*Qualitative methods in KPZ UNIVERSALITY*

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# Mean-field directed polymers on a complete graph.

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$\diamondsuit$

Based on collaborations with
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Discrete polymer models in random environment

- ☞ Compact space (= finite, cardinality = $N$)
- ☞ Mean-field type (to find exactly solvable models)

Approximation of non-compact case $N \to \infty$

# Contents

# Mean-Field Model on Complete graph with $N$ sites

- For $1 \le i, j \le N$, paths starting at $(0, i)$ ending at $(t, j)$,

  $J_N(0, i; t, j) = \{\mathbf{j} = (j_0, \cdots, j_t) : 1 \le j_s \le N, \forall_{0 \le s \le t-1}, j_0 = i, j_t = j\}$,

- Let $\{\omega_{i,j}(t) : 1 \le i, j \le N, t \ge 0\}$ i.i.d. $> 0$. P2P partition function

$$Z_N(0, i; t, j) = \sum_{\mathbf{j} \in J_N(0,i;t,j)} \prod_{s=1}^{t} \omega_{j_{s-1}, j_s}(s).$$

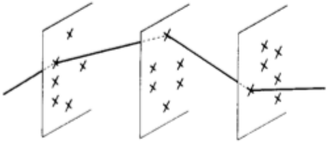

Figure: From Cook-Derrida 1990

- Finite transverse space for polymer: Derrida (+Cook'90; + Brunet'04)
  Free energy (= Lyapunov exp.) Eckman-Wayne 1989
  Last-passage percolation: C-Quastel-Ramirez 2015

Let $\mathfrak{X}(t) = [\omega_{i,j}(t)]_{i,j}$ $N \times N$ positive matrix

$$\Pi(s,t) = \mathfrak{X}(s+1)\dots\mathfrak{X}(t), \qquad \Pi(t,t) = \mathrm{I}_N, \qquad \Pi(t) = \Pi(0,t).$$

so

$$\boxed{Z_N(0,i;t,j) = \Pi(0,t)_{i.j}}$$

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☎ P2L partition function: set $J_N(0,i;t,\star) = \bigcup_{j=1}^N J_N(0,i;t,j)$, etc....

$$Z_N(0,i;t,\star) = \sum_{\mathbf{j}\in J_N(0,i;t,\star)} \prod_{s=1}^{t} \omega_{j_{s-1},j_s}(s) = \left(\Pi(t)\mathbf{1}\right)_i.$$

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✆ L2P partition function

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✄ We single out the L2P p.f. (column vector) $Z_N(t) = (Z(t,1),\cdots,Z(t,N))^*$

$$Z_N(t,\star) = \sum_{j=1}^{N} Z_N(t,j) = ||Z_N(t)||_1,$$

$$Z_N(t)^* = Z_N(t-1)^*\,\mathfrak{X}(t),$$

# Fondamental results on products of random matrices

Products of i.i.d. random matrices. Fustenberg, Kesten'60. Lyapunov exp.
Fix $\alpha > 0$; the $\alpha$-norm of $v \in \mathbb{R}_+^N$ is $||v||_\alpha = (\sum_{j=1}^N v_i^\alpha)^{1/\alpha}$.

$$\bar{B}_\alpha = \{v \in \mathbb{R}_+^N : \ ||v||_\alpha = 1\} \qquad (\alpha\text{−symplex}),$$

Define projection $\Psi_\alpha : \mathbb{R}_+^N \setminus \{0\} \to \bar{B}_\alpha$, $\Psi_\alpha(v) = \frac{v}{||v||_\alpha}$

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✂ Normalized product (action on directions)

$$\mathfrak{X} \overset{\alpha}{\cdot} v := \frac{\mathfrak{X} v}{||\mathfrak{X} v||_\alpha} \in \bar{B}_\alpha.$$

(drop the subscripts from notation when $\alpha = 1$, e.g.
$\bar{B} := \bar{B}_1, \quad \mathfrak{X} \cdot v := \mathfrak{X} \overset{1}{\cdot} v$.) Finally, define

$$X_{N,\alpha}(t) := \Psi_\alpha(Z_N(t)) = \frac{Z_N(t)}{||Z_N(t)||_\alpha} \in \bar{B}_\alpha,$$

by the recursion and homogeneity,

$$X_{N,\alpha}(t) = \Psi_\alpha\left(\mathfrak{X}(t)^* Z_{N,\alpha}(t-1)\right) = \Psi_\alpha\left(\mathfrak{X}(t)^* X_{N,\alpha}(t-1)\right),$$

thus, $\{X_{N,\alpha}(t) : \ t \geq 0\}$ is a Markov chain.

$\forall \alpha > 0$, $(X_{N,\alpha}(t))_{t \geq 0}$ is a Markov chain in $\bar{B}_\alpha$.

