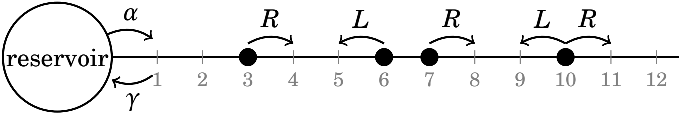
ASEP on a half-space with an open boundary and the KPZ equation in a half-space

Guillaume Barraquand

Joint work with Alexei Borodin, Ivan Corwin and Michael Wheeler.

The model

Let $R > L \geqslant 0$, and consider the asymmetric simple exclusion process on the positive integers with open boundary condition:

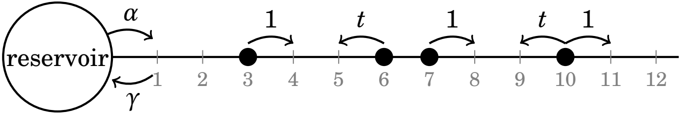


Notations

- Without loss of generality, one can assume $R = 1$.
- We denote the parameter L by $t \in [0,1)$.
- Denote time by τ.
- We are interested in the probability distribution of

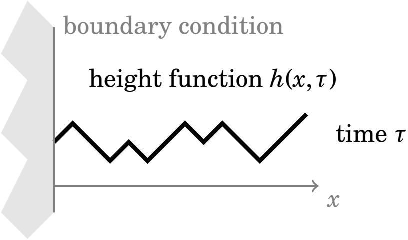
$$N_x(\tau) = \#\text{particles on the right of } x \text{ at time } \tau,$$

at large times τ.



Motivations

1. **KPZ growth in a half-space.** $N_x(\tau)$ can be seen as a height function.



   One knows the fluctuations of TASEP in a half-space (equivalently last-passage percolation in a half-quadrant). Are those of ASEP similar?

2. **KPZ equation on the positive reals.** Weakly asymmetric scaling limit of ASEP (Corwin-Shen 2016) suggests that a natural boundary condition is of Neumann type:

$$\begin{cases} \partial_\tau h = \frac{1}{2}\Delta h + \left(\partial_x h\right)^2 + \dot{\mathscr{W}} \\ \partial_x h(x,\tau)\Big|_{x=0} = a \in \mathbb{R}. \end{cases}$$

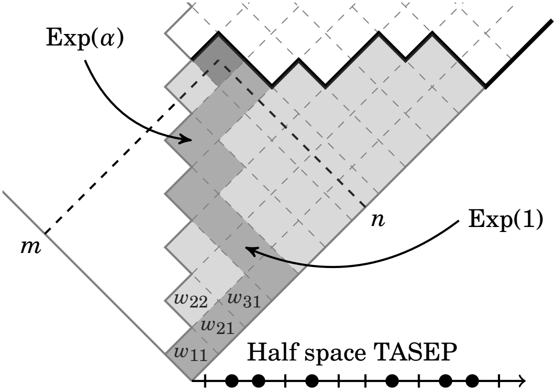
   What is the law of the solution?

Plan of the talk

1. The totally asymmetric case is equivalent to **LPP in a half-quadrant**, which is the simplest benchmark model for understanding KPZ growth in a half space.
2. New results on **half-line ASEP**: Tracy-Widom GOE asymptotics of the current at the origin.
3. **KPZ equation** on $\mathbb{R}_{>0}$.
4. Ideas of the **proof** using 3 ingredients:
    - Half-space stochastic six-vertex model (cf Amol's talk).
    - Half-space Macdonald processes.
    - Pfaffian point processes.

Last Passage Percolation in a half quadrant

Let w_{ij} a family of i.i.d. exponential random variables with rate 1 when $i > j$ and α when $i = j$.



