Multi-time distribution of periodic TASEP

Jinho Baik
University of Michigan

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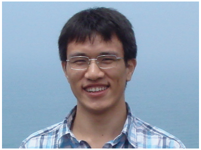
Joint work with Zhipeng Liu (Courant Institute)



1. (Baik, Liu) Fluctuations of TASEP on a ring in relaxation time scale (arXiv:1605.07102)
2. (Liu) Height fluctuations of stationary TASEP on a ring in relaxation time scale (arXiv:1610.04601)
3. (Baik, Liu) Multi-time, multi-location distribution of periodic TASEP (in preparation)

Introduction

- height fluctuations, spatial correlations, time correlations 1:2:3
- Height function $H(s,t)$

$$h_\epsilon(\gamma,\tau) := \frac{H(\epsilon^{-2/3}\gamma, \epsilon^{-1}\tau) - \langle H(t^{2/3}\gamma, t\tau)\rangle}{\epsilon^{-1/3}}$$

- What is the limiting two–dimensional process?

$$(\gamma,\tau) \mapsto h(\gamma,\tau) = \lim_{\epsilon\to 0} h_\epsilon(\gamma,\tau)$$

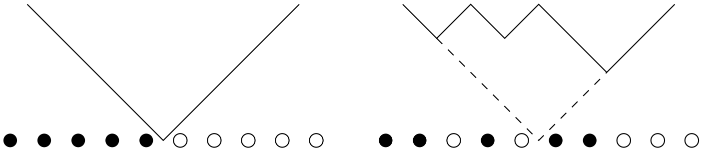
- One-point distribution: Tracy–Widom distributions

- Fix τ and consider $\gamma \mapsto h(\gamma, \tau)$
- Depends on the initial condition
- Airy$_2$ process for step initial condition
- Airy$_1$ process for flat initial condition
- Does not depend on τ (after a simple scale)
- Proved for TASEP and some zero temperature directed polymers (but not for ASEP, positive temperature directed polymers and KPZ equation yet)
- Prähofer, Spohn, Johansson, Sasamoto, Borodin, Ferrari, Matetski, Quastel, Remenik, ...

- Slow decorrelation [Ferrari 2008]
- Two-time distribution (not rigorous) [Dotsenko 2013]
- Two-time distribution (Brownian directed last passage percolation) [Johansson 2016]
- Short time ($\tau_2/\tau_1 \to 1$) and long time ($\tau_2/\tau_1 \to 0$) asymptotics of time covariance $\mathrm{Cov}(h(0,\tau_1), h(0,\tau_2))$ [Ferrari, Spohn 2016]
- Tail of two-time distribution: $p_{\tau_2/\tau_1}(x_1, x_2)$ for large positive x_1 and arbitrary x_2 as $\tau_2/\tau_1 \to 1$ and $\to 0$ [de Nardis, Le Doussal 2016]

This talk: Multi-time distribution for periodic TASEP

Associate ● ○ with ╲╱ and associate ○ ● with ╱╲



Periodic TASEP (TASEP on a ring)

- L period
- N number of particles per period
- $\rho = \frac{N}{L}$ particle density (ρ fixed, L, N large)

- t not too large: infinite TASEP (KPZ dynamics)
- t too large: finite TASEP (equilibrium dynamics)
- crossover: relaxation time scale $t = O(L^{3/2})$

- Gwa and Spohn 1992
- Derrida and Lebowitz 1998
- Priezzhev, Povlotsky, Golinelli, Mallick
- Prolhac 2013–2016

Results (Periodic step initial condition)

