Large deviations for certain inhomogeneous corner growth models

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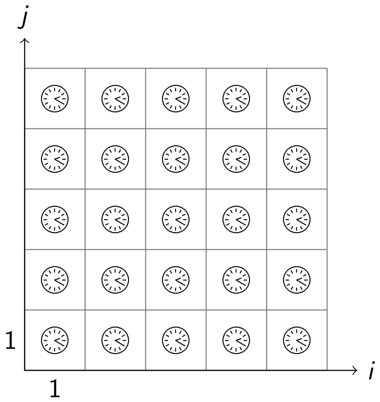
(joint work with Elnur Emrah)

Outline

1. The (totally asymmetric) corner growth model
   1. Questions: general homogeneous shape theorem and large deviations
   2. Inhomogeneous exponential model
   3. Model with random parameters
2. Results and proof sketches
   1. The shape function: appearance of linear regions.
   2. Overview of quenched and annealed large deviation results.
   3. Sketch of quenched rate function computation.

Goal: To better understand what can happen in inhomogeneous models in the KPZ class at the level of large deviations through a solvable example.

(Homogeneous) Last passage percolation



Take $W(i,j) \geqslant 0$, $(i,j) \in \mathbb{N}^2$ i.i.d.

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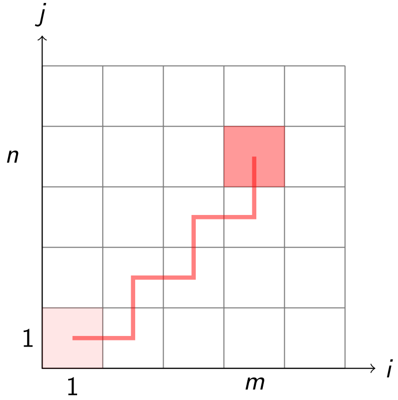


Take $W(i,j) \geqslant 0$, $(i,j) \in \mathbb{N}^2$ i.i.d.

$$G(m,n) = \max_{\substack{\text{up-right paths} \\ \pi:(1,1)\to(m,n)}} \sum_{(i,j)\in\pi} W(i,j)$$

Homogeneous shape theorem

Theorem (Martin, '04)

Suppose the family $\{W(i,j)\}$ *are positive, i.i.d. random variables with*

$$\int_0^\infty \sqrt{P(W(1,1) > r)}dr < \infty.$$

Then there exists a finite, concave, homogeneous function $g : (0,\infty)^2 \to \mathbb{R}_+$ *such that*

$$\lim_{n\to\infty} n^{-1} G(\lfloor ns \rfloor, \lfloor nt \rfloor) = g(s,t).$$

Homogeneous right tail large deviations

Theorem

Suppose the family $\{W(i,j)\}$ *are positive, i.i.d. random variables with* $E[e^{\lambda W(1,1)}] < \infty$ *for some* $\lambda > 0$ *and that* $P(W(1,1) > r) > 0$ *for all* $r > 0$. *Then there exists a finite, convex function* $J_{s,t}(r) : \mathbb{R} \to \mathbb{R}_+$ *such that*

$$\lim_{n\to\infty} -n^{-1} \log P\left(G(\lfloor ns \rfloor, \lfloor nt \rfloor) \geqslant nr\right) = J_{s,t}(r).$$

Homogeneous left tail large deviations

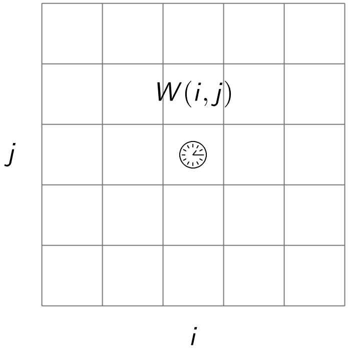
Theorem

Suppose the family $\{W(i,j)\}$ are non-negative, non-degenerate, i.i.d. random variables with $E[e^{\lambda W(1,1)}] < \infty$ for some $\lambda > 0$. Then there exist constants $C > 0$ such that for $r \in (0, g(s,t))$

$$P\left(G(\lfloor ns \rfloor, \lfloor nt \rfloor) \leqslant nr\right) \leqslant C^{-1} e^{-Cn^2}.$$

A lower bound with the same rate holds for all $r \in (0, g(s,t))$ if $P(W(1,1) \in [0,\epsilon)) > 0$ for all $\epsilon > 0$.

Inhomogeneous exponential last passage percolation



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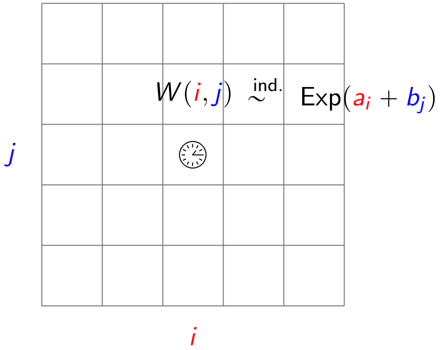

