Qualitative methods in KPZ UNIVERSALITY

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Mean-field directed polymers on a complete graph.

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Based on collaborations with
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Discrete polymer models in random environment

- Compact space (= finite, cardinality = $N$)
- Mean-field type (to find exactly solvable models)

Approximation of non-compact case $N \rightarrow \infty$
Contents

1 Mean-Field Model
2 Integrable model on Complete graph
3 Large $N$ asymptotics
Mean-Field Model on Complete graph with $N$ sites

- For $1 \leq i, j \leq N$, paths starting at $(0, i)$ ending at $(t, j)$,
  \[
  J_N(0, i; t, j) = \{j = (j_0, \cdots, j_t) : 1 \leq j_s \leq N, \forall 0 \leq s \leq t-1, j_0 = i, j_t = j\},
  \]
- Let $\{\omega_{i,j}(t) : 1 \leq i, j \leq N, t \geq 0\}$ i.i.d. $> 0$. P2P partition function
  \[
  Z_N(0, i; t, j) = \sum_{j \in J_N(0,i,t,j)} \prod_{s=1}^{t} \omega_{j_{s-1},j_s}(s).
  \]

Figure: From Cook-Derrida 1990

- Finite transverse space for polymer: Derrida (+Cook’90; + Brunet’04)
  Free energy (= Lyapunov exp.) Eckman-Wayne 1989
  Last-passage percolation: C-Quastel-Ramirez 2015
Let $\mathcal{X}(t) = [\omega_{i,j}(t)]_{i,j} N \times N$ positive matrix

$$\Pi(s, t) = \mathcal{X}(s + 1) \ldots \mathcal{X}(t), \quad \Pi(t, t) = I_N, \quad \Pi(t) = \Pi(0, t).$$

so

$$Z_N(0, i; t, j) = \Pi(0, t)_{i,j}$$
Let $\mathcal{X}(t) = [\omega_{i,j}(t)]_{i,j} N \times N$ positive matrix

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$$Z_N(0, i; t, j) = \Pi(0, t)_{i,j}$$

*P2L partition function:* set $J_N(0, i; t, \star) = \bigcup_{j=1}^N J_N(0, i; t, j)$, etc.

$$Z_N(0, i; t, \star) = \sum_{j \in J_N(0, i; t, \star)} \prod_{s=1}^t \omega_{j_{s-1}, j_s}(s) = (\Pi(t) \mathbf{1})_i.$$
Let $\mathcal{X}(t) = [\omega_{i,j}(t)]_{i,j} N \times N$ positive matrix

$$\Pi(s, t) = \mathcal{X}(s + 1) \ldots \mathcal{X}(t), \quad \Pi(t, t) = I_N, \quad \Pi(t) = \Pi(0, t).$$

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**P2L partition function:** set $J_N(0, i; t, \star) = \bigcup_{j=1}^{N} J_N(0, i; t, j)$, etc....

$$Z_N(0, i; t, \star) = \sum_{j \in J_N(0, i; t, \star)} \prod_{s=1}^{t} \omega_{j_{s-1},j_{s}}(s) = (\Pi(t)1)_{i}.$$

**L2P partition function**

$$Z_N(t, j) = Z_N(0, \star; t, j) = \sum_{j \in J_N(0, \star; t, j)} \prod_{s=1}^{t} \omega_{j_{s-1},j_{s}}(s) = j\text{-th coord. of } 1^\star \Pi(t).$$
Let $\chi(t) = [\omega_{i,j}(t)]_{i,j} \in \mathbb{R}^{N \times N}$ be a positive matrix

$$\Pi(s, t) = \chi(s + 1) \ldots \chi(t), \quad \Pi(t, t) = I_N, \quad \Pi(t) = \Pi(0, t).$$

so

$$Z_N(0, i; t, j) = \Pi(0, t)_{i,j}$$

- P2L partition function: set $J_N(0, i; t, *) = \bigcup_{j=1}^{N} J_N(0, i; t, j)$, etc.

