Current Fluctuations of the Stationary ASEP and Six-Vertex Model

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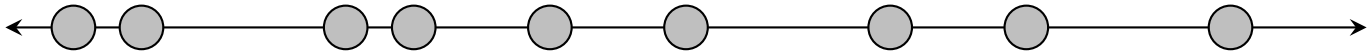
Outline

1. ASEP
   - Definition of model
   - Statement of the result
2. Stochastic six-vertex model
   - Definition of model
   - Statement of the result
3. Ferroelectric Symmetric Six-Vertex Model
   - Definition of model
   - Informal statement of the result
   - Context and previous predictions
   - More formal statement of the result
4. On the proofs

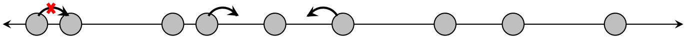
The Asymmetric Simple Exclusion Process (ASEP)

Place particles on $\mathbb{Z}$ such that at most one particle occupies any site.



Particles

- jump to the left with exponential rate L,
- jump to the right with exponential rate R,
- so that jumps to occupied locations are suppressed.

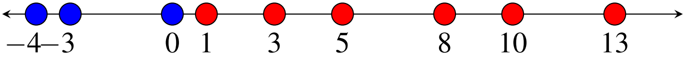


Assume $R > L \geq 0$ to force a drift to the right.

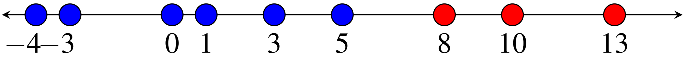
Current of the ASEP

We would like to analyze the *current* $J_t(x)$ of the ASEP. To define it,

- color all particles to weakly the left of 0 blue;
- color all particles to strictly the right of 0 blue.



- Run the ASEP for time t.



- Define $J_t(x) =$ (Number of Blue Particles strictly to the right of x) – (Number of red particles weakly to the left of x).
- Above, $J_t(4) = 1$.

Asymptotics of the Current

- We are interested in the asymptotic fluctuations of the current, meaning the behavior of $J_t(x) - \mathbb{E}\big[J_t(x)\big]$ for large t.
- Provable results are **only available for certain classes of initial data**, including **step** and **step-Bernoulli** (Tracy-Widom).
- Of widespread interest is *stationary* (equilibrium): particles occupy sites independently with fixed probability $\rho \in (0, 1)$; this is invariant under the ASEP dynamics.
- Since the ASEP is a discretization of the Kardar-Parisi-Zhang (KPZ) equation, one expects **the fluctuations of $J_T(xT)$ to be of order $T^{1/3}$ and to converge to the long-time height fluctuations of the stationary KPZ equation**, for some value of $x = (R - L)(1 - 2\rho)$ (called the *characteristic velocity*).
  - The latter fluctuations have been studied and are known to converge the *Baik-Rains distribution* (Borodin-Corwin-Ferrari-Vető).
  - We expect the same for the ASEP current $J_T(xT)$.

New Results

Our first result is a **precise fluctuation theorem** for the current of the stationary ASEP along the characteristic line, that confirms this prediction **on the level of exact statistics**.

Theorem (A., 2016)

Assume that $R > L$ and set $\delta = R - L$. There exist explicit $c, \chi \in \mathbb{R}$ so that

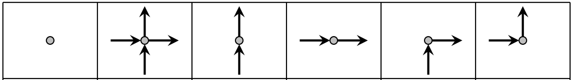
$$\lim_{T \to \infty} \mathbb{P}\left[\frac{J_T\big(\delta(1-2\rho)T\big) - cT}{\chi T^{1/3}} \geq s\right] = \Phi(s),$$

for any $s \in \mathbb{R}$, where $\Phi(s)$ is the Baik-Rains distribution.

- Previous studies of the stationary ASEP by Ferrari-Spohn (2006) and Balász-Seppäläinen (2010), as well as many others.
- Proof of the above theorem uses an analysis of the *six-vertex model*.

Asymmetric Six-Vertex Model

- Consider a two-dimensional lattice (torus, square, $\mathbb{Z}^2$, quadrant).
- Give each vertex one of the six following edge configurations, weighted as below.