## Proposition (Hennion 1997, Hennion-Hervé 2008)

1. $\exists$ *event* $\Omega_0$ *with* $\mathbb{P}(\Omega_0) = 1$ *such that the (random) limit*

$$X_{N,\alpha}^{\infty} = \lim_{t \to \infty} \Pi(t) \overset{\alpha}{\cdot} v,$$

*exists for all* $\alpha > 0, \omega \in \Omega_0$ *and does not depend on* $v \in \mathbb{R}_+^N$.

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*exists for all* $\alpha > 0, \omega \in \Omega_0$ *and does not depend on* $v \in \mathbb{R}^N_+$. *Moreover,* $X^\infty_{N,\alpha} = \Psi_\alpha(X^\infty_{N,\beta})$ *for all* $\alpha, \beta > 0$.

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   *exists for all $\alpha > 0, \omega \in \Omega_0$ and does not depend on $v \in \mathbb{R}^N_+$.*
   *Moreover, $X^\infty_{N,\alpha} = \Psi_\alpha(X^\infty_{N,\beta})$ for all $\alpha, \beta > 0$.*
2. *Let $m_{N,\alpha}$ denote the law of $X^\infty_{N,\alpha}$.*
   *The chain $(X_{N,\alpha}(t))_{t\geq 0}$ with initial law $m_{N,\alpha}$ is stationary and ergodic.*

$\forall \alpha > 0$, $(X_{N,\alpha}(t))_{t\geq 0}$ is a Markov chain in $\bar{B}_\alpha$.

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2. *Let $m_{N,\alpha}$ denote the law of $X^\infty_{N,\alpha}$.*
*The chain $(X_{N,\alpha}(t))_{t\geq 0}$ with initial law $m_{N,\alpha}$ is stationary and ergodic.*

3. *Denote by $\theta_s$ the shift on $\Omega$ by $s \in \mathbb{Z}$, $\theta_s\omega(t) = \omega(s+t)$, and set*

$$X^\infty_{N,\alpha}(s) := X^\infty_{N,\alpha} \circ \theta_s = \lim_{t\to\infty} \Pi(s,t) \overset{\alpha}{\cdot} v,$$

*(In particular, $X^\infty_{N,\alpha}(0) = X^\infty_{N,\alpha}$.) Then,*

$$\mathfrak{X}(0) \overset{\alpha}{\cdot} X^\infty_{N,\alpha} = X^\infty_{N,\alpha}(-1)$$

# Infinite volume polymer measure

☞ P2L polymer measure = probability measure on $J_N(0,i;T,\star)$ :

$$P^{\omega}_{0,,i;T,\star}(\mathbf{j}=(j_0,\cdots,j_t)) = \frac{\mathbf{1}_{j_0=i}}{Z_N(0,i;T,\star)} \prod_{s=1}^{t} \omega_{j_{s-1},j_s}(s) \ .$$

☞ Similarly, there exists an almost-sure limit to the backward product

$$\overleftarrow{X}^{\infty}_{N,\alpha}(s) = \lim_{t\to\infty} \Pi(-t,s-1)^{*} \overset{\alpha}{\cdot} v$$

which does not depend on $v \in \mathbb{R}^N_+$. Since $\mathfrak{X}(0)^{*} \overset{\text{law}}{=} \mathfrak{X}(0)$, we have

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# Infinite volume polymer measure

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☞ Define the random probability measure $\nu_N(t,\cdot)$ on $\{1,\ldots N\}$ by $(X^{\infty}_{N,\alpha}(s,j) = j$-th component of $X^{\infty}_{N,\alpha}(s))$

$$\nu_N(t,j) = \frac{\overleftarrow{X}^{\infty}_{N,\alpha}(t,j) X^{\infty}_{N,\alpha}(t,j)}{\sum_{k=1}^{N} \overleftarrow{X}^{\infty}_{N,\alpha}(t,k) X^{\infty}_{N,\alpha}(t,k)} .$$

Result: existence of an infinite volume polymer measure and a co-variant law.

# Infinite volume polymer measure

## Proposition

1. *For almost every environment $\omega$, the polymer measure $P^{\omega}_{0,,i;T,\star}$ converges as $T \to \infty$ to the (time-inhomogeneous) Markov chain with $P^{\omega}(j_0 = i) = 1$ and transition probabilities given by*

$$P^{\omega}(j_{t+1} = \ell \big| j_t = k) = \frac{\omega_{k,\ell}(t+1) X_N^{\infty}(t+1,\ell)}{\sum_{\ell'=1}^{N} \omega_{k,\ell'}(t+1) X_N^{\infty}(t+1,\ell')} \qquad (1)$$

   *for $t \geq 0, k, \ell \in \{1, \dots N\}$.*

2. *Let $\omega \in \Omega_0$. For the chain with transition (1) starting at time $s$ with law $\nu(s,\cdot)$, we have for $t \geq s$,*

$$P^{\omega}(j_t = \ell) = \nu_N(t,\ell) , \qquad \ell = 1, \dots N.$$

The co-variant law is proportional to the doubly infinite sum of weights over polymers (from times $-\infty$ to $+\infty$) which take the value $j$ at time $t$.