$$Z_N(0, i; t, *) = \sum_{j \in J_N(0, i; t, *)} \prod_{s=1}^{t} \omega_{j_{s-1}, j_s}(s) = (\Pi(t) 1)_{i}.$$  

- L2P partition function

$$Z_N(t, j) = Z_N(0, *; t, j) = \sum_{j \in J_N(0, *; t, j)} \prod_{s=1}^{t} \omega_{j_{s-1}, j_s}(s) = j-$th coord. of $1^* \Pi(t).$$

We single out the L2P p.f. (column vector) $Z_N(t) = (Z(t, 1), \ldots, Z(t, N))^*$

$$Z_N(t, *) = \sum_{j=1}^{N} Z_N(t, j) = \|Z_N(t)\|_1,$$

$$Z_N(t)^* = Z_N(t - 1)^* \chi(t).$$
Fondamental results on products of random matrices

Products of i.i.d. random matrices. Fustenberg, Kesten’60. Lyapunov exp.
Fix $\alpha > 0$; the $\alpha$-norm of $\mathbf{v} \in \mathbb{R}^N$ is $||\mathbf{v}||_\alpha = (\sum_{j=1}^N v_i^\alpha)^{1/\alpha}$.

$$B_\alpha = \{ \mathbf{v} \in \mathbb{R}_+^N : ||\mathbf{v}||_\alpha = 1 \} \quad (\alpha\text{-symplex}),$$

Define projection $\psi_\alpha : \mathbb{R}_+^N \setminus \{0\} \to B_\alpha$, $\psi_\alpha(\mathbf{v}) = \frac{\mathbf{v}}{||\mathbf{v}||_\alpha}$.
Fondamental results on products of random matrices

Products of i.i.d. random matrices. Fustenberg, Kesten’60. Lyapunov exp.

Fix $\alpha > 0$;

the $\alpha$-norm of $v \in \mathbb{R}_+^N$ is $\|v\|_\alpha = \left(\sum_{j=1}^N v_j^\alpha\right)^{1/\alpha}$.

$$\bar{B}_\alpha = \{v \in \mathbb{R}_+^N : \|v\|_\alpha = 1\} \quad (\alpha{-}\text{symplex}),$$

Define projection $\psi_\alpha : \mathbb{R}_+^N \setminus \{0\} \to \bar{B}_\alpha$, $\psi_\alpha(v) = \frac{v}{\|v\|_\alpha}$

Normalized product (action on directions)

$\bar{x} \cdot v := \frac{\bar{x}v}{\|\bar{x}v\|_\alpha} \in \bar{B}_\alpha$.

(drop the subscripts from notation when $\alpha = 1$, e.g. $\bar{B} := \bar{B}_1$, $\bar{x} \cdot v := \bar{x}^1 \cdot v$.)

Finally, define

$$X_{N,\alpha}(t) := \psi_\alpha(Z_N(t)) = \frac{Z_N(t)}{\|Z_N(t)\|_\alpha} \in \bar{B}_\alpha,$$

by the recursion and homogeneity,

$$X_{N,\alpha}(t) = \psi_\alpha(\bar{x}(t)^* Z_{N,\alpha}(t - 1)) = \psi_\alpha(\bar{x}(t)^* X_{N,\alpha}(t - 1)),$$

thus, $\{X_{N,\alpha}(t) : t \geq 0\}$ is a Markov chain.
\(\forall \alpha > 0, (X_{N,\alpha}(t))_{t \geq 0}\) is a Markov chain in \(\bar{B}_\alpha\).

**Proposition (Hennion 1997, Hennion-Hervé 2008)**

\[
\exists \text{ event } \Omega_0 \text{ with } \mathbb{P}(\Omega_0) = 1 \text{ such that the (random) limit } \quad X_{N,\alpha}^\infty = \lim_{t \to \infty} \Pi(t)^\alpha \cdot v,
\]

exists for all \(\alpha > 0, \omega \in \Omega_0\) and does not depend on \(v \in \mathbb{R}_+^N\).
∀α > 0, (X_{N,α}(t))_{t≥0} is a Markov chain in \( \bar{B}_α \).