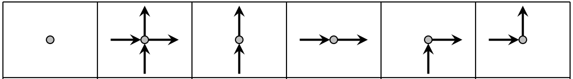


a_1	a_2	b_1	b_2	c_1	c_2

- When $a_1 = a = a_2$; $b_1 = b = b_2$; and $c_1 = c = c_2$, this becomes the *symmetric six-vertex model*.
  - Introduced independently by Pauling (1935) and Slater (1941).
  - Studied by Lieb, Baxter, and many others (1967 to present).
- When $a_1 = 1 = a_2$; $b_1 + c_1 = 1$; and $b_2 + c_2 = 1$, this becomes the *stochastic six-vertex model*.
  - Introduced by Gwa-Spohn (1992).
  - Studied by Borodin-Corwin-Gorin and Reshetikhin-Sridhar (2016).

Stochastic Six-Vertex Model as a Markov Process

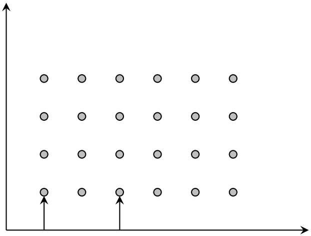
Consider the stochastic six-vertex model on the positive quadrant with weights as below.



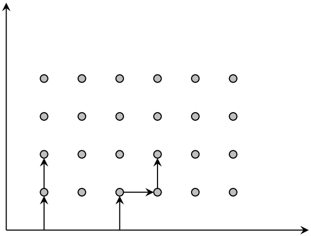
1	1	δ_1	δ_2	$1-\delta_1$	$1-\delta_2$

- The stochastic six-vertex model can be sampled row by row.
- Under this sampling, the y-axis tracks time evolution.

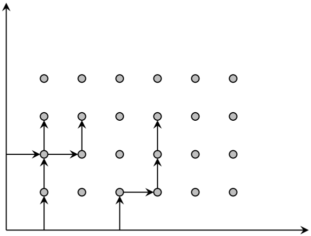
Stochastic Six-Vertex Model as a Markov Process



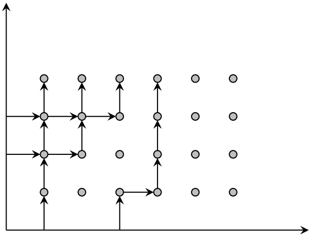
Stochastic Six-Vertex Model as a Markov Process



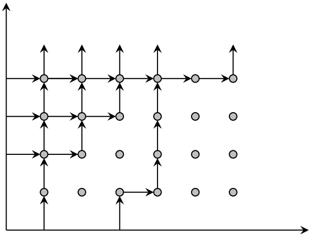
Stochastic Six-Vertex Model as a Markov Process



Stochastic Six-Vertex Model as a Markov Process



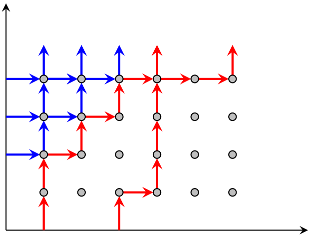
Stochastic Six-Vertex Model as a Markov Process



Height Function of the Six-Vertex Model

We are interested in asymptotics of the height function $H(X, Y)$ of the six-vertex model. To define it,

- Color all paths emanating from the x-axis red.
- Color all paths emanating from the y-axis blue.



Let $H(X, Y) =$ (Number of blue paths strictly to the right of (X, Y)) $-$ (Number of red paths weakly to the left of (X, Y)).

Height Function of the Six-Vertex Model

We are interested in asymptotics of the height function $H(X, Y)$ of the six-vertex model. To define it,

- Color all paths emanating from the x-axis blue.
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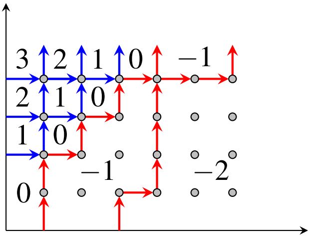


Let $H(X, Y) =$ (Number of blue paths strictly to the right of (X, Y)) − (Number of red paths weakly to the left of (X, Y)).

