
$$

(i.e. there are rate $n$ annealed left tail large deviations)
Proposition

For any $s, t > 0$, let $\zeta \in [-\alpha, \beta]$ solve (uniquely) $g_{\zeta}(s, t) = g(s, t)$. As $\epsilon \downarrow 0$, there are explicit values $C_1, C_2, C_3$ depending on $s, t, \zeta$ such that

$$J_{s, t}(g(s, t) + \epsilon) = \begin{cases} C_1(s, t) \epsilon^2 + o(\epsilon^2) & (s, t) \in S_1 \\ C_2(s, t, \zeta) \epsilon^2 + o(\epsilon^2) & (s, t) \in S, l_1, l_2 \\ C_3(s, t) \epsilon^2 + o(\epsilon^2) & (s, t) \in S_2 \end{cases}$$
Proposition

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$S_1$, $S_2$, $l_1$, $l_2$
Theorem

For any $s, t > 0$ and $r \geq g(s, t)$,

$$\mathbb{J}_{s,t}(r) = \inf_{\nu_1, \nu_2} \left\{ I_{s,t}^{\nu_1,\nu_2}(r) + s H(\nu_1|\alpha) + t H(\nu_2|\beta) \right\}.$$
Theorem

For any \( s, t > 0 \) and \( r \geq g(s, t) \),

\[
J_{s,t}(r) = \inf_{\nu_1, \nu_2} \left\{ I_{s,t}^{\nu_1, \nu_2}(r) + s H(\nu_1 | \alpha) + t H(\nu_2 | \beta) \right\}.
\]

A unique minimizing pair \((\nu_1, \nu_2)\) exists.
Theorem

For any $s, t > 0$ and $r \geq g(s, t)$,

$$\mathbb{J}_{s,t}(r) = \inf_{\nu_1, \nu_2} \left\{ I_{s,t}^{\nu_1, \nu_2}(r) + s H(\nu_1 | \alpha) + t H(\nu_2 | \beta) \right\}. $$

A unique minimizing pair $(\nu_1, \nu_2)$ exists. The equality

$$\mathbb{J}_{s,t}(r) = I_{s,t}^{\nu_1, \nu_2}(r) + s H(\nu_1 | \alpha) + t H(\nu_2 | \beta)$$

holds if and only if

$$\frac{d\nu_1}{d\alpha}(a) \propto \frac{a + z_* + \lambda_*}{a + z_*}, \quad \frac{d\nu_2}{d\beta}(b) \propto \frac{b - z_*}{b - z_* - \lambda_*}.$$
Variational connection for right tail

**Theorem**

For any $s, t > 0$ and $r \geq g(s, t)$,

$$\mathbb{J}_{s, t}(r) = \inf_{\nu_1, \nu_2} \left\{ I_{s, t}^{\nu_1, \nu_2}(r) + s H(\nu_1 | \alpha) + t H(\nu_2 | \beta) \right\}.$$

A unique minimizing pair $(\nu_1, \nu_2)$ exists. The equality

$$\mathbb{J}_{s, t}(r) = I_{s, t}^{\nu_1, \nu_2}(r) + s H(\nu_1 | \alpha) + t H(\nu_2 | \beta)$$

holds if and only if

$$\frac{d\nu_1}{d\alpha}(a) \propto \frac{a + z_* + \lambda_*}{a + z_*}, \quad \frac{d\nu_2}{d\beta}(b) \propto \frac{b - z_*}{b - z_* - \lambda_*}$$

where $z_*$ and $\lambda_*$ are the unique $z_*, \lambda_*$ with $\lambda_* \in [0, \alpha + \beta], z_* \in [-\alpha, \beta - \lambda_*]$ satisfying

$$\mathbb{J}_{s, t}(r) = r \lambda_* - s \log E^\alpha \left[ \frac{a + z_* + \lambda_*}{a + z_*} \right] - t \log E^\beta \left[ \frac{b - z_*}{b - z_* - \lambda_*} \right].$$
Denote by $\theta_{i,j}$ the shift on the environment by $(i,j)$. We have

$$
\hat{G}_z(n, n) = \max\left\{ \max_{1 \leq k \leq n} \{ G(n - k + 1, n) \circ \theta_{k-1,0} + \hat{G}_z(k, 0), \right. \\
\left. \max_{1 \leq \ell \leq n} \{ G(n, n - \ell + 1) \circ \theta_{0,\ell-1} + \hat{G}_z(0, \ell) \} \right\}.
$$
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\max_{1 \leq \ell \leq n} \{ G(n, n - \ell + 1) \circ \theta_{0,\ell-1} + \hat{G}_z(0, \ell) \} \}.$$
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\max_{1 \leq \ell \leq n} \{ G(n, n - \ell + 1) \circ \theta_{0,\ell-1} + \hat{G}_z(0, \ell) \} \}.$$
Proof sketch: finite $n$ variational problem