**Proposition (Hennion 1997, Hennion-Hervé 2008)**

∃ event \( Ω_0 \) with \( P(Ω_0) = 1 \) such that the (random) limit

\[
X_{N,α}^∞ = \lim_{t→∞} Π(t)^α \cdot v,
\]

exists for all \( α > 0, ω ∈ Ω_0 \) and does not depend on \( v ∈ R_+^N \).
Moreover, \( X_{N,α}^∞ = Ψ_α(X_{N,β}^∞) \) for all \( α, β > 0 \).
∀ α > 0, \((X_{N,\alpha}(t))_{t\geq 0}\) is a Markov chain in $\bar{B}_\alpha$.

**Proposition (Hennion 1997, Hennion-Hervé 2008)**

1. \(\exists\) event $\Omega_0$ with $\mathbb{P}(\Omega_0) = 1$ such that the (random) limit

   \[ X_{N,\alpha}^\infty = \lim_{t\to\infty} \Pi(t)^\alpha \cdot v, \]

   exists for all $\alpha > 0$, $\omega \in \Omega_0$ and does not depend on $v \in \mathbb{R}^N$.

   Moreover, $X_{N,\alpha}^\infty = \Psi_\alpha(X_{N,\beta}^\infty)$ for all $\alpha, \beta > 0$.

2. Let $m_{N,\alpha}$ denote the law of $X_{N,\alpha}^\infty$.

   The chain $(X_{N,\alpha}(t))_{t\geq 0}$ with initial law $m_{N,\alpha}$ is stationary and ergodic.
∀α > 0, \((X_{N,\alpha}(t))_{t\geq 0}\) is a Markov chain in \(\bar{B}_{\alpha}\).

**Proposition (Hennion 1997, Hennion-Hervé 2008)**

1. \(\exists\) event \(\Omega_0\) with \(\mathbb{P}(\Omega_0) = 1\) such that the (random) limit
   \[
   X_{N,\alpha}^\infty = \lim_{t \to \infty} \Pi(t)^\alpha \cdot v,
   \]
   exists for all \(\alpha > 0, \omega \in \Omega_0\) and does not depend on \(v \in \mathbb{R}^N_+\).
   Moreover, \(X_{N,\alpha}^\infty = \Psi_\alpha(X_{N,\beta}^\infty)\) for all \(\alpha, \beta > 0\).

2. Let \(m_{N,\alpha}\) denote the law of \(X_{N,\alpha}^\infty\).
   The chain \((X_{N,\alpha}(t))_{t\geq 0}\) with initial law \(m_{N,\alpha}\) is stationary and ergodic.

3. Denote by \(\theta_s\) the shift on \(\Omega\) by \(s \in \mathbb{Z}\), \(\theta_s(\omega)(t) = \omega(s + t)\), and set
   \[
   X_{N,\alpha}(s) := X_{N,\alpha}^\infty \circ \theta_s = \lim_{t \to \infty} \Pi(s, t)^\alpha \cdot v,
   \]
   (In particular, \(X_{N,\alpha}^\infty(0) = X_{N,\alpha}^\infty\).) Then,
   \[
   \mathfrak{X}(0)^\alpha X_{N,\alpha}^\infty = X_{N,\alpha}^\infty(-1)
   \]
P2L polymer measure = probability measure on $J_N(0, i; T, \star)$:

$$P_{0,,i,T,\star}^\omega(j = (j_0, \cdots, j_t)) = \frac{1_{j_0=i}}{Z_N(0, i; T, \star)} \prod_{s=1}^{t} \omega_{j_{s-1},j_s}(s).$$

Similarly, there exists an almost-sure limit to the backward product

$$\hat{X}_{N,\alpha}^\infty(s) = \lim_{t \to \infty} \Pi(-t, s - 1)^\star \alpha \cdot \nu$$

which does not depend on $\nu \in \mathbb{R}^N_+$. Since $\hat{x}(0)^\star \overset{\text{law}}{=} x(0)$, we have

$$\hat{X}_{N,\alpha}^\infty(s) \overset{\text{law}}{=} X_{N,\alpha}^\infty.$$
Infinite volume polymer measure