Relationship With the ASEP

- Consider the stochastic six-vertex model on the positive quadrant.
- There is a limit degeneration from the stochastic six-vertex model to the ASEP.
  - Let $b_1 = \varepsilon L$, $b_2 = \varepsilon R$; scale time by ε^{-1}; and observe on the diagonal.
- This can be used to degenerate exact identities for the stochastic six-vertex model to the ASEP.
- We will consider the stochastic six-vertex model with *double-sided* (ρ_1, ρ_2)*-Bernoulli initial data.*
  - This means that arrows enter through the y-axis with probability ρ_1 and x-axis with probability ρ_2.

New Results for the Stochastic Six-Vertex Model

Our second result is a **precise fluctuation theorem** for the height function of the stochastic six-vertex model with certain boundary data.

Theorem (A., 2016)

Assume that $\delta_2 > \delta_1$. Consider the stochastic six-vertex model with (ρ_1, ρ_2)-Bernoulli initial data, where

$$\frac{\rho_1}{1-\rho_1} = \left(\frac{1-\delta_1}{1-\delta_2}\right) \frac{\rho_2}{1-\rho_2}.$$

Then, there exist (explicit) $c, v, \mathcal{F} \in \mathbb{R}$ such that

$$\lim_{X \to \infty} \mathbb{P}\left[\frac{H(X, vX) - cX}{\mathcal{F} X^{1/3}} \geq s\right] = \Phi(s),$$

for any $s \in \mathbb{R}$.

A similar statement holds for the ferroelectric symmetric six-vertex model.

Predictions for the Translation-Invariant Stochastic Six-Vertex Model

- The stochastic six-vertex model with double-sided (ρ_1, ρ_2)-Bernoulli initial data was considered by Gwa and Spohn (1992) in the case

$$\frac{\rho_1}{1-\rho_1} = \left(\frac{1-\delta_1}{1-\delta_2}\right) \frac{\rho_2}{1-\rho_2}.$$

- In this case, they observed that the stochastic six-vertex model should be **translation-invariant**.
- Predicted that the **height fluctuations of the stochastic six-vertex model should be of order $T^{1/3}$** along an explicit characteristic line.
- From KPZ universality, they predicted that the rescaled fluctuations converge to the long-time statistics of the stationary KPZ equation, **which is the Baik-Rains distribution**.

New Results for the Stochastic Six-Vertex Model

Theorem (A., 2016)

Assume that $\delta_2 > \delta_1$. *Consider the stochastic six-vertex model with* (ρ_1, ρ_2)*-Bernoulli initial data, where*

$$\frac{\rho_1}{1 - \rho_1} = \left(\frac{1 - \delta_1}{1 - \delta_2} \right) \frac{\rho_2}{1 - \rho_2}.$$

Then, there exist (explicit) $c, v, \mathcal{F} \in \mathbb{R}$ *such that*

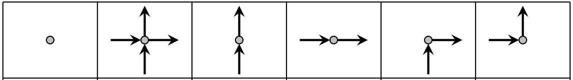
$$\lim_{X \to \infty} \mathbb{P} \left[\frac{H(X, vX) - cX}{\mathcal{F} X^{1/3}} \geq s \right] = \Phi(s),$$

for any $s \in \mathbb{R}$.

- **Confirms the Gwa-Spohn prediction**, on the level of exact statistics.
- Similar statement holds for the ferroelectric symmetric six-vertex model.

Phases of the Six-Vertex Model

The symmetric six-vertex model has the following weights.



a	a	b	b	c	c

- Properties of the six-vertex model are dependent on the parameter

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}.$$

  - Ferroelectric phase $\Delta > 1$.
  - Disordered phase $\Delta \in (-1, 1)$.
  - Anti-ferroelectric phase $\Delta < -1$.
- We will be interested in the ferroelectric phase.