Periodic step initial condition ●●●○○○○●●●○○○○●●●○○○○

1. Multi-time, multi-position joint distribution in the limit $t = O(L^{3/2})$

2. A discussion on the one-point distribution

** One-point distribution for three (step, flat, stationary) initial conditions: Prolhac & Baik–Liu, independently, 2016

- $t, L, N \to \infty$ with $t = O(L^{3/2})$ and $\rho = N/L$ fixed
- There are shocks. In this talk, assume $\rho = 1/2$
- Joint height distribution $\mathbb{P}\left(\cap_{j=1}^{m}\{H(s_j, t_j) \le h_j\}\right)$
- Position $s_j = \gamma_j L$ with $\gamma_i \in [0, 1]$
- Time $t_j = 2\tau_j L^{3/2}$ satisfying $0 < \tau_1 < \cdots < \tau_m$
- Height $h_j = \frac{1}{2} t_j - x_j L^{1/2}$ with $x_j \in \mathbb{R}$

$$\mathbb{P}\left(\cap_{j=1}^{m}\{H(s_j, t_j) \le h_j\}\right) \to \mathbf{F}(x_1, \cdots, x_m; (\gamma_1, \tau_1), \cdots, (\gamma_m, \tau_m))$$

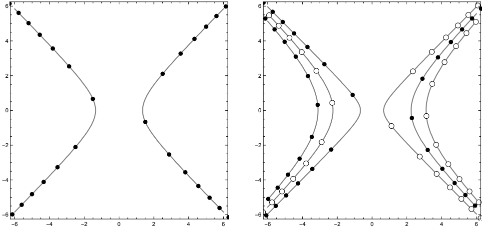
Limiting joint distribution

- $\mathbf{F}(x_1, \cdots, x_m) = \frac{1}{(2\pi \mathrm{i})^m} \oint \cdots \oint \mathbf{C}(\mathbf{z})\mathbf{D}(\mathbf{z}) \prod_{i=1}^m \frac{\mathrm{d}z_i}{z_i}$
- Nested circles $|z_m| < \cdots < |z_1| < 1$
- $\mathbf{C}(\mathbf{z})$ has simple poles at $z_i = z_{i+1}$
- $\mathbf{D}(\mathbf{z})$ has an isolated singularity at $z_i = 0$, and $\mathbf{D}(\mathbf{z}) = \det(\mathbf{1} - \mathbf{K})$

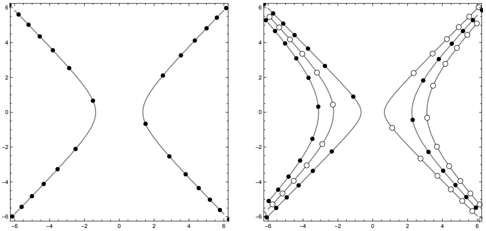
$$\mathbf{C}(\mathbf{z}) = \left[\prod_{i=1}^{m-1} \frac{z_i}{z_{i+1} - z_i}\right] \left[\prod_{i=1}^{m} \frac{\mathbf{A}_i(z_i)}{\mathbf{A}_{i-1}(z_i)}\right] \mathbf{Q}(\mathbf{z})$$

where $\mathbf{A}_i(z) = e^{-\sqrt{\frac{2}{\pi}}\left(x_i \operatorname{Li}_{3/2}(z) + \tau_i \operatorname{Li}_{5/2}(z)\right)}$. $\mathbf{Q}(\mathbf{z})$ is analytic, $\mathbf{Q}(0) \neq 0$, and it does not depend on x_i, τ_i, γ_i.

- $\mathbf{D}(\mathbf{z}) = \det(\mathbf{1} - \mathbf{K})$ where $\mathbf{K} = \mathbf{K_1}\mathbf{K_2}$
- Give $|z| < 1$, consider the zeros of the equation $e^{-w^2/2} = z$
- Denote the set of zeros by $L_z \cup R_z$.