Consider directed paths π from the box $(1,1)$ to (n,m) in the half quadrant. We define the last passage percolation time $H(n,m)$ by

$$H(n,m) = \max_{\pi} \sum_{(i,j)\in\pi} w_{ij}.$$

Passage-times on the diagonal

Theorem (Baik-Rains 2001 / Baik-B.-Corwin-Suidan 2016)

- *When* $\alpha > 1/2$,
$$\frac{H(n,n)-4n}{2^{4/3}n^{1/3}} \Longrightarrow \mathscr{L}_{\mathrm{GSE}},$$
- *When* $\alpha = 1/2$,
$$\frac{H(n,n)-4n}{2^{4/3}n^{1/3}} \Longrightarrow \mathscr{L}_{\mathrm{GOE}},$$
- *When* $\alpha < 1/2$,
$$\frac{H(n,n)-cn}{c'n^{1/2}} \Longrightarrow \mathscr{N},$$

In particular, if $N_0(\tau)$ is the current in half-line TASEP (right jump rate 1, insertion of particles at rate $\alpha = 1/2$, no particle moving to the left), starting from the empty initial condition,

$$\frac{N_0(\tau)-\frac{\tau}{4}}{2^{-4/3}\tau^{1/3}} \xrightarrow[\tau\to\infty]{} -\mathscr{L}_{GOE}.$$

Understanding of the phase transition

- The fact that $H(n,n) \sim 4n$ shows that the weights along the optimal path have size 2 in average. Thus, the disorder on the boundary becomes competitive when it has average at least 2, hence the transition at $\alpha = 1/2$.
- Algebraic considerations show that for any α, the law of $H(n,n)$ in the model with weight $\mathrm{Exp}(\alpha)$ on the diagonal is the same as the law of $H(n,n)$ in a modified model where the weights on the boundary are $\mathrm{Exp}(1)$ and the weights on the first row are $\mathrm{Exp}(\alpha)$.

  Open question: Is there a probabilistic proof?

  Open question: In the critical case, geodesics take $\mathcal{O}(n^{1/3})$ weights on the diagonal. Where?

Passage times away from the boundary

Theorem (Sasamoto-Imamura 2005/Baik-B.-Corwin-Suidan 2016)

For $\kappa \in (0,1)$ *and* $\alpha > \sqrt{\kappa}/(1+\sqrt{\kappa})$,

$$\frac{H(n,\kappa n) - (1+\sqrt{\kappa})^2 n}{\sigma n^{1/3}} \Longrightarrow \mathscr{L}_{\mathrm{GUE}}.$$

- One recovers the exact same result as for LPP in a full quadrant. The boundary has no influence as long as the boundary weights are not too big.
- If α decreases (i.e. boundary weights increase) the fluctuations should transition between GUE Tracy-Widom and Gaussian, with F^2_{GOE} fluctuations when $\alpha = \sqrt{\kappa}/(1+\sqrt{\kappa})$. This is the Baik-Ben Arous-Péché (2005) phase transition also arising in the full space case.

Crossovers

Condider two parameters $\varpi \in \mathbb{R}, \eta > 0$.

Theorem (Baik-B.-Corwin-Suidan 2016)

When the boundary parameter scales as

$$\alpha = \frac{1}{2} + 2^{-4/3} \varpi n^{-1/3},$$

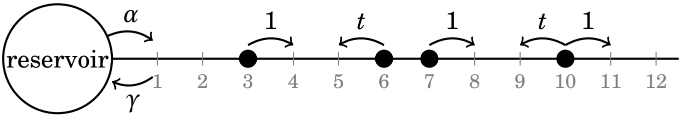
and one consider passage times at distance $\eta\, n^{2/3}$ *from the boundary,*

$$H_n(\eta, \varpi) := \frac{H\left(n + 2^{2/3}\eta n^{2/3}, n - 2^{2/3}\eta n^{2/3}\right) - 4n + n^{1/3} 2^{4/3} \eta^2}{2^{4/3} n^{1/3}},$$

The (multipoint) limiting distribution of $H_n(\eta, \varpi)$ *is a new two-parametric distribution that interpolates between GUE, GOE and GSE Tracy-Widom distribution.*

- It is related to RMT models interpolating between Unitary, Orthogonal and Symplectic Gaussian ensembles.

ASEP: previous results



- Liggett 1975 classified the stationary measures when

$$\alpha + \frac{\gamma}{t} = 1$$

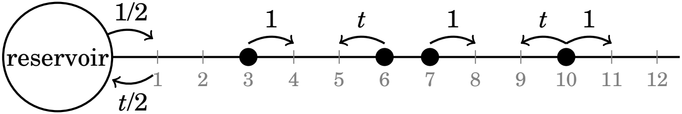
  There is a **phase transition** at $\alpha = 1/2$ between product-form Bernoulli measure and spatially correlated stationary measures. The parameter α is the average density enforced at the boundary.