New Results for the Ferroelectric Six-Vertex Model (Informally)

Theorem (A., 2016, Informal Version)

Let $a, b, c > 0$ satisfy $\Delta = (a^2 + b^2 - c^2)/2ab > 1$. There exists a one-parameter family of translation-invariant Gibbs measures for the symmetric, ferrolectric six-vertex model with weights (a, b, c) such that the following holds. For any $s \in \mathbb{R}$, we have that

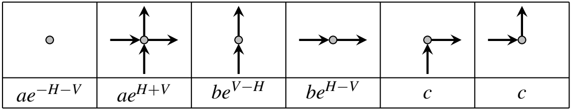
$$\lim_{X \to \infty} \mathbb{P}\left[\frac{H(X, vX) - cX}{\mathcal{F} X^{1/3}} \geq s \right] = \Phi(s),$$

for some (explicit) $c, v, \mathcal{F} \in \mathbb{R}$.

Before explaining the above theorem more carefully, we begin with some context.

Free Energy Profile

- Analysis of the *free energy* of the symmetric six-vertex model dates back to Lieb and Sutherland-Yang-Yang (1967).
- In 1995, Bukman and Shore analyzed the free energy profile $F(H, V)$ of the ferroeletric, symmetric six-vertex model in the presence of a magnetic field (H, V).



Free Energy Profile

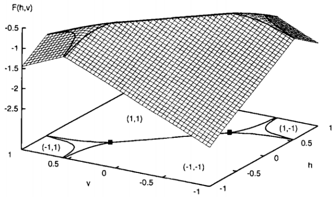


Figure: Figure 3 of D. J. Bukman and J. D. Shore, The Conical Point in the Ferroelectric Six-Vertex Model, J. Stat. Phys. **78**, 1277–1309, 1995.

Conical Singularity

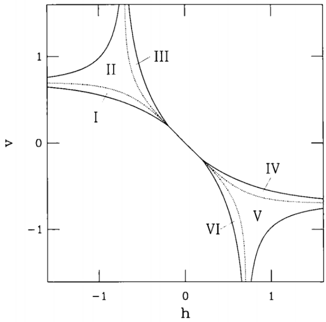
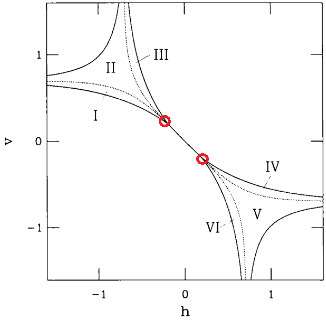


Figure: Figure 2 of D. J. Bukman and J. D. Shore, The Conical Point in the Ferroelectric Six-Vertex Model, J. Stat. Phys. **78**, 1277–1309, 1995.

Conical Singularity



- They found that the free energy exhibits a singularity at the corners above, called *conical singularities* (or *tricritical points*).
- These were missed in the original Sutherland-Yang-Yang analysis almost 30 years earlier.

The Conical Singularity in Terms of Slopes

- Associated with any translation-invariant six-vertex model is a slope (x, y), where

$$x = \mathbb{E}\big[H(X+1, Y) - H(X, Y)\big];$$
$$y = \mathbb{E}\big[H(X, Y+1) - H(X, Y)\big].$$

- What is the slope corresponding to the conical singularity?
- The slope (x, y) can usually be recovered from the magnetic field (H, V) by taking the Legendre dual.
  - This expresses x and y in terms of derivatives of $F(H, V)$.
- However, F is not differentiable at the conical singularity, so this does not apply.

The Conical Singularity in Terms of Slopes

Instead, the conical singularity corresponds to a one-parameter family of slopes $(x, y) = (\rho_1, \rho_2)$, where ρ_1 and ρ_2 satisfy

$$\frac{\rho_1}{1-\rho_1} = \kappa \frac{\rho_2}{1-\rho_2},$$

for some κ that is explicit in terms of a, b, and c.

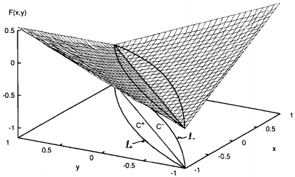


Figure: Figure 5 of D. J. Bukman and J. D. Shore, The Conical Point in the Ferroelectric Six-Vertex Model, J. Stat. Phys. **78**, 1277–1309, 1995.