- $W(i,j) \overset{ind.}{\sim} \text{Exp}(a_i + b_j)$

  $\mathbf{a} = (a_n)_{n\geqslant 1}$ and $\mathbf{b} = (b_n)_{n\geqslant 1}$
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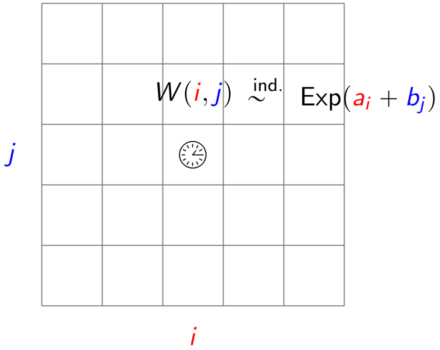

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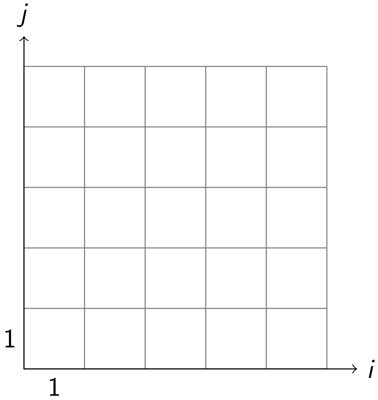
- Model introduced by Johansson '01.

Extend environment to include $(i,0),(0,j)$, $i,j \geqslant 0$, $W(0,0)=0$.



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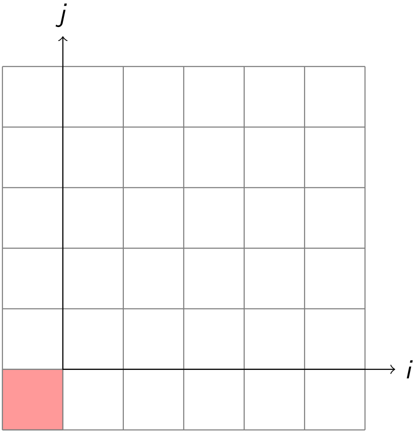

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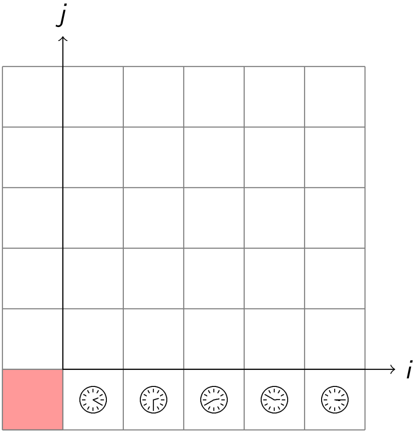

Take z: $-a_i < z < b_j$, $i, j \in \mathbb{N}$

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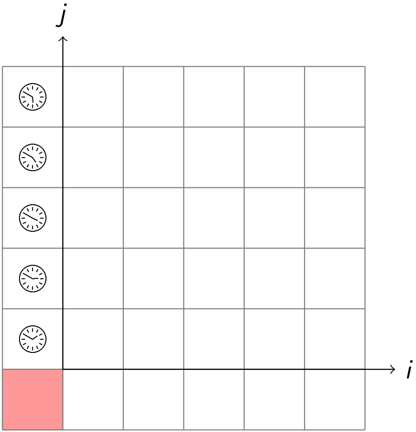

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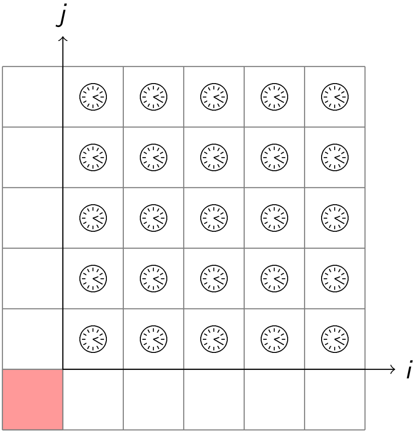

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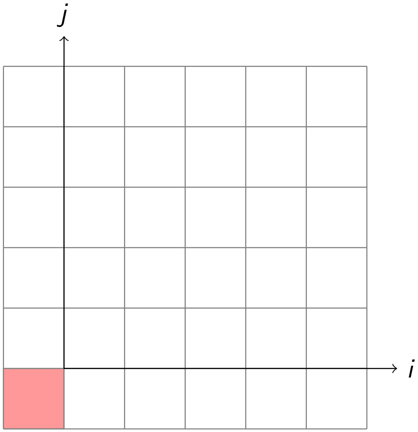

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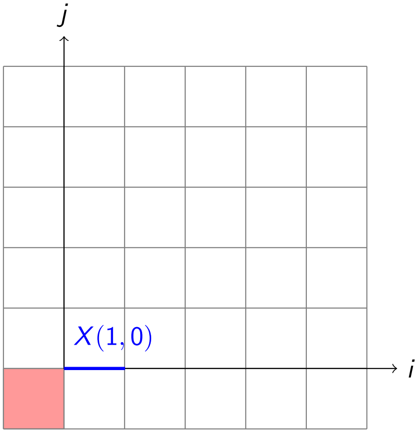

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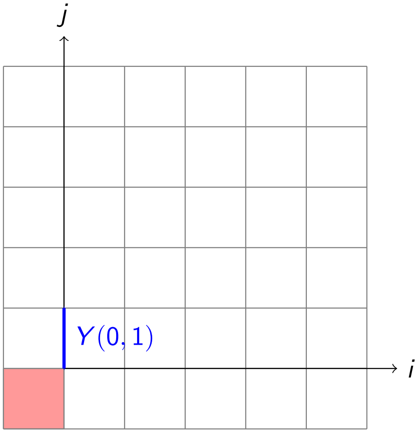