▫ P2L polymer measure = probability measure on $J_N(0, i; T, \star)$:

$$P_{0,i;T,\star}^{\omega}(j = (j_0, \cdots, j_t)) = \frac{1_{j_0 = i}}{Z_N(0, i; T, \star)} \prod_{s=1}^{t} \omega_{j_{s-1}, j_s}(s).$$

▫ Similarly, there exists an almost-sure limit to the backward product

$$\hat{X}_{N,\alpha}^{\infty}(s) = \lim_{t \to \infty} \prod(-t, s - 1)^* \cdot \nu$$

which does not depend on $\nu \in \mathbb{R}^{N}_+$. Since $\mathcal{X}(0)^* \overset{\text{law}}{=} \mathcal{X}(0)$, we have

$$\hat{X}_{N,\alpha}^{\infty}(s) \overset{\text{law}}{=} X_{N,\alpha}^{\infty}.$$

▫ Define the random probability measure $\nu_N(t, \cdot)$ on $\{1, \ldots N\}$ by

$$(X_{N,\alpha}^{\infty}(s, j) = j\text{-th component of } X_{N,\alpha}^{\infty}(s))$$

$$\nu_N(t, j) = \frac{\hat{X}_{N,\alpha}^{\infty}(t, j)X_{N,\alpha}^{\infty}(t, j)}{\sum_{k=1}^{N} \hat{X}_{N,\alpha}^{\infty}(t, k)X_{N,\alpha}^{\infty}(t, k)}.$$

Result: existence of an infinite volume polymer measure and a co-variant law.
Infinite volume polymer measure

Proposition

1. For almost every environment \( \omega \), the polymer measure \( P^\omega_{0,i;T,*} \) converges as \( T \to \infty \) to the (time-inhomogeneous) Markov chain with \( P^\omega(j_0 = i) = 1 \) and transition probabilities given by

\[
P^\omega(j_{t+1} = \ell | j_t = k) = \frac{\omega_k,\ell(t + 1)X_N^\infty(t + 1, \ell)}{\sum_{\ell' = 1}^N \omega_k,\ell'(t + 1)X_N^\infty(t + 1, \ell')}
\]

for \( t \geq 0, k, \ell \in \{1, \ldots N\} \).

2. Let \( \omega \in \Omega_0 \). For the chain with transition (1) starting at time \( s \) with law \( \nu(s, \cdot) \), we have for \( t \geq s \),

\[
P^\omega(j_t = \ell) = \nu_N(t, \ell) \ , \quad \ell = 1, \ldots N.
\]

The co-variant law is proportional to the doubly infinite sum of weights over polymers (from times \(-\infty \) to \( +\infty \)) which take the value \( j \) at time \( t \).
Infinite volume polymer measure: sketch of proof

□ Finite horizon P2L polymer measure: for \( j \in J_N(0, i; T, \star) \),

\[
P_{0,i;T,\star}^\omega(j) = \frac{\prod_{t=1}^T \omega_{j_{t-1},j_t}(t)}{Z_N(0, i; T, \star)}
\]

is time-inhomogeneous Markov chain on \( \{1, \ldots, N\} \). Transition for \( 0 \leq t < T \):

\[
P_{0,i;T,\star}^\omega(j_{t+1} = \ell | j_t = k) = \frac{\omega_{k,\ell}(t+1)Z_N(t+1, \ell; T, \star)}{\sum_{1 \leq m \leq N} \omega_{k,m}(t+1)Z_N(t+1, m; T, \star)}
\]

But, a.s.,

\[
\prod(t+1, T) \cdot 1 \longrightarrow X_N^\infty(t+1)
\]

so above LHS converges

\[
P_{0,i;T,\star}^\omega(j_{t+1} = \ell | j_t = k) \longrightarrow \frac{\omega_{k,\ell}(t+1)X_N^\infty(t+1, \ell)}{\sum_{\ell' = 1}^{N} \omega_{k,\ell'}(t+1)X_N^\infty(t+1, \ell')}
\]
Free energy and Gaussian fluctuation at fixed $N$