Bukman-Shore Predictions

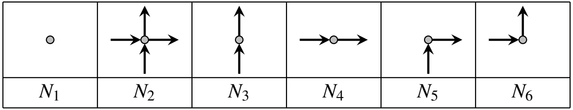
- Bukman and Shore analyzed the free energy $F(x, y)$ of the six-vertex model (on the $N \times N$ torus) with slope (x, y).
- They observed that a second derivative of $F(x, y)$ is singular when $(x, y) = (\rho_1, \rho_2)$ lies on the conical singularity, a **second-order phase transition.**
- After more careful analysis, they found that the second derivative of $F(x, y)$ is of order $\gamma^{-1/3}$, where γ is the distance from (x, y) to the conical singularity.
- Unfortunately, turning their physical heuristics into a mathematical proof **at the moment seems inaccessible**
- Still, it suggests that the two-point function of the six-vertex model should decay as $N^{-2/3}$ at the tricritical point, leading to their prediction of **KPZ fluctuations at the conical singularity**.
- We will assess this prediction from the infinite-volume viewpoint.

Gibbs Measures for the Six-Vertex Model

- Let Ω denote the set of all six-vertex configurations on $\mathbb{Z}^2$.
- For any $\omega \in \Omega$, let $\omega|_\Lambda$ denote the restriction of ω to Λ.

Definition

A probability measure μ on Ω is said to have the *Gibbs property* if the following holds. For any finite subset $\Lambda \subset \mathbb{Z}^2$, the probability $\mu_\Lambda(\omega)$ of selecting $\omega \in \Omega$, conditioned on $\omega|_{\mathbb{Z}^2 \setminus \Lambda}$, is proportional to $a_1^{N_1} a_2^{N_2} b_1^{N_3} b_2^{N_4} c_1^{N_5} c_2^{N_6}$.



Gibbs Measures and Translation Invariance

Lemma (A., 2016)

Let $a, b, c > 0$ *such that* $\Delta = (a^2 + b^2 - c^2)/2ab > 1$. *Define*

$$\delta_1 = \frac{b}{a}\left(\Delta - \sqrt{\Delta^2 - 1}\right); \quad \delta_2 = \frac{b}{a}\left(\Delta + \sqrt{\Delta^2 - 1}\right);$$

and let $\rho_1, \rho_2 \in (0, 1)$ *be any positive real numbers satisfying*

$$\frac{\rho_1}{1 - \rho_1} = \left(\frac{1 - \delta_1}{1 - \delta_2}\right) \frac{\rho_2}{1 - \rho_2}.$$

Then, the stochastic six-vertex model with double-sided (ρ_1, ρ_2)*-Bernoulli initial data is a translation-invariant Gibbs measure for the ferroelectric, symmetric six-vertex model with weights* (a, a, b, b, c, c).

New Results for the Ferroelectric Six-Vertex Model

Theorem (A., 2016)

Let $a, b, c > 0$ satisfy $\Delta = (a^2 + b^2 - c^2)/2ab > 1$. Let $\rho_1, \rho_2 \in (0, 1)$ satisfy

$$\frac{\rho_1}{1-\rho_1} = \kappa \frac{\rho_2}{1-\rho_2}.$$

Consider the infinite-volume ferroelectric six-vertex model with weights (a, b, c) under the previously defined Gibbs measure with slope (ρ_1, ρ_2). Then, there exist (explicit) $c, v, \mathcal{F} \in \mathbb{R}$ such that

$$\lim_{X \to \infty} \mathbb{P}\left[\frac{H(X, vX) - cX}{\mathcal{F} X^{1/3}} \geq s\right] = \Phi(s),$$

for any $s \in \mathbb{R}$.