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$$Y(i,j) = \hat{G}_z(i,j) - \hat{G}_z(i,j-1)$$

Lemma

Suppose that (X, Y, W) *are mutually independent exponential random variables with means* $(a+z)^{-1}, (b-z)^{-1}, (a+b)^{-1}$ *and define*

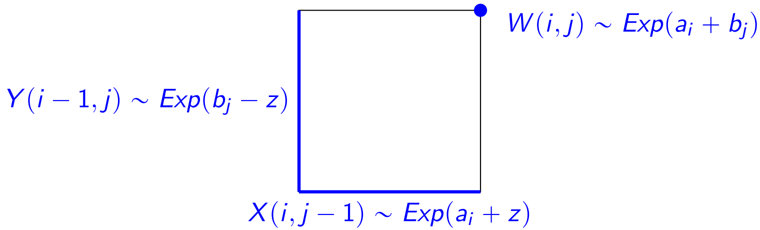
$$\check{X} = (X - Y)^+ + W, \qquad \check{Y} = (Y - X)^+ + W, \qquad \check{W} = X \wedge Y.$$

Then

$$(\check{X}, \check{Y}, \check{W}) \stackrel{d}{=} (X, Y, W).$$

Proof: Compute the Laplace transform of $(\check{X}, \check{Y}, \check{W})$.

Stationary model - corner flipping



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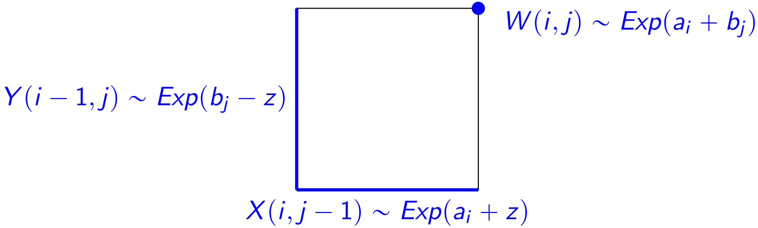

$$X(i,j) = (X(i,j-1) - Y(i-1,j))^+ + W(i,j)$$
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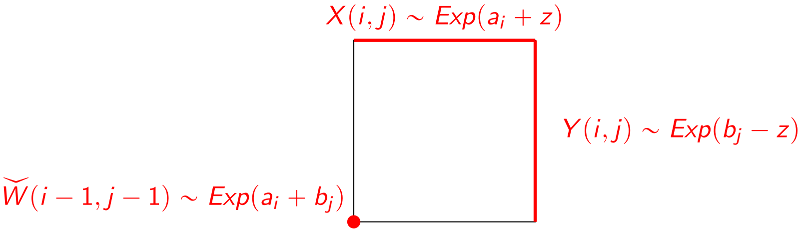

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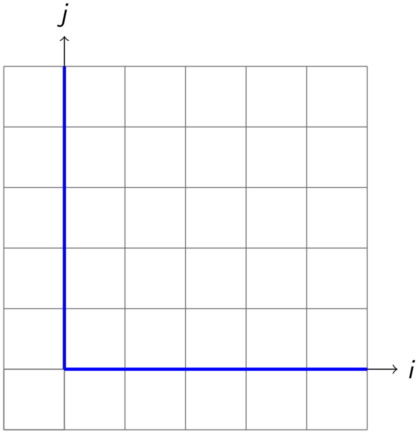
Stationary model - down right paths



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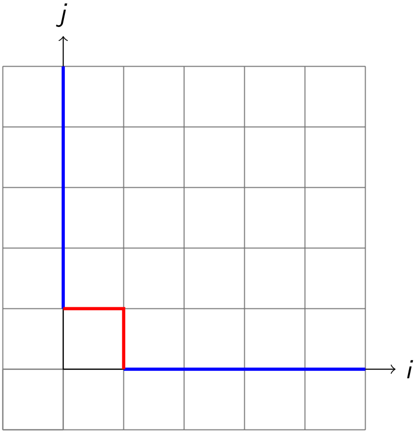

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By induction, mutual independence and distributions are preserved along down-right **edge** paths.

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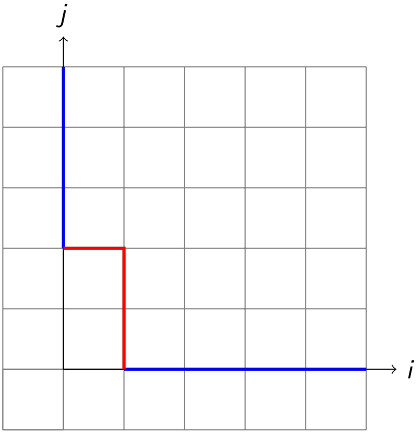

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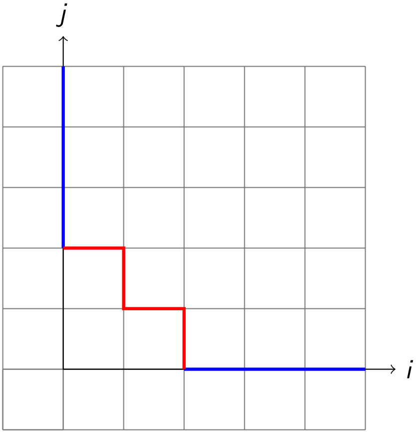