- Tracy-Widom 2013 used **Bethe ansatz** to find formulas for the transition probabilities, not amenable to asymptotic analysis though.
- A way to analyze ASEP is through a half space version of the **stochastic six-vertex model**, that will be defined later. (analogously as in the full-space Borodin-Corwin-Gorin 2014, Aggarwal-Borodin 2016, Aggarwal 2016, Borodin-Olshanski 2016)

Main result on half-line ASEP

We assume

1. Ligget's condition.
2. The boundary enforces a density of particles $\alpha = 1/2$ at the origin.



Theorem (B.-Borodin-Corwin-Wheeler 2017)

For any $t \in [0,1)$, starting from the empty initial condition,

$$\frac{N_0\left(\frac{T}{1-t}\right) - \frac{T}{4}}{2^{-4/3}T^{1/3}} \xrightarrow[T\to\infty]{} -\mathscr{L}_{GOE}.$$

Recall $N_x(\tau)$ is the number of particles on the right of site x at time τ.

- Based on the prediction that ASEP fluctuations are the same as TASEP modulo a rescaling by the asymmetry, one expects diffusive scaling in the low density phase $\alpha < 1/2$ and GSE fluctuations in the high density phase $\alpha > 1/2$.

KPZ equation in a half-space

Consider

$$(SHE) \quad \begin{cases} \partial_\tau Z = \frac{1}{2}\Delta Z + Z\dot{W} \\ \partial_x Z(x,\tau)\Big|_{x=0} = a\ Z(\tau,0) \end{cases}$$

on $\mathbb{R}_+$ with delta initial data at the origin, in the mild sense:

$$Z(x,\tau) = p^a_\tau(x,0) + \int_0^\tau \int_0^\infty p^a_{\tau-s}(x,y) Z(y,s) \mathrm{d}W_s(\mathrm{d}Y)$$

where the last integral is the Itô integral with respect to Wiener process W, and p^a is the heat kernel satisfying the Robin boundary condition

$$\partial_x p^a_\tau(0,y) = a\ \ p^a_\tau(0,y) \qquad (\forall \tau > 0, y > 0)\,.$$

One can show that a.s. $Z(x,\tau) > 0$ and we define the solution of the KPZ equation

$$(KPZ) \quad \begin{cases} \partial_\tau h = \frac{1}{2}\Delta h + \left(\partial_x h\right)^2 + \dot{W} \\ \partial_x h(x,\tau)\Big|_{x=0} = a. \end{cases}$$

in the Cole-Hopf sense, i.e. as $h = \log(Z)$. (see also Hairer-Gerencsér 2017)

Weakly asymmetric scaling of ASEP

Theorem (B.-Borodin-Corwin-Wheeler 2017)

Under the scalings

$$t = e^{-\epsilon}, \qquad \tau = \frac{8\epsilon^{-3}\tilde{\tau}}{1-t} \approx 8\epsilon^{-4}\tilde{\tau},$$

the random variable

$$\mathcal{Z}_\epsilon(\tilde{\tau}) = \frac{4\exp\left[-\epsilon N(\tau) - 2\epsilon^{-2}\hat{\tau}\right)\right]}{1-t^2}$$

weakly converges as $\epsilon \to 0$ to a positive random variable $\mathcal{Z}(\tilde{\tau})$. For any $z > 0$,

$$\mathbb{E}\left[\exp\left(\frac{-z}{4}\mathcal{Z}(\tau)\right)\right] = \mathbb{E}\left[\prod_{i=1}^{+\infty} \frac{1}{\sqrt{1 + z\exp\left((\tau/2)^{1/3}\mathfrak{a}_i^{\mathrm{GOE}}\right)}}\right],$$

where $\{\mathfrak{a}_i^{\mathrm{GOE}}\}_{i=1}^{\infty}$ forms the GOE point process (i.e. the sequence of rescaled eigenvalues of a large Gaussian real symmetric matrix).