New Results for the Ferroelectric Six-Vertex Model

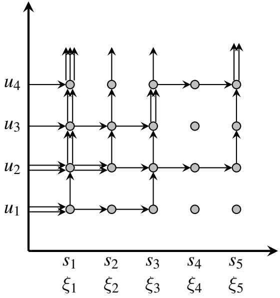
- **Establishes KPZ growth** of the ferroelectric six-vertex model at any (ρ_1, ρ_2) at the conical singularity.
  - This **confirms the Bukman-Shore prediction**.
- Proves that the **height fluctuations converge to the Baik-Rains distribution** (**not predicted** by Bukman-Shore).
  - Exact statistics for height fluctuations of the translation-invariant six-vertex model are **very rare**.
  - To the best of our knowledge, they have only **been proven in dimer-type specializations** of the six-vertex model (Kenyon, Johansson, Kenyon-Okounkov-Sheffield, Okounkov-Reshetikhin).
  - These can be mapped to free-fermionic degenerations ($\Delta \in \{0, \infty\}$) of the six-vertex model, where a wealth of determinantal methods (Kasteleyn matrix, Schur processes) exist.
  - Such methods **do not apply for the generic ferroelectric six-vertex model**.

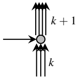
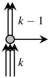
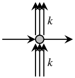
New Results for the Ferroelectric Six-Vertex Model

- Proves that the fluctuations of the height function of the six-vertex model are in the KPZ universality class **only along a single characteristic line** (also **not predicted** by Bukman-Shore).
- Gaussian fluctuations of $H(X, v'X)$ of order $X^{1/2}$, for all $v' \neq v$.
- Implies that spin-spin correlations decay as an inverse power of the distance (exponent $2/3$) along a single characteristic line and exponentially elsewhere (predicted by Reshetikhin-Sridhar).
- Power-law decay of correlations without rotational symmetry is an **unusual phenomenon from the viewpoint of classical spin systems**.

The Inhomogeneous Stochastic Higher Spin Vertex Model

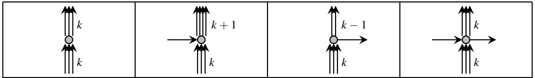
Paths enter through the x-axis and y-axis and then move up and right according to the probabilities below. This produces a random ensemble of paths called the *inhomogeneous stochastic higher spin vertex model*.



			
$\frac{1 - q^k s_x \xi_x u_y}{1 - s_x \xi_x u_y}$	$\frac{1 - q^k s_x^2}{1 - s_x \xi_x u_y}$	$\frac{(1 - q^k) s_x \xi_x u_y}{1 - s_x \xi_x u_y}$	$\frac{q^k s_x^2 - s_x \xi_x u_y}{1 - s_x \xi_x u_y}$

- The $\{u_y\}$ are *spectral parameters*.
- The $\{s_x\}$ are *spin parameters*.
- The $\{\xi_x\}$ are *inhomogeneity parameters*.

Degeneration to the Stochastic Six-Vertex Model

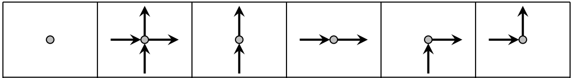


$\dfrac{1 - q^k s_x \xi_x u_y}{1 - s_x \xi_x u_y}$	$\dfrac{1 - q^k s_x^2}{1 - s_x \xi_x u_y}$	$\dfrac{(1 - q^k) s_x \xi_x u_y}{1 - s_x \xi_x u_y}$	$\dfrac{q^k s_x^2 - s_x \xi_x u_y}{1 - s_x \xi_x u_y}$

Fix $0 < \delta_1 < \delta_2 < 1$. Setting

$$q = \frac{\delta_1}{\delta_2} < 1; \quad \kappa = \frac{1 - \delta_1}{1 - \delta_2} > 1; \quad s_1 = s_2 = \cdots = s = q^{-1/2};$$
$$\xi_1 = \xi_2 = \cdots = \xi = 1; \quad u_1 = u_2 = \cdots = u = \kappa s,$$

the probabilities above become the stochastic six-vertex probabilities below.