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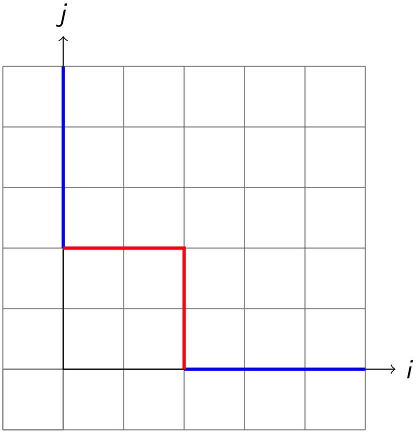

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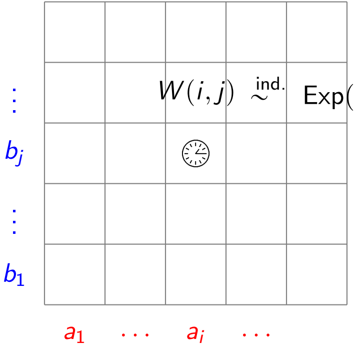

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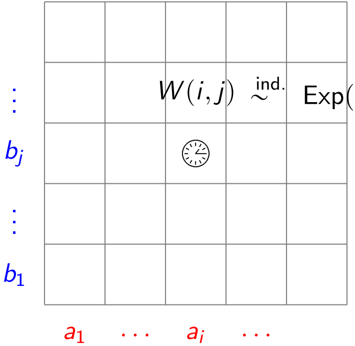

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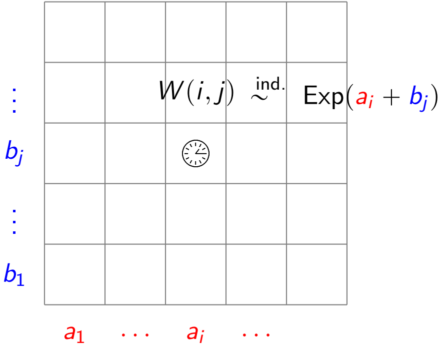
Random environment



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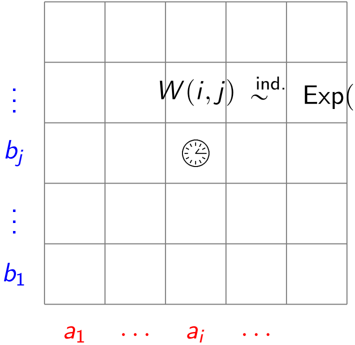
$\mathbb{P}$: average $\mathbf{P}_{\mathbf{a},\mathbf{b}}$ over $(\mathbf{a}, \mathbf{b})$: $\mathbb{P}(\cdot) = \mathrm{E}[\mathbf{P}_{\mathbf{a},\mathbf{b}}(\cdot)]$.

Random environment



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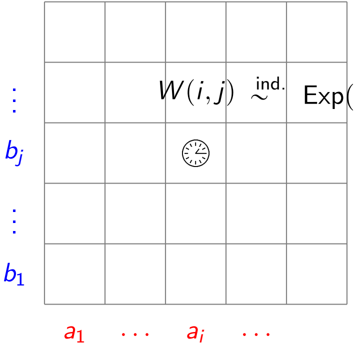

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if $i \neq i'$ or $j \neq j'$

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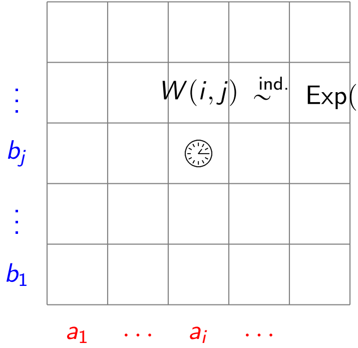

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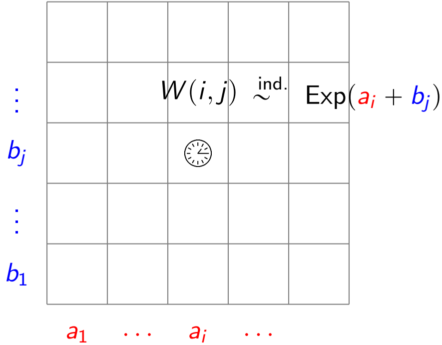

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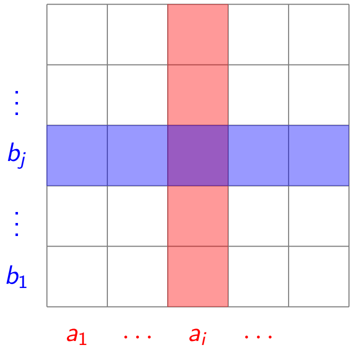

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If $i = i'$ or $j = j'$

$$\mathrm{Cov}(W(i,j), W(i',j')) \neq 0.$$

Stationary shape function

Lemma

For $s, t > 0$, $\mathbb{P}^z$ *almost surely and for almost all* $(\mathbf{a}, \mathbf{b})$ $\mathbf{P}^z_{\mathbf{a},\mathbf{b}}$ *almost surely*

$$g_z(s,t) := \lim_{n\to\infty} \frac{1}{n} \hat{G}_z(\lfloor ns \rfloor, \lfloor nt \rfloor) = s\,\mathrm{E}\left[\frac{1}{a_1 + z}\right] + t\,\mathrm{E}\left[\frac{1}{b_1 - z}\right].$$