Denote by $\theta_{i,j}$ the shift on the environment by $(i,j)$. We have

$$\hat{G}_z(n,n) = \max \{ \max_{1 \leq k \leq n} \{ G(n-k+1,n) \circ \theta_{k-1,0} + \hat{G}_z(k,0), \} \} \quad \text{and} \quad \max_{1 \leq \ell \leq n} \{ G(n,n-\ell+1) \circ \theta_{0,\ell-1} + \hat{G}_z(0,\ell) \}.$$
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$$\max_{1 \leq \ell \leq n} \{ G(n, n - \ell + 1) \circ \theta_{0,\ell-1} + \hat{G}_z(0, \ell)\}\}. $$
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\max_{1 \leq \ell \leq n} \{ G(n, n - \ell + 1) \circ \theta_{0,\ell-1} + \hat{G}_z(0, \ell) \} \} \}.$$
Proof sketch: telescoping sum

\[ \hat{G}_z(n, n) = \sum_{1 \leq i \leq n} \hat{G}_z(i, 0) - \hat{G}_z(i - 1, 0) + \sum_{1 \leq j \leq n} \hat{G}_z(n, j) - \hat{G}_z(n, j - 1) \]
Proof sketch: telescoping sum

\[ \hat{G}_z(n, n) = \sum_{1 \leq i \leq n} \hat{G}_z(i, 0) - \hat{G}_z(i - 1, 0) + \sum_{1 \leq j \leq n} \hat{G}_z(n, j) - \hat{G}_z(n, j - 1) \]

\[ = \sum_{1 \leq i \leq n} X(i, 0) + \sum_{1 \leq j \leq n} Y(n, j) \text{ (sums not indep.)}. \]
Equate two expressions for $\hat{G}_z(n, n)$ and subtract the $\sum_{i=1}^n X(i, 0)$ terms:

$$\sum_{1 \leq j \leq n} Y(n, j) = \max_{1 \leq k \leq n} \{ \max \{ G(n - k + 1, n) \circ \theta_{k-1, 0} - \sum_{1 \leq i \leq n} X(i, 0),

G(n, n - k + 1) \circ \theta_{0,k-1} - \sum_{1 \leq i \leq n} X(i, 0) + \sum_{1 \leq j \leq k} Y(0, j) \} \}.$$
Proof sketch: variational problem for $J_{s,t}$

Equate two expressions for $\hat{G}_z(n, n)$ and subtract the $\sum_{i=1}^{n} X(i, 0)$ terms:

$$\sum_{1 \leq j \leq n} Y(n, j) = \max_{1 \leq k \leq n} \left\{ \max \{ G(n - k + 1, n) \circ \theta_{k-1,0} - \sum_{k < i \leq n} X(i, 0), 
G(n, n - k + 1) \circ \theta_{0,k-1} - \sum_{1 \leq i \leq n} X(i, 0) + \sum_{1 \leq j \leq k} Y(0, j) \} \right\}. $$
Proof sketch: variational problem for $J_{s,t}$

Equate two expressions for $\hat{G}_z(n,n)$ and subtract the $\sum_{i=1}^n X(i,0)$ terms:

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$$G(n,n-k+1) \circ \theta_{0,k-1} - \sum_{1 \leq i \leq n} X(i,0) + \sum_{1 \leq j \leq k} Y(0,j).$$

For each $z \in (-\alpha, \beta)$ of the summands within the maxima are $P^{z}_{a,b}$-indep. Fix $\lambda > 0$ and $z \in (-\alpha, \beta - \lambda)$. For each $k$, we have

$$E^{z}_{a,b} \left[ e^{\lambda \sum Y(n,j)} \right] \geq E^{z}_{a,b} \left[ e^{\lambda G(n-k+1,n) \circ \theta_{k-1,0}} \right] E^{z}_{a,b} \left[ e^{-\lambda \sum_{k<i \leq n} X(i,0)} \right]$$

$$E^{z}_{a,b} \left[ e^{\lambda \sum Y(n,j)} \right] \geq E^{z}_{a,b} \left[ e^{\lambda G(n,n-k+1) \circ \theta_{0,k-1}} \right] E^{z}_{a,b} \left[ e^{-\lambda \sum_{1 \leq i \leq n} X(i,0)} \right]$$

$$\times E^{z}_{a,b} \left[ e^{\lambda \sum_{1 \leq j \leq k} Y(0,j)} \right]$$

so LHS $\geq$ max of RHS over $k$. 