1	1	δ_1	δ_2	$1 - \delta_1$	$1 - \delta_2$

Integral Identities

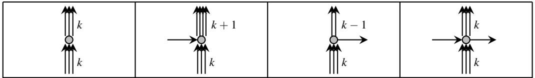
- What makes the inhomogeneous stochastic higher spin vertex model accessible is that its weights satisfy the **Yang-Baxter equation**.
- Using this fact, Borodin-Petrov established the contour integral identity for the height function of the inhomogeneous stochastic higher spin vertex model, run with **step initial data** (one path enters through each vertex of the y-axis, and no paths enter through the x-axis),

$$\mathbb{E}[q^{k\mathfrak{h}_t(x)}] = \frac{q^{\binom{k}{2}}}{(2\pi\mathbf{i})^k} \oint \cdots \oint \prod_{i=1}^{k} \left(\prod_{j=1}^{x-1} \frac{s_j\xi_j - s_j^2 w_i}{s_j\xi_j - w_i} \prod_{j=1}^{t} \frac{1 - qu_jw_i}{1 - u_jw_i} \right) \times \prod_{1 \le i < j \le k} \frac{w_i - w_j}{w_i - qw_j} \prod_{i=1}^{k} \frac{dw_i}{w_i},$$

for suitable contours for the w_j.

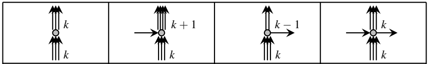
Half-Stationary Initial Data

- Their identities **only hold for step initial data**.
- Recall that the stochastic six-vertex model was obtained from a *homogeneous degeneration* of the higher spin model, that is, by setting all $\{u_y\}$, $\{s_x\}$, and $\{\xi_x\}$ equal.
- Instead, we can try altering the values of s_1 and ξ_1, while keeping the $\{u_y\}_{y\geq 1}$ and other $\{s_x\}_{x\geq 2}$ and $\{\xi_x\}_{x\geq 2}$ as in the homogeneous six-vertex case.
  - In particular, let us set $\xi_1 = \dfrac{b_1}{s_1 u(1-b_1)}$, and then let s_1 tend to 0.
  - The stochastic weights in the first column become the following.



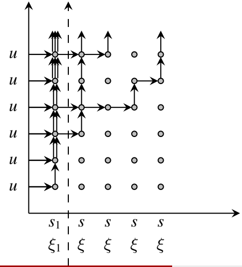
$1 - b_1 + q^k b_1$	$1 - b_1$	$(1 - q^k) b_1$	b_1

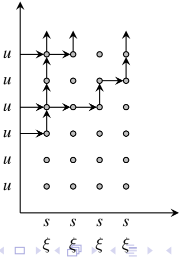
Half-Stationary Initial Data



$1 - b_1 + q^k b_1$	$1 - b_1$	$(1 - q^k) b_1$	b_1

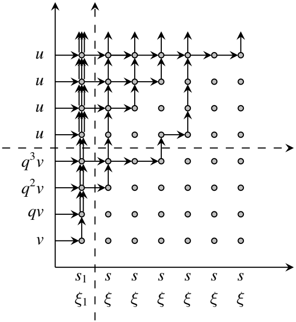
- Due to the step initial data, the red probabilities above are irrelevant.
- The blue probabilities produce a “filter” at the first column, out of which paths exit randomly and independently, with probability b_1.
- Shifting the diagram to the left yields the stochastic six-vertex model with *step-Bernoulli*, or *half-stationary*, initial data (analyzed by AA-Borodin).

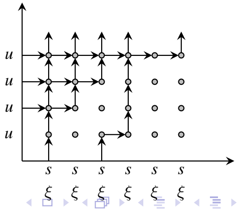




Analytic Continuation and Horizontal Initial Data

- Altering the first spin and inhomogeneity parameters alters the boundary data at the y-axis. We also would like to alter the boundary data through the x-axis.
- To that end, we change the first J spectral parameters. Specifically, set $u_1 = v$, $u_2 = qv$, $\dots$, $u_J = q^{J-1}v$ (*fusion*), and $u_{J+1} = u_{J+2} = \cdots = u$, where $v = (b_2 - 1)/b_2 s$.
- Shift the model up J coordinates. Then, J arrows enter through the x-axis.
- The entrance law of these arrows will be analytic in q^{-J}. Set $J = -\infty$ so that $q^{-J} = 0$.
- The result will be that arrows independently enter through the x-axis with probability b_2.





A Fredholm Determinant Identity

A suitable combination of this analytic continuation in q^{-J} with the contour integral identities of Borodin-Petrov yields the following Fredholm determinant identity.