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Proof:

$$\begin{aligned} \widehat{G}_z(n,n) &= \sum_{1\leqslant i\leqslant n} \left[\widehat{G}_z(i,0) - \widehat{G}_z(i-1,0)\right] + \sum_{1\leqslant j\leqslant n} \left[\widehat{G}_z(n,j) - \widehat{G}_z(n,j-1)\right] \\ &= \sum_{1\leqslant i\leqslant n} X(i,0) + \sum_{1\leqslant j\leqslant n} Y(n,j). \end{aligned}$$

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These sums are marginally i.i.d. under $\mathbb{P}^z$ (not mutually indep. under $\mathbf{P}^z_{\mathbf{a},\mathbf{b}}$ or $\mathbb{P}^z$).

Shape function

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Theorem (Emrah '15)

For $s, t > 0$, $\mathbb{P}$ almost surely and for almost all $(\mathbf{a}, \mathbf{b})$ $\mathbf{P}_{\mathbf{a},\mathbf{b}}$ almost surely

$$g(s,t) := \lim_{n\to\infty} \frac{1}{n} G(\lfloor ns \rfloor, \lfloor nt \rfloor) = \min_{-\underline{\alpha} \leqslant z \leqslant \underline{\beta}} \left\{ s\,\mathrm{E}\left[\frac{1}{a_1 + z}\right] + t\,\mathrm{E}\left[\frac{1}{b_1 - z}\right] \right\}$$

Shape function

Notation: $\underline{\alpha} = \text{essinf}\{a_1\}$, $\underline{\beta} = \text{essinf}\{b_1\}$.

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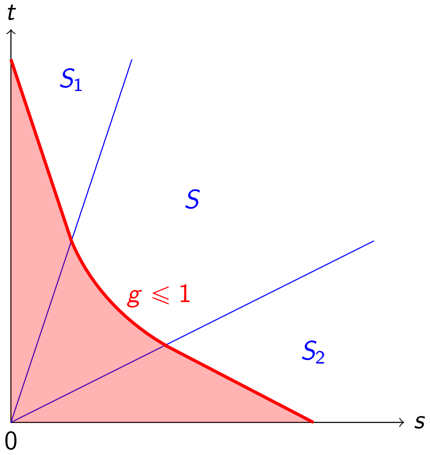
Remark:

- These results also hold if **a** and **b** are both separately ergodic, rather than a pair of independent i.i.d. sequences. The formulas only depend on marginal distributions of a_1 and b_1 separately.

Asymptotic shape of the cluster



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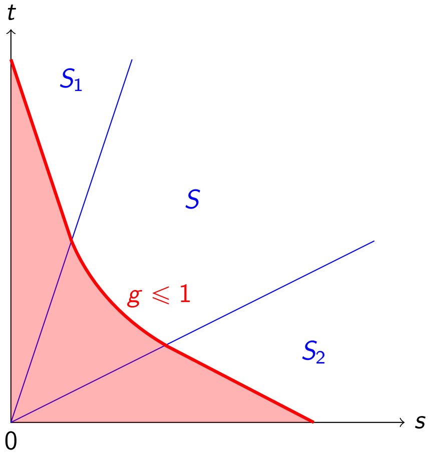

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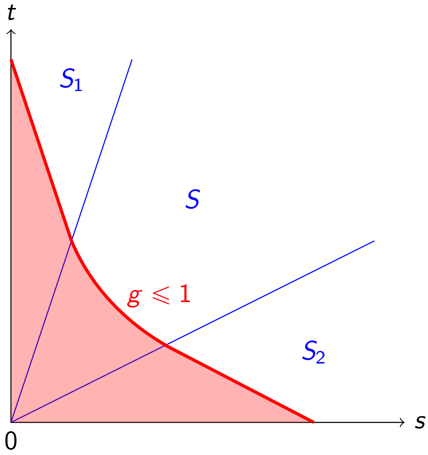

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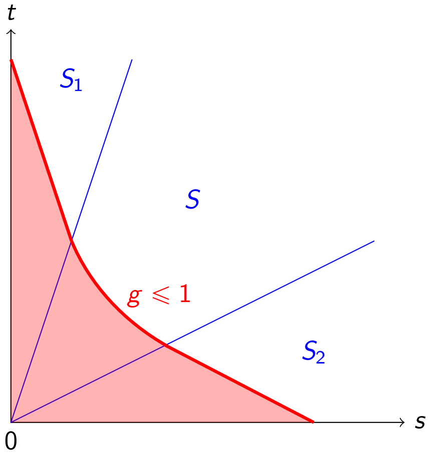

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$$E[(a_1 - \underline{\alpha})^{-2}] < \infty \quad (S_1)$$
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Quenched right tail rate function

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For almost all $(\mathbf{a}, \mathbf{b})$*, for any* $s, t > 0$ *and* $r \geqslant g(s,t)$

$$\mathbf{J}_{s,t}(r) = \lim_{n\to\infty} -n^{-1} \log \mathbf{P}_{\mathbf{a},\mathbf{b}}\left(n^{-1} G(\lfloor ns \rfloor, \lfloor nt \rfloor) \geqslant r\right)$$