Interpretation

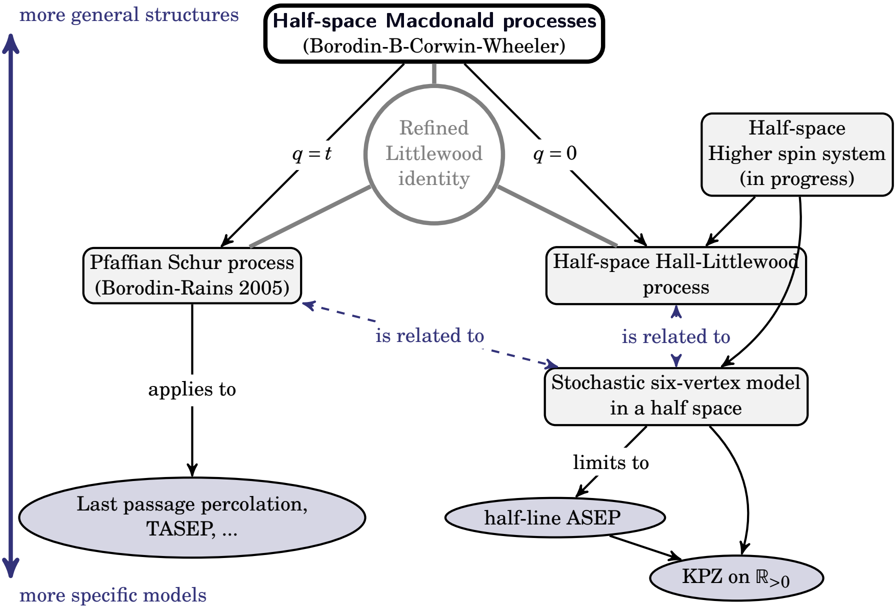
- Using results from Corwin-Shen 2016, $\log \mathcal{Z}(\tau) - \tau/24$ is expected to have the law of the solution to KPZ equation $h(0,\tau)$ with boundary parameter $a = -1/2$ (though Corwin-Shen work with $a \geqslant 0$).
- The result should be compared with the analogous full-space result (Amir-Corwin-Quastel, Calabrese-Le Doussal-Rosso, Dotsenko, Sasamoto-Spohn 2011, Borodin-Gorin 2016) where

$$\mathbb{E}\left[\exp\left(\frac{-z}{4}\mathcal{Z}(\tau)\right)\right] = \mathbb{E}\left[\prod_{i=1}^{+\infty} \frac{1}{1+z\exp\left((\tau/2)^{1/3}\mathfrak{a}_i^{\mathrm{GUE}}\right)}\right],$$

$$\mathbb{E}\left[\exp\left(\frac{-z}{4}\mathcal{Z}(\tau)\right)\right] = \mathbb{E}\left[\prod_{i=1}^{+\infty} \frac{1}{\sqrt{1+z\exp\left((\tau/2)^{1/3}\mathfrak{a}_i^{\mathrm{GOE}}\right)}}\right],$$

- In the cases $a = +\infty$ (Le Doussal-Gueudre 2012) and $a = 0$ (Borodin-Bufetov-Corwin 2015) there exist non rigorous results about the law of $\log(Z(\tau))$, though only when $\tau \to \infty$.

Roadmap of integrable structures at play



Half-space Macdonald measures

Skew Macdonald polynomials $P_{\lambda/\mu}, Q_{\lambda/\mu}$ are symmetric polynomials in many variables whose coefficients are rational functions in two parameters $q,t \in (0,1)$. They degenerate to skew Schur functions $s_{\lambda/\mu}$ when $q=t$.
For two sets of variables $a_1,\dots,a_n$ and $b_1,\dots,b_m$ in $(0,1)$, we consider the **Pfaffian Macdonald measure**

$$\mathbb{P}(\lambda) \propto P_\lambda(a)\, \mathscr{E}_\lambda(b),$$

where

$$\mathscr{E}_\lambda = \sum_{\mu' \text{even}} b_\lambda^{\text{el}} Q_{\lambda/\mu}.$$

In the following, we set $b_i \equiv 0$, so that the measure depends only on parameters $a_1,\dots,a_n$.

- It's a variant of the Macdonald measure (Borodin-Corwin 2011) which is a (q,t)-generalization of the Schur measure.
- As in the full-space case, one can define more general **half-space Macdonald processes**.
- Half-space Macdonald degenerate when $q=t$ to Pfaffian Schur processes.