# Infinite volume polymer measure: sketch of proof

□ Finite horizon P2L polymer measure: for $\mathbf{j} \in J_N(0, i; T, \star)$,

$$P^{\omega}_{0,i;T,\star}(\mathbf{j}) = \frac{\prod_{t=1}^{T} \omega_{j_{t-1},j_t}(t)}{Z_N(0, i; T, \star)}$$

is time-inhomogeneous Markov chain on $\{1, \ldots, N\}$. Transition for $0 \leq t < T$:

$$P^{\omega}_{0,i;T,\star}(j_{t+1} = \ell | j_t = k) = \frac{\omega_{k,\ell}(t+1) Z_N(t+1, \ell; T, \star)}{\sum_{1 \leq m \leq N} \omega_{k,m}(t+1) Z_N(t+1, m; T, \star)}$$

But, a.s.,

$$\Pi(t+1, T) \cdot \mathbf{1} \longrightarrow X_N^{\infty}(t+1)$$

so above LHS converges

$$P^{\omega}_{0,i;T,\star}(j_{t+1} = \ell | j_t = k) \longrightarrow \frac{\omega_{k,\ell}(t+1) X_N^{\infty}(t+1, \ell)}{\sum_{\ell'=1}^{N} \omega_{k,\ell'}(t+1) X_N^{\infty}(t+1, \ell')}$$

■

# Free energy and Gaussian fluctuation at fixed $N$

From general results on products of random matrices:

**Theorem**

*Fix N and assume that* $\mathbb{E}|\ln \omega_{i,j}|^{2+\delta} < \infty$. *Then, there exist* $v_N$ *and* $\sigma_N > 0$ *[assuming* $\omega$ *is not constant] such that, for all* $j = 1, \ldots N$,

$$\lim_{t\to\infty} \frac{1}{t} \ln Z_N(t,j) = v_N \qquad \text{a.s.},$$

*and*

$$\frac{1}{\sqrt{t}}\big(\ln Z_N(t,j) - v_N t\big) \xrightarrow{law} \mathcal{N}(0,\sigma_N^2) \qquad \text{as } t \to \infty.$$

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*Furthermore,*

$$v_N = \mathbb{E}\left[ \ln ||\mathfrak{X}(0) X^{\infty}_{N,\alpha}||_\alpha \right].$$

# Contents

# $\alpha$-Stable Environments

$$\omega \sim \mathcal{S}_\alpha$$

stable law of index $\alpha \in (0,1)$,

$$\mathbb{E}e^{-\lambda S} = e^{-\lambda^\alpha}, \qquad \lambda \geq 0.$$

( For independent $\mathcal{S}_\alpha$-distributed r.v.s, $\sum_{i=1}^N a_i S_i \overset{\text{law}}{=} \mathcal{S}_\alpha$ if $\sum_{i=1}^N a_i^\alpha = 1$.) Let

$$S_N(t,j) := \frac{Z_N(t,j)}{||Z_N(t-1)||_\alpha}, \quad \varphi_N(t) := \log ||Z_N(t-1)||_\alpha,$$

so that

$$\log Z_N(t,j) = \log S_N(t,j) + \varphi_N(t).$$

$\alpha$-norm describes the mean height of the polymer.

# $\alpha$-Stable Environments

**Theorem**

*Suppose* $\{\omega_{i,j}(t)\}$ *i.i.d. ,* $\mathcal{S}_\alpha$*-distributed. Then,*

1. $\{S_N(t,j) : t \geq 1, 1 \leq j \leq N\}$ *is i.i.d.* $\mathcal{S}_\alpha$*-distributed.*

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2. $\{\varphi_N(t) : t \geq 1\}$ *is a random walk with i.i.d jumps* $\{\Upsilon_N(t) : t \geq 1\}$

$$\Upsilon_N \stackrel{law}{=} \ln \|S_N\|_\alpha,$$

*where* $S_N$ *is an i.i.d. family of* $\mathcal{S}_\alpha$*-distributed random variables.*

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4. *The invariant law* $m_{N,\alpha} =$ *law of* $\frac{S_N}{||S_N||_\alpha}$.