From general results on products of random matrices:

**Theorem**

Fix $N$ and assume that $\mathbb{E} | \ln \omega_{i,j}|^{2+\delta} < \infty$. Then, there exist $v_N$ and $\sigma_N > 0$ [assuming $\omega$ is not constant] such that, for all $j = 1, \ldots, N$,

$$\lim_{t \to \infty} \frac{1}{t} \ln Z_N(t, j) = v_N \quad \text{a.s.,}$$

and

$$\frac{1}{\sqrt{t}} \left( \ln Z_N(t, j) - v_N t \right) \xrightarrow{\text{law}} \mathcal{N}(0, \sigma_N^2) \quad \text{as } t \to \infty.$$
Free energy and Gaussian fluctuation at fixed $N$

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$$\lim_{t \to \infty} \frac{1}{t} \ln Z_N(t, j) = v_N \quad \text{a.s.},$$

*and*

$$\frac{1}{\sqrt{t}} \left( \ln Z_N(t, j) - v_N t \right) \xrightarrow{\text{law}} \mathcal{N}(0, \sigma_N^2) \quad \text{as } t \to \infty.$$ 

*Furthermore,*

$$v_N = \mathbb{E} \left[ \ln \| \mathcal{K}(0) X_{N,\alpha}^\infty \|_\alpha \right].$$
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1 Mean-Field Model
2 Integrable model on Complete graph
3 Large $N$ asymptotics
\( \omega \sim S_\alpha \)

stable law of index \( \alpha \in (0, 1) \),

\[
\mathbb{E} e^{-\lambda S} = e^{-\lambda^\alpha}, \quad \lambda \geq 0.
\]

(For independent \( S_\alpha \)-distributed r.v.s, \( \sum_{i=1}^{N} a_i S_i \overset{\text{law}}{=} S_\alpha \) if \( \sum_{i=1}^{N} a_i^\alpha = 1 \).) Let

\[
S_N(t, j) := \frac{Z_N(t, j)}{\|Z_N(t - 1)\|_\alpha}, \quad \varphi_N(t) := \log \|Z_N(t - 1)\|_\alpha,
\]

so that

\[
\log Z_N(t, j) = \log S_N(t, j) + \varphi_N(t).
\]

\( \alpha \)-norm describes the mean height of the polymer.
Theorem

Suppose \( \{\omega_{i,j}(t)\} \) i.i.d., \( S_{\alpha} \)-distributed. Then,

\[
\{S_N(t,j) : t \geq 1, 1 \leq j \leq N\} \text{ is i.i.d. } S_{\alpha} \text{-distributed.}
\]
**Theorem**

Suppose \( \{\omega_{i,j}(t)\} \) i.i.d. , \( S_\alpha \)-distributed. Then,

1. \( \{S_N(t,j) : t \geq 1, 1 \leq j \leq N\} \) is i.i.d. \( S_\alpha \)-distributed.
2. \( \{\varphi_N(t) : t \geq 1\} \) is a random walk with i.i.d jumps \( \{\Upsilon_N(t) : t \geq 1\} \)

\[ \Upsilon_N \overset{\text{law}}{=} \ln \|S_N\|_\alpha, \]

where \( S_N \) is an i.i.d. family of \( S_\alpha \)-distributed random variables.
\(\alpha\)-Stable Environments

**Theorem**

Suppose \(\{\omega_{i,j}(t)\}\) i.i.d., \(S_\alpha\)-distributed. Then,

1. \(\{S_N(t,j) : t \geq 1, 1 \leq j \leq N\}\) is i.i.d. \(S_\alpha\)-distributed.
2. \(\{\varphi_N(t) : t \geq 1\}\) is a random walk with i.i.d jumps \(\{\gamma_N(t) : t \geq 1\}\)

\[\gamma_N \overset{\text{law}}{=} \ln \|S_N\|_\alpha,\]

where \(S_N\) is an i.i.d. family of \(S_\alpha\)-distributed random variables.