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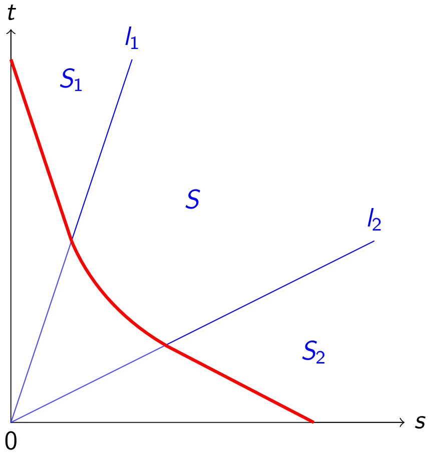
Remarks:

- The same theorem also holds if $(\mathbf{a},\mathbf{b})$ is totally ergodic rather than a pair of independent i.i.d. sequences. Again, these formulas only depend on marginal distributions of a_1 and b_1 separately.
- Rate n LDP for $n^{-1}G(\lfloor ns\rfloor,\lfloor nt\rfloor)$ under $\mathbf{P}_{\mathbf{a},\mathbf{b}}$ with rate function

$$\mathbf{I}_{s,t}(r) = \mathbf{J}_{s,t}(r)1_{\{r\geqslant g(s,t)\}} + \infty 1_{\{r<g(s,t)\}}.$$

Expected fluctuations



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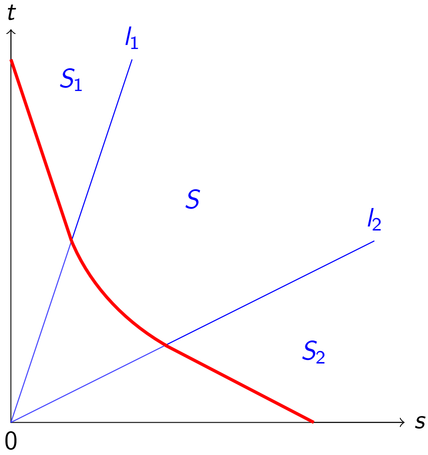

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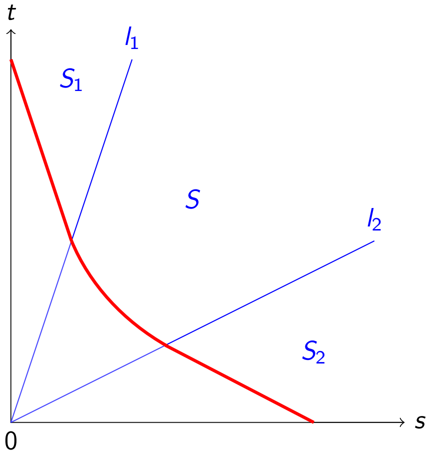

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For any $s, t > 0$*, let* $\zeta \in [-\underline{\alpha}, \underline{\beta}]$ *solve (uniquely)* $g_\zeta(s,t) = g(s,t)$*. As* $\epsilon \downarrow 0$*, there are explicit values* C_1, C_2, C_3 *depending on* s, t, ζ *such that*

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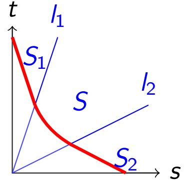

Scaling and the quenched rate functions

Proposition

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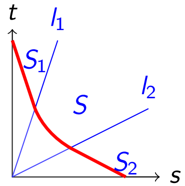

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which agrees with the leading order TW_{GUE} right tail.

Annealed large deviations

Theorem

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Remark:

- We do not have the rate n left tail rate function in this case, but we can show existence of $r \in (0, g(s,t))$ with

$$\limsup -\frac{1}{n}\mathbb{P}\left(n^{-1}G(\lfloor ns \rfloor, \lfloor nt \rfloor) \leqslant r\right) < \infty.$$

(i.e. there are rate n annealed left tail large deviations)

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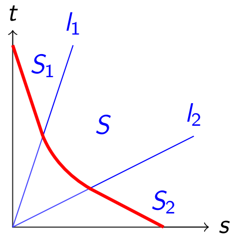

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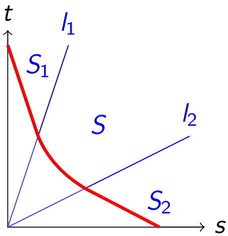

Variational connection for right tail

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For any $s, t > 0$ *and* $r \geqslant g(s,t)$,

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Proof sketch: finite n variational problem

Denote by $\theta_{i,j}$ the shift on the environment by (i,j). We have

$$\widehat{G}_z(n,n) = \max\{ \max_{1\leqslant k\leqslant n} \{G(n-k+1,n)\circ\theta_{k-1,0} + \widehat{G}_z(k,0),$$
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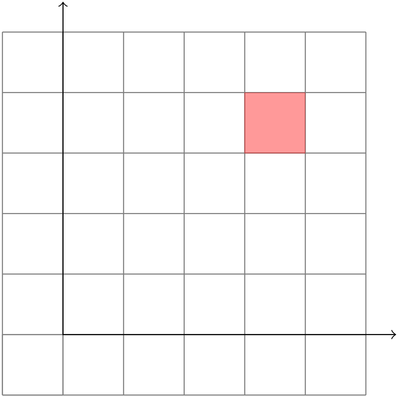