- $(m = 3)$ $\mathbf{K_1} : \ell^2(R_{z_1}) \oplus \ell^2(L_{z_2}) \oplus \ell^2(R_{z_3}) \to \ell^2(L_{z_1}) \oplus \ell^2(R_{z_2}) \oplus \ell^2(L_{z_3})$



- Using $\xi_i \in L_{z_i}$ and $\eta_i \in R_{z_i}$, the matrix kernel is of form (for $m = 5$)

$$\mathbf{K_1} = \begin{bmatrix} \mathbf{K_1}(\xi_1, \eta_1) & \mathbf{K_1}(\xi_1, \xi_2) & & & \\ \mathbf{K_1}(\eta_2, \eta_1) & \mathbf{K_1}(\eta_2, \xi_2) & & & \\ & & \mathbf{K_1}(\xi_3, \eta_3) & \mathbf{K_1}(\xi_3, \xi_4) & \\ & & \mathbf{K_1}(\eta_4, \eta_3) & \mathbf{K_1}(\eta_4, \xi_4) & \\ & & & & \mathbf{K_1}(\xi_5, \eta_5) \end{bmatrix}$$

$$\mathbf{K}_2 = \begin{bmatrix} \mathbf{K}_2(\eta_1,\xi_1) & & & & \\ & \mathbf{K}_2(\eta_2,\xi_2) & \mathbf{K}_2(\xi_2,\xi_3) & & \\ & \mathbf{K}_2(\eta_3,\eta_2) & \mathbf{K}_2(\eta_3,\xi_3) & & \\ & & & \mathbf{K}_2(\xi_4,\eta_4) & \mathbf{K}_2(\xi_4,\xi_5) \\ & & & \mathbf{K}_2(\eta_5,\eta_4) & \mathbf{K}_2(\eta_5,\xi_5) \end{bmatrix}$$

Set $\mathbf{F}_i(w) = \exp\left(-\frac{1}{3}\tau_i w^3 + \frac{1}{2}\gamma_i w^2 + x_i w\right)$

The 2×2 blocks are ($(\mathrm{Re}(\xi) < 0$ and $\mathrm{Re}(\eta) > 0)$

$$\begin{bmatrix} \mathbf{K_1}(\xi, \eta) & \mathbf{K_1}(\xi, \xi') \\ \mathbf{K_1}(\eta', \eta) & \mathbf{K_1}(\eta', \xi') \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\mathbf{F}_i(\xi)}{\mathbf{F}_{i-1}(\xi)} & 0 \\ 0 & \frac{\mathbf{F}_i(\eta')}{\mathbf{F}_{i+1}(\eta')} \end{bmatrix} \begin{bmatrix} f(\xi) & 0 \\ 0 & g(\eta') \end{bmatrix} \begin{bmatrix} \frac{1}{\xi - \eta} & \frac{1}{\xi - \xi'} \\ \frac{1}{\eta' - \eta} & \frac{1}{\eta' - \xi'} \end{bmatrix} \begin{bmatrix} h(\xi) & 0 \\ 0 & j(\eta') \end{bmatrix}$$

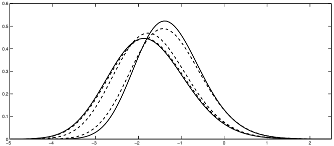
where f, g, h, j depend also on z, z' but do not depend on x_i, τ_i, γ_i

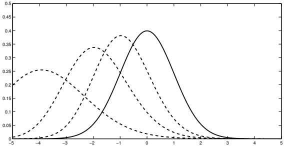
$$h(\eta) = e^{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\eta} \mathrm{Li}_{1/2}(z' e^{(w^2 - y^2)/2}) \mathrm{d}y} \left(\frac{z'}{z} - 1\right)$$

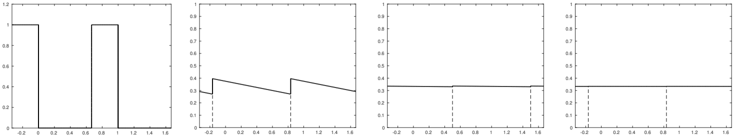
One-point distribution $\mathbf{F}(x;(\gamma,\tau))$

Formal computation shows:

- $\tau \to 0$: $\mathbf{F}(\tau^{1/3}x + \frac{\gamma^2}{4\tau^{2/3}};(\gamma,\tau)) \to \begin{cases} \mathbf{F}_{GUE}(x) & \gamma \neq 1/2 \\ \mathbf{F}_{GUE}(x)^2 & \gamma = 1/2 \end{cases}$
- $\tau \to \infty$: $\mathbf{F}(\frac{\sqrt{2}\tau^{1/6}}{\pi^{1/4}}(x+\tau);(\gamma,\tau)) \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\mathrm{d}y$







Discontinuity $O(Lt^{-1})$

Infinite TASEP with $O(1)$ discontinuity: Ferrari, Nejjar 2015

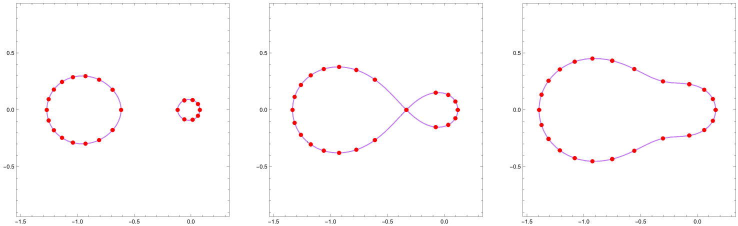
Very brief discussion on the proof (finite time formula)

- The limit is obtained from an exact finite time formula, which has a parallel structure
- TASEP in the configuration space $\mathcal{X}_{L,N} = \{x_N < \cdots < x_1 < x_N + L\}$
- Coordinate Bethe ansatz method
- Schütz (1997): Computed transition probability for TASEP
- Rákos and Schütz (2005): Using Schütz's formula, reproduced Johansson's result (the Fredholm determinant formula for the 1-point distribution for step initial condition)
- Borodin, Ferrari, Prähofer and Sasamoto (2007–2008): Using Schütz's formula, obtained Fredholm determinant formula for equal-time processes (and space-like points)
- Tracy and Widom (ASEP) (2008–2009): ASEP, 1-point distribution

For X and Y in $\{x_N < \cdots < x_1 < x_N + L\}$,

$$\mathbb{P}_Y(X;t) = \oint \det\left[\frac{1}{L}\sum_w \frac{w^{i-j+1}(w+1)^{-x_i+y_j-i+j}e^{tw}}{w+\rho}\right]_{N\times N} \frac{\mathrm{d}z}{2\pi \mathrm{i} z}$$

Sum over the roots of $w^N(w+1)^{L-N} = z^L$



Obtained by solving the Kolmogorov forward equation using coordinate Bethe ansatz

$$\mathbb{P}_Y\left(\cap_{i=1}^m \{x_{k_i}(t_i) \geq a_i\}\right)$$
$$= \sum \cdots \sum \mathbb{P}_Y(X^{(1)}; t_1)\, \mathbb{P}_{X^{(1)}}(X^{(2)}; t_2 - t_1) \cdots \mathbb{P}_{X^{(m-1)}}(X^{(m)}; t_m - t_{m-1})$$

The sums are over all $x_N^{(i)} < \cdots < x_1^{(i)} < x_N^{(i)} + L$ satisfying $x_{k_i}^{(i)} \geq a_i$. It becomes

$$\frac{1}{(2\pi \mathrm{i})^m} \oint \cdots \oint \mathcal{C}(\mathbf{z}, \mathbf{k}) \mathcal{D}_Y(\mathbf{z}, \mathbf{k}, \mathbf{t}, \mathbf{a}) \prod_{i=1}^m \frac{\mathrm{d}z_i}{z_i}$$

where

$$\mathcal{D}_Y(\mathbf{z}) = \det \left[\sum_{w_1, \cdots, w_m} \frac{w_1^{-i} (w_1 + 1)^{y_i + i - 1} w_m^{-j}}{\prod_{\ell=2}^m (w_\ell - w_{\ell-1})} \prod_{\ell=1}^m g_\ell(w_\ell) \right]_{N \times N}$$