3. \(\nu_N = E[\gamma_N], \sigma_N^2 = \text{Var}[\gamma_N].\)
**Theorem**

Suppose \( \{\omega_{i,j}(t)\} \) i.i.d. , \( S_\alpha \)-distributed. Then,  
1. \( \{S_N(t,j) : t \geq 1, 1 \leq j \leq N\} \) is i.i.d. \( S_\alpha \)-distributed.  
2. \( \{\varphi_N(t) : t \geq 1\} \) is a random walk with i.i.d jumps \( \{\gamma_N(t) : t \geq 1\} \)  
   \[ \gamma_N \overset{law}{=} \ln \|S_N\|_\alpha, \]  
   where \( S_N \) is an i.i.d. family of \( S_\alpha \)-distributed random variables.  
3. \( \nu_N = \mathbb{E}[\gamma_N], \sigma_N^2 = \text{Var}[\gamma_N]. \)  
4. The invariant law \( m_{N,\alpha} = \text{law of } \frac{S_N}{\|S_N\|_\alpha}. \)
**Theorem**

Suppose \( \{\omega_{i,j}(t)\} \) i.i.d., \( S_\alpha \)-distributed. Then,

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2. \( \{\varphi_N(t) : t \geq 1\} \) is a random walk with i.i.d jumps \( \{\gamma_N(t) : t \geq 1\} \)

\[ \gamma_N \overset{\text{law}}{=} \ln ||S_N||_\alpha, \]

where \( S_N \) is an i.i.d. family of \( S_\alpha \)-distributed random variables.

3. \( v_N = \mathbb{E}[\gamma_N], \sigma_N^2 = \text{Var}[\gamma_N]. \)
4. The invariant law \( m_{N,\alpha} = \text{law of } \frac{S_N}{||S_N||_\alpha}. \)
5. \( \{X_{N,\alpha}^\infty(t,j) : t \geq 1, 1 \leq j \leq N\} \) is i.i.d. \( m_{N,\alpha} \)-distributed.
$\alpha$-Stable Environments

Some comments:

- Lyapunov exponents are usually not explicit. Some exceptions: Cohen-Newman 1984
- Polymer height function = moving front

$$\log Z_N(t, j) = \log S_N(t, j) + \varphi_N(t).$$

(height function remains concentrated around a ballistic motion)
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Lyapunov exponent for $\alpha$-Stable Environments

As the velocity and variance are explicit $\alpha$-Stable environments, we obtain asymptotics as $N$ grows to $\infty$. Observe

$$\gamma_N = \frac{1}{\alpha} \ln \sum_{j=1}^{N} S_N(j)^\alpha$$

where

$$\sum_{j=1}^{N} S_N(j)^\alpha \overset{\text{law}}{=} c_\alpha N \ln N + NS_1^+ + o(N),$$

since $(S_\alpha)^\alpha \in \text{Dom}(S_1^+)$ totally asymmetric, Cauchy distribution.
Lyapunov exponent for $\alpha$-Stable Environments

As the velocity and variance are explicit $\alpha$-Stable environments, we obtain asymptotics as $N$ grows to $\infty$. Observe

$$\gamma_N = \frac{1}{\alpha} \ln \sum_{j=1}^{N} S_N(j)^{\alpha} \quad \text{where} \quad \sum_{j=1}^{N} S_N(j)^{\alpha} \xrightarrow{\text{law}} c_\alpha N \ln N + N S_1^+ + o(N),$$

since $(S_\alpha)^\alpha \in \text{Dom}(S_1^+)$ totally asymmetric, Cauchy distribution.

Proposition

Assume $\{\omega_{i,j}(t) : t \geq 1, 1 \leq i, j \leq N\}$ is an i.i.d., $S_\alpha$-distributed family. Then, as $N \to \infty$,

$$v_N = \alpha^{-1} \left( \ln N + \ln \ln N + \ln c_\alpha \right) + o(1), \quad (2)$$

$$\sigma_N^2 = \frac{\pi^2}{3\alpha^2 \ln N} + o\left( \frac{1}{\ln N} \right). \quad (3)$$
Front profile for $\alpha$-Stable Environments

Front profile $U_N(t, \cdot)$ at time $t :=$ random distribution of the log – P2P polymer partition function:

$$U_N(t, x) = \frac{1}{N} \sum_{j=1}^{N} 1_{\{\ln Z_N(t, j) > x\}}.$$

Let $u_\alpha(x) = P(S_\alpha > e^x), x \in \mathbb{R}$.