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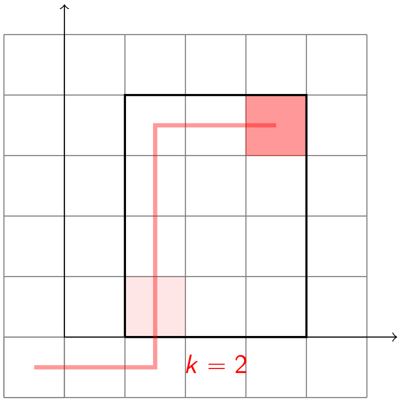

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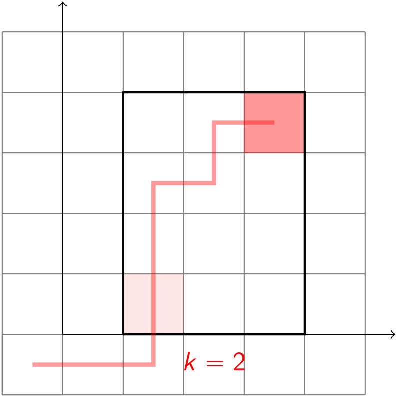

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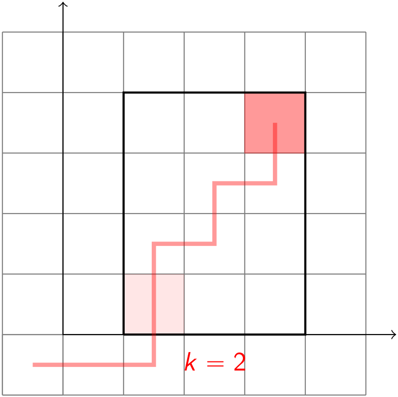

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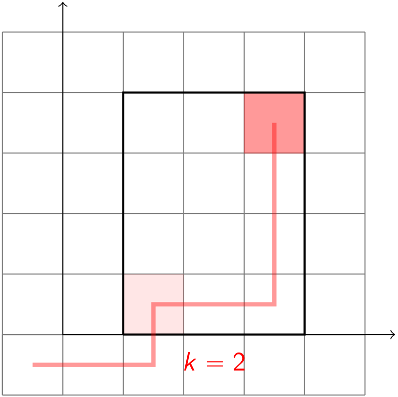

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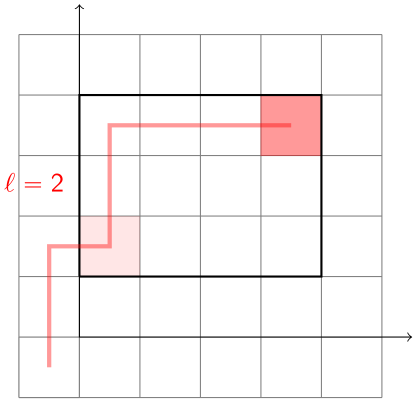

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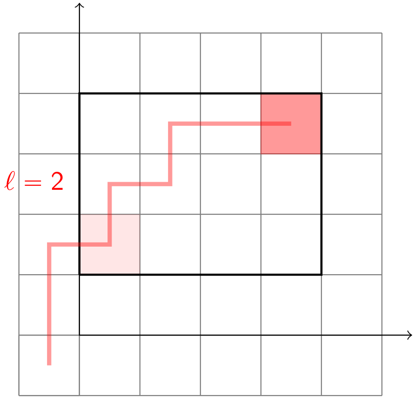

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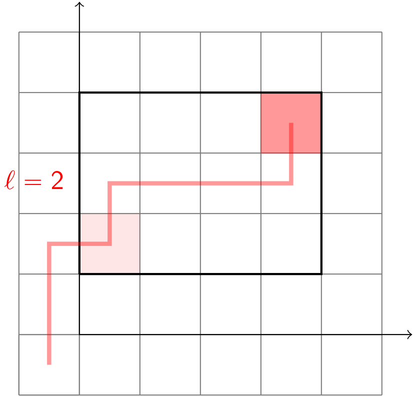
$$\widehat{G}_z(n,n) = \max\{ \max_{1\leqslant k\leqslant n}\{G(n-k+1,n)\circ\theta_{k-1,0} + \widehat{G}_z(k,0),$$
$$\max_{1\leqslant \ell\leqslant n}\{G(n,n-\ell+1)\circ\theta_{0,\ell-1} + \widehat{G}_z(0,\ell)\}\}.$$



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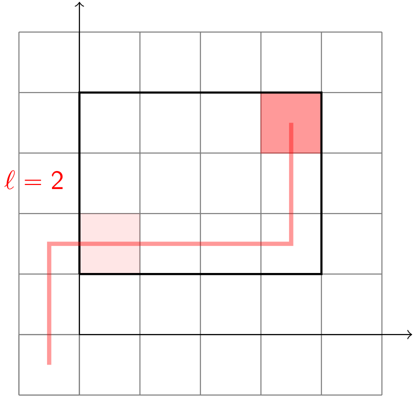
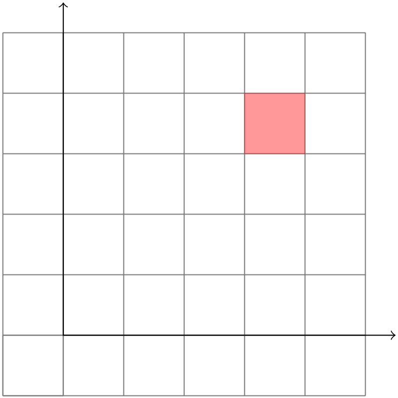