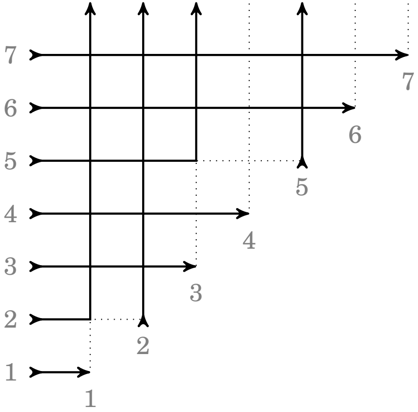
Half-space Hall-Litllewood measure

When $q = 0$, Macdonald polynomials degenerate to Hall-Littlewood polynomials

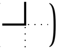
$$P_\lambda(x_1,\dots,x_n;t) = c(\lambda) \sum_{\sigma \in \mathcal{S}_n} \sigma\left(x_1^{\lambda_i} \dots, x_n^{\lambda_n} \prod_{i<j} \frac{x_i - t x_j}{x_i - x_j} \right).$$

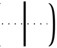
- Hall Littlewood polynomials have been recently connected to the six-vertex model and spin systems (Korff 2011 , Borodin 2014, Wheeler-Zinn-Justin 2014 & 2015).
- For the stochastic six vertex-model in a rectangular domain, the connection is very precise (Borodin 2016, Borodin-Bufetov-Wheeler 2016). One can use a spin model representation of Hall-Littlewood functions to relate half-space Hall-Littlewood processes to a stochastic six-vertex model in a quadrant.
- We adapt this to the half-space case using half-space Hall-Littlewood processes.

Stochastic six vertex model in a half space



$\mathbb{P}\Big($  $\Big) = \dfrac{1-a_x a_y}{1-t a_x a_y}$,

$\mathbb{P}\Big($  $\Big) = \dfrac{(1-t) a_x a_y}{1-t a_x a_y}$,

$\mathbb{P}\Big($  $\Big) = \dfrac{t(1-a_x a_y)}{1-t a_x a_y}$,

$\mathbb{P}\Big($  $\Big) = \dfrac{1-t}{1-t a_x a_y}$,

Proposition (B.-Borodin-Corwin-Wheeler 2017)

$$\mathbb{P}\big(\mathfrak{h}(n,n)=k\big) = \mathbb{P}\big(\ell(\lambda)=k\big),$$

where $\mathfrak{h}(n,n)$ is the number of outgoing vertical arrows from the vertices on the left of (n,n), and $\ell(\lambda)$ is the number of nonzero components in a partition λ following the Pfaffian Hall-Littlewood measure.

Relation Hall-Littlewood and Schur

A refined Littlewood identity for Macdonald functions (Rains 2015) shows that certain observables of half-space Macdonald measures do not depend on q.
Comparing the $q=0$ and $q=t$ cases yields identities relating functionals of Schur and Hall-Littlewood random partitions:

Proposition (B.-Borodin-Corwin-Wheeler 2017)

For any $x\in\mathbb{R}$, $n\in 2\mathbb{Z}_{>0}$, and $(a_1,\dots,a_n)\in(0,1)$ and $b\equiv 0$,

$$\mathbb{E}^{HL}\left[\frac{1}{(-t^{x+n-\ell(\lambda)},t^2)_\infty}\right]=\mathbb{E}^{Schur}\left[\prod_{p\in\mathbb{Z}\setminus\Lambda}\left(1+\mathsf{f}_x(p)\right)\right]=\mathrm{Pf}\left[\mathsf{J}+\mathsf{f}_x\cdot\mathsf{K}^{\complement}\right]_{\ell^2(\mathbb{Z}_{\geqslant 0})},$$

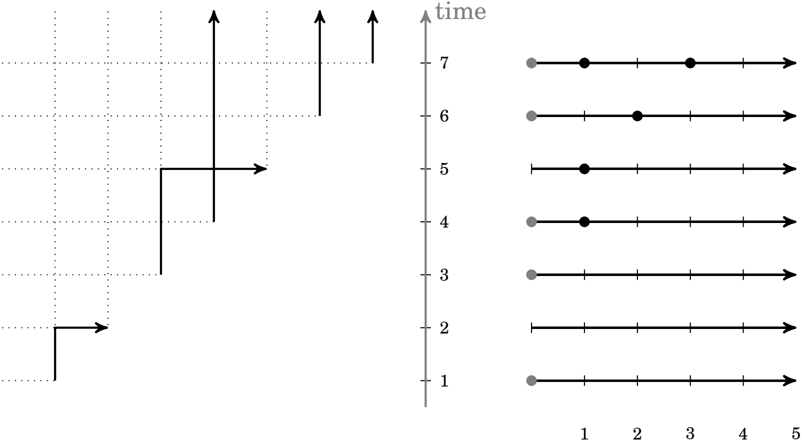
where $\mathsf{K}^{\complement}=\mathsf{J}-\mathsf{K}^{Schur}$ is the correlation kernel of the complement of the Pfaffian Schur point process $\Lambda:=\{\lambda_i-i\}_i$,