Proposition

Fix $t \geq 1$, we have

1. Conditionally on $F_t$, we have

$$U_N(t, x + \phi_N(t - 1)) \rightarrow u_\alpha(x),$$

as $N \rightarrow \infty$, uniformly in $x$.

2. As $N \rightarrow \infty$,

$$c_\alpha \ln N \times \left[ U_N(t, x + (t - 1)v_N + \phi_N(0)) - u_\alpha(x) \right] \xrightarrow{\text{law}} u'_\alpha(x) X,$$

where $X$ is distributed as a sum of $t$ independent $S_\alpha + 1$ random variables.
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Let

$$u_\alpha(x) = \mathbb{P}(S_\alpha > e^x), \quad x \in \mathbb{R}.$$

**Proposition**

*Fix $t \geq 1$, we have*

1. *Conditionally on $\mathcal{F}_t$, we have*

$$U_N(t, x + \varphi_N(t - 1)) \to u_\alpha(x), \quad \text{a.s.},$$

*as $N \to \infty$, uniformly in $x$.***
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Front profile $U_N(t, \cdot)$ at time $t := \text{random distribution of the log – P2P polymer partition function}$:

$$U_N(t, x) = \frac{1}{N} \sum_{j=1}^{N} 1\{\ln Z_N(t,j) > x\}.$$ 

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**Proposition**

*Fix $t \geq 1$, we have*

1. *Conditionally on $\mathcal{F}_t$, we have*

   $$U_N(t, x + \varphi_N(t - 1)) \to u_\alpha(x), \quad \text{a.s.},$$

   *as $N \to \infty$, uniformly in $x$.*

2. *As $N \to \infty$,*

   $$c_\alpha \ln N \times \left[ U_N(t, x + (t - 1) v_N + \varphi_N(0)) - u_\alpha(x) \right] \xrightarrow{\text{law}} u'_\alpha(x) \mathcal{X}.$$

   *where $\mathcal{X}$ is distributed as a sum of $t$ independent $S_1^+$ random variables.*
Perturbative results

Environments close to $\alpha$-stable

Environments that are perturbations of the $S_\alpha$ laws. Suppose

$$1 - \mathbb{E} \exp(it\omega) \sim t^\alpha, \quad t \sim 0,$$

for some $\alpha \in (0, 1)$.

Claim (in progress)

For $t \geq 2$,

$$\frac{Z_N(t, i)}{\|Z_N(t - 1)\|_\alpha} \xrightarrow{\text{law}} S_\alpha.$$

For any sequence $K_N \subset \{1, \cdots, N\}$ with fixed size $|K_N| = k$,

$$\left\{ \frac{Z_N(t, i)}{\|Z_N(t - 1)\|_\alpha} : i \in K_N \right\} \xrightarrow{\text{law}} S_\alpha^{\otimes k}.$$

For $t \geq 2$, we have:

$$U_N(t, x + \varphi_N(t - 1)) \rightarrow u_\alpha(x), \quad \text{a.s.},$$

as $N \rightarrow \infty$, uniformly in $x$. 
Reason:

\[ \sum_{i=1}^{N} a_i \omega_{i,j}(t) \sim S_\alpha \]

in law, if

\[ \sum_{i=1}^{N} a_i^\alpha = 1 \quad \text{and} \quad a_i \quad \text{small.} \]

Can be checked with \( a_i = \frac{Z(t-1,i)}{\|Z_N(t-1)\|^\alpha} \) after 1 step of the dynamics.

Note: It appears \( S_\alpha \), not \( S_1^+ \).
Other questions (in progress)

- $(\alpha$-stable case) Asymptotics of the invariant measure
- $(\alpha$-stable case) Scaling limit
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THANK YOU!