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4. *The invariant law* $m_{N,\alpha} =$ *law of* $\frac{S_N}{||S_N||_\alpha}$.
5. $\{X^\infty_{N,\alpha}(t,j) : t \geq 1, 1 \leq j \leq N\}$ *is i.i.d.* $m_{N,\alpha}$*-distributed.*

# $\alpha$-Stable Environments

Some comments:

- Lyapunov exponents are usually not explicit.
  Some exceptions: Cohen-Newman 1984
- polymer height function = moving front

$$\log Z_N(t,j) = \log S_N(t,j) + \varphi_N(t).$$

(height function remains concentrated around a ballistic motion)

# Contents

# Lyapunov exponent for $\alpha$-Stable Environments

As the velocity and variance are explicit $\alpha$-Stable environments, we obtain asymptotics as $N$ grows to $\infty$. Observe

$$\Upsilon_N = \frac{1}{\alpha} \ln \sum_{j=1}^{N} S_N(j)^{\alpha} \quad \text{where} \quad \sum_{j=1}^{N} S_N(j)^{\alpha} \stackrel{\text{law}}{=} c_\alpha N \ln N + N \mathcal{S}_1^{+} + o(N),$$

since $(\mathcal{S}_\alpha)^\alpha \in \mathrm{Dom}(\mathcal{S}_1^{+})$ totally asymmetric, Cauchy distribution.

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since $(\mathcal{S}_\alpha)^\alpha \in \mathrm{Dom}(\mathcal{S}_1^+)$ totally asymmetric, Cauchy distribution.

## Proposition

*Assume $\{\omega_{i,j}(t) : t \geq 1, 1 \leq i, j \leq N\}$ is an i.i.d., $\mathcal{S}_\alpha$-distributed family. Then, as $N \to \infty$,*

$$v_N = \alpha^{-1}\big(\ln N + \ln\ln N + \ln c_\alpha\big) + o(1), \tag{2}$$

$$\sigma_N^2 = \frac{\pi^2}{3\alpha^2 \ln N} + o\Big(\frac{1}{\ln N}\Big). \tag{3}$$

# Front profile for $\alpha$-Stable Environments

Front profile $U_N(t,\cdot)$ at time $t :=$ random distribution of the log – P2P polymer partition function :

$$U_N(t,x) = \frac{1}{N}\sum_{j=1}^{N} \mathbf{1}_{\{\ln Z_N(t,j) > x\}}.$$

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Let

$$u_\alpha(x) = \mathbb{P}(\mathcal{S}_\alpha > e^x)\ , \quad x \in \mathbb{R}$$

## Proposition

*Fix $t \geq 1$, we have*

1. *Conditionally on $\mathcal{F}_t$, we have*

$$U_N(t, x + \varphi_N(t-1)) \to u_\alpha(x), \quad a.s.,$$

*as $N \to \infty$, uniformly in $x$.*

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   *as $N \to \infty$, uniformly in $x$.*

2. *As $N \to \infty$,*

$$c_\alpha \ln N \times \Big[U_N\big(t, x + (t-1)\, v_N + \varphi_N(0)\big) - u_\alpha(x)\Big] \xrightarrow{law} u'_\alpha(x)\mathcal{X}.$$

   *where $\mathcal{X}$ is distributed as a sum of $t$ independent $\mathcal{S}_1^+$ random variables.*

## Perturbative results Environments close to $\alpha$-stable

Environments that are perturbations of the $\mathcal{S}_\alpha$ laws. Suppose

$$1 - \mathbb{E}\exp(it\omega) \sim t^\alpha, \quad t \sim 0,$$

for some $\alpha \in (0, 1)$.

### Claim (in progress)

*For $t \geq 2$,*

$$\frac{Z_N(t,i)}{||Z_N(t-1)||_\alpha} \xrightarrow{law} \mathcal{S}_\alpha.$$

*For any sequence $K_N \subset \{1, \cdots, N\}$ with fixed size $|K_N| = k$,*

$$\left\{ \frac{Z_N(t,i)}{||Z_N(t-1)||_\alpha} : i \in K_N \right\} \xrightarrow{law} \mathcal{S}_\alpha^{\otimes k}.$$

*For $t \geq 2$, we have:*

$$U_N\big(t, x + \varphi_N(t-1)\big) \to u_\alpha(x), \quad a.s.,$$

*as $N \to \infty$, uniformly in $x$.*

# Perturbative results Environments close to $\alpha$-stable

Reason:

$$\sum_{i=1}^{N} a_i \omega_{i,j}(t) \simeq \mathcal{S}_\alpha$$

in law, if

$$\sum_{i=1}^{N} a_i^\alpha = 1 \quad \text{and} \quad a_i \quad \text{small.}$$

Can be checked with $a_i = \frac{Z(t-1,i)}{||Z_N(t-1)||_\alpha}$ after 1 step of the dynamics.

Note: It appears $\mathcal{S}_\alpha$, not $\mathcal{S}_1^+$.

## Other questions (in progress)

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*THANK YOU !*