The sum is over the roots $w_i^N (w_i + 1)^{L-N} = z_i^L$. The function

$$g_\ell(w) = \frac{w(w+1)}{L(w+\rho)} \frac{w^{k_\ell} (w+1)^{-a_\ell - k_\ell - 1} e^{t_\ell w}}{w^{k_{\ell-1}} (w+1)^{-a_{\ell-1} - k_{\ell-1} - 1} e^{t_{\ell-1} w}}$$

Set $y_i = -i + 1$. Then

$$\mathcal{D}_Y(\mathbf{z}) = \det \left[\sum_{w_1, \cdots, w_m} \frac{w_1^{-i} w_m^{-j}}{\prod_{\ell=2}^{m} (w_\ell - w_{\ell-1})} \prod_{\ell=1}^{m} g_\ell(w_\ell) \right]_{N \times N}$$

This simplifies to a Fredholm determinant. Here we need to take $|z_i| < r_0$ for all i.

Slightly longer discussion

Inserting the Schütz-like formual from Step 1

$$P_Y(X;t) = \oint \det\left[\frac{1}{L}\sum_w \frac{w^{i-j+1}(w+1)^{-x_i+y_j-i+j}e^{tw}}{w+\rho}\right]_{N\times N} \frac{\mathrm{d}z}{2\pi \mathrm{i}z}$$

into

$$\begin{aligned}&\mathbb{P}_Y\left(\cap_{i=1}^m \{x_{k_i}(t_i)\geq a_i\}\right)\\ &= \sum\cdots\sum \mathbb{P}_Y(X^{(1)};t_1)\,\mathbb{P}_{X^{(1)}}(X^{(2)};t_2-t_1)\cdots\mathbb{P}_{X^{(m-1)}}(X^{(m)};t_m-t_{m-1})\end{aligned}$$

(sums over all $x_N^{(i)} < \cdots < x_1^{(i)} < x_N^{(i)} + L$ satisfying $x_{k_i}^{(i)} \geq a_i$), we need to evaluate

$$\sum_{\{x_N<\cdots<x_1<x_N+L\}\cap\{x_k\geq a\}} \det\left[w_i^j(w_i+1)^{-x_j-j}\right]\det\left[(w_i')^{-j}(w_i'+1)^{x_j+j}\right]$$

where $w_i^N(w_i+1)^{L-N} = z^L$, and $(w_i')^N(w_i'+1)^{L-N} = (z')^L$

Key lemma: It is equal to

$$\left(\frac{z'}{z}\right)^{(k-1)L}\left(1-\left(\frac{z}{z'}\right)^{L}\right)^{N-1}\left[\prod_{j=1}^{N}\left(\frac{w_j'}{w_j}\right)^{N-k+1}\frac{(w_j'+1)^{a-1-N+k}}{(w_j+1)^{a-2-N+k}}\right]\det\left[\frac{1}{w_{i'}'-w_i}\right]$$

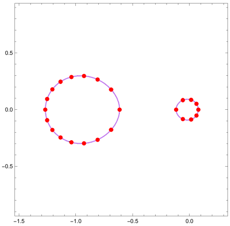
when $w_i^N(w_i+1)^{L-N}=z^L$, and $(w_i')^N(w_i'+1)^{L-N}=(z')^L$

Step 3

From Step 2, and using step initial condition,

$$\mathcal{D}_Y(\mathbf{z}) = \det\left[\sum_{w_1,\cdots,w_m} \frac{w_1^{-i} w_m^{-j}}{\prod_{\ell=2}^{m}(w_\ell - w_{\ell-1})} \prod_{\ell=1}^{m} g_\ell(w_\ell)\right]_{N\times N}$$

where the sum is over all roots $w_i^N(w_i+1)^{L-N} = z_i^L$.



- Take $|z_i| < r_0$
- Expand the det of the sum as sums of dets
- Sums are over N-tuples of roots $w_i^{(j)}$, $j = 1, \cdots, N$.
- For $w_i^{(j)}$ on the right circle, use hole-particle duality.
- The result is the series expansion of a Fredholm determinant.

The end