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Proof sketch: variational problem for $J_{s,t}$

Equate two expressions for $\hat{G}_z(n, n)$ and subtract the $\sum_{i=1}^{n} X(i, 0)$ terms:

$$
\sum_{1 \leq j \leq n} Y(n, j) = \max_{1 \leq k \leq n} \{ \max \{ G(n - k + 1, n) \circ \theta_{k-1,0} - \sum_{k<i\leq n} X(i, 0),
\}
$$

$$
G(n, n - k + 1) \circ \theta_{0,k-1} - \sum_{1\leq i \leq n} X(i, 0) + \sum_{1 \leq j \leq k} Y(0, j) \}.
$$

For each $z \in (-\alpha, \beta)$ of the summands within the maxima are $P_{a,b}^z$-indep.

Fix $\lambda > 0$ and $z \in (-\alpha, \beta - \lambda)$. For each $k$, we have

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$$

$$
E_{a,b}^z \left[ e^{\lambda \sum Y(n,j)} \right] \geq E_{a,b}^z \left[ e^{\lambda \sum G(n,n-k+1) \circ \theta_{0,k-1}} \right] E_{a,b}^z \left[ e^{-\lambda \sum_{1\leq i \leq n} X(i,0)} \right]
$$

$$
\times E_{a,b}^z \left[ e^{\lambda \sum_{1 \leq j \leq k} Y(0,j)} \right]
$$

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Equate two expressions for $\hat{G}_z(n, n)$ and subtract the $\sum_{i=1}^{n} X(i, 0)$ terms:

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$$

$$G(n, n - k + 1) \circ \theta_{0,k-1} - \sum_{1 \leq i \leq n} X(i, 0) + \sum_{1 \leq j \leq k} Y(0, j) \} \}.$$

For each $z \in (-\alpha, \beta)$ of the summands within the maxima are $P_{a,b}^z$-indep.

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$$\times E_{a,b}^z \left[ e^{\lambda \sum_{1 \leq j \leq k} Y(0,j)} \right]$$

so LHS $\geq$ max of RHS over $k$. 
Proof sketch: variational problem for $J_{s,t}$

Define

$$L_{s,t}(\lambda) = \lim_{n \to \infty} \frac{1}{n} \log E_{a,b} \left[ e^{\lambda G(\lfloor ns \rfloor, \lfloor nt \rfloor)} \right]$$

For $\lambda \in (0, \beta - z)$, if we apply $\lim_{n \to \infty} n^{-1} \log E_{a,b}^z[e^{\lambda \cdot}]$, the previous inequality and another coming from $\max \leq \sum$ gives

$$E \left[ \log \frac{b_1 - z}{b_1 - z - \lambda} \right] = \sup_{0 \leq t \leq 1} \left\{ \max \left\{ L_{1,t}(\lambda) - t E \left[ \log \frac{a_1 + z + \lambda}{a_1 + z} \right] \right\} \right\},$$

$$L_{1,t}(\lambda) = E \left[ \log \frac{a_1 + z + \lambda}{a_1 + z} \right] + (1 - t) E \left[ \log \frac{b_1 - z}{b_1 - z - \lambda} \right].$$
This variational problem can be inverted to solve for $L_{s,t}(\lambda) =$

$$
\begin{cases}
\lambda g(s, t) & \lambda \leq 0 \\
\min_{-\alpha \leq z \leq \beta - \lambda} \left\{ s \log \left( \frac{a_1 + z + \lambda}{a_1 + z} \right) + t \log \left( \frac{b_1 - z - \lambda}{b_1 - z} \right) \right\} & \lambda \in (0, \alpha + \beta] \\
\infty & \lambda > \alpha + \beta
\end{cases}
$$

Cannot use Gärtner-Ellis (steepness can fail), but can verify that the Legendre transform gives the rate function.
Thanks!