$$\mathsf{f}_x(j)=\frac{(-t^{x+j+1};t^2)_\infty}{(-t^{x+j};t^2)_\infty}-1.$$

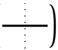
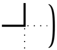
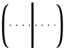
where

$$(a;t^2)_\infty=(1-a)(1-at^2)(1-at^4)\dots$$

Limit to ASEP on a half space

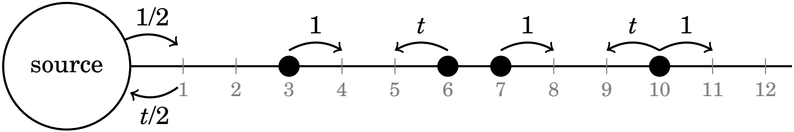


If the parameters are scaled such that $a_x \equiv 1 - \frac{(1-t)\epsilon}{2}$,

$\mathbb{P}\Big($  $\Big) \approx t\epsilon$, $\mathbb{P}\Big($  $\Big) \approx 1 - t\epsilon$, $\mathbb{P}\Big($  $\Big) \approx \epsilon$, $\mathbb{P}\Big($  $\Big) \approx 1 - \epsilon$.

and paths will almost always zig-zag and do soomething else at rates 1 and t.

Laplace transform of ASEP current



Recall that $N_0(\tau)$ denotes the total number of particles in the system at time τ.

Theorem (B.-Borodin-Corwin-Wheeler 2017)

For any time $\tau > 0$ *and* $x \in \mathbb{R}$,

$$\mathbb{E}\left[\frac{1}{(-t^{x+N_0(\tau)}, t^2)_\infty}\right] = \mathrm{Pf}\left[\mathsf{J} + \mathsf{f}_x \cdot \mathsf{K}^{\mathrm{ASEP}}\right]_{\ell^2(\mathbb{Z}_{\geqslant 0})}$$

where $\mathsf{K}^{\mathrm{ASEP}}$ *is a limit of* $\mathsf{K}^{\complement}$ *from the previous slides, which can be expressed exactly as contour integrals.*

The L.H.S of the equation should be thought of as a deformed Laplace transform.

Asymptotic analysis

$$\mathbb{E}\left[\frac{1}{(-t^{x+N_0(\tau)},t^2)_\infty}\right] = \mathrm{Pf}\left[\mathsf{J} + \mathsf{f}_x \cdot \mathsf{K}^{\mathrm{ASEP}}\right]_{\ell^2(\mathbb{Z}_{\geqslant 0})}$$

Theorem (B.-Borodin-Corwin-Wheeler 2017)

For any $t \in [0,1)$, starting from the empty initial condition,

$$\lim_{T\to\infty} \mathbb{P}\left(\frac{N_0\left(\frac{T}{1-t}\right) - \frac{T}{4}}{2^{-4/3}T^{1/3}} > -x\right) = \mathrm{Pf}\left[\mathsf{J} - \mathsf{K}^{\mathrm{GOE}}\right]_{\mathbb{L}^2(x,\infty)} = F^{\mathrm{GOE}}(x).$$

- For fixed t, in the scaling limit above, $\mathsf{K}^{\mathrm{ASEP}}$ goes to the correlation kernel of the GOE, and the function f goes to $-\mathbb{1}_{\cdot > x}$.
- In the scaling limit leading to the KPZ equation, $\mathsf{K}^{\mathrm{ASEP}}$ still go to the correlation kernel of the GOE, and f converges to another function. Hence, the Laplace transform of the solution to the KPZ equation equals a multiplicative functional of the GOE.

Further directions

- **More general boundary conditions**. This will probably require going higher in the hierarchy of integrable structures.
- Other interesting models are coming from Pfaffian Macdonald processes: **Log gamma directed polymer in a half space**. (in preparation)
- Ultimately one hopes to prove that the Laplace transform of **KPZ equation in a half space at any space point and for general boundary condition** is a multiplicative functional of a certain point process corresponding to the two-dimensional crossover kernel obtained in LPP.

Thank you