Proof sketch: telescoping sum

$$\widehat{G}_z(n,n) = \sum_{1\leqslant i\leqslant n} \widehat{G}_z(i,0) - \widehat{G}_z(i-1,0) + \sum_{1\leqslant j\leqslant n} \widehat{G}_z(n,j) - \widehat{G}_z(n,j-1)$$



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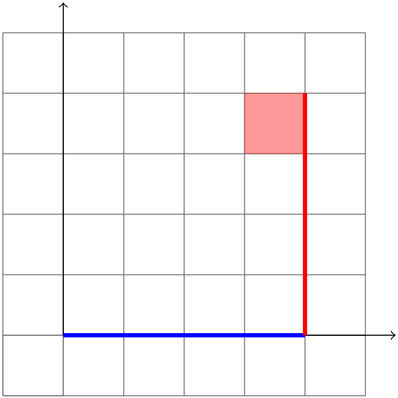
$$= \sum_{1\leqslant i\leqslant n} X(i,0) + \sum_{1\leqslant j\leqslant n} Y(n,j) \text{ (sums not indep.).}$$



Equate two expressions for $\hat{G}_z(n,n)$ and subtract the $\sum_{i=1}^{n} X(i,0)$ terms:

$$\sum_{1\leqslant j\leqslant n} Y(n,j) = \max_{1\leqslant k\leqslant n}\{\max\{G(n-k+1,n)\circ\theta_{k-1,0} - \sum_{k<i\leqslant n} X(i,0),$$
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Proof sketch: variational problem for $\mathbf{J}_{s,t}$

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For each $z \in (-\underline{\alpha}, \underline{\beta})$ of the summands within the maxima are $\mathbf{P}^z_{\mathbf{a},\mathbf{b}}$-indep.
Fix $\lambda > 0$ and $z \in (-\underline{\alpha}, \underline{\beta} - \lambda)$. For each k, we have

$$\mathbf{E}^z_{\mathbf{a},\mathbf{b}}\left[e^{\lambda\sum Y(n,j)}\right] \geqslant \mathbf{E}^z_{\mathbf{a},\mathbf{b}}\left[e^{\lambda G(n-k+1,n)\circ\theta_{k-1,0}}\right]\mathbf{E}^z_{\mathbf{a},\mathbf{b}}\left[e^{-\lambda\sum_{k<i\leqslant n} X(i,0)}\right]$$
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$$\times\,\mathbf{E}^z_{\mathbf{a},\mathbf{b}}\left[e^{\lambda\sum_{1\leqslant j\leqslant k} Y(0,j)}\right]$$

so LHS $\geqslant$ max of RHS over k.

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For each $z\in(-\underline{\alpha},\underline{\beta})$ of the summands within the maxima are $\mathbf{P}^z_{\mathbf{a},\mathbf{b}}$-indep. Fix $\lambda>0$ and $z\in(-\underline{\alpha},\underline{\beta}-\lambda)$. For each k, we have

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so LHS $\geqslant$ max of RHS over k.

Define

$$\mathbf{L}_{s,t}(\lambda) = \lim_{n\to\infty} \frac{1}{n} \log \mathbf{E}_{\mathbf{a},\mathbf{b}} \left[e^{\lambda G(\lfloor ns \rfloor, \lfloor nt \rfloor)} \right]$$

For $\lambda \in (0, \underline{\beta} - z)$, if we apply $\lim_{n\to\infty} n^{-1} \log \mathbf{E}^{z}_{\mathbf{a},\mathbf{b}}[e^{\lambda \cdot}]$, the previous inequality and another coming from $\max \leqslant \sum$ gives

$$\mathrm{E}\left[\log \frac{b_1 - z}{b_1 - z - \lambda}\right] = \sup_{0 \leqslant t \leqslant 1} \left\{ \max\left\{ \mathbf{L}_{1,t}(\lambda) - t\,\mathrm{E}\left[\log \frac{a_1 + z + \lambda}{a_1 + z}\right] \right\}, \right.$$
$$\left. \mathbf{L}_{1,t}(\lambda) - \mathrm{E}\left[\log \frac{a_1 + z + \lambda}{a_1 + z}\right] + (1 - t)\,\mathrm{E}\left[\log \frac{b_1 - z}{b_1 - z - \lambda}\right] \right\}$$

This variational problem can be inverted to solve for $\mathbf{L}_{s,t}(\lambda) =$

$$\begin{cases} \lambda g(s,t) & \lambda \leqslant 0 \\ \min\limits_{-\underline{\alpha}\leqslant z \leqslant \underline{\beta}-\lambda} \left\{ s\,\mathrm{E}\left[\log \dfrac{a_1+z+\lambda}{a_1+z}\right] + t\,\mathrm{E}\left[\log \dfrac{b_1-z}{b_1-z-\lambda}\right]\right\} & \lambda \in (0, \underline{\alpha}+\underline{\beta}] \\ \infty & \lambda > \underline{\alpha}+\underline{\beta} \end{cases}$$

Cannot use Gärtner-Ellis (steepness can fail), but can verify that the Legendre transform gives the rate function.

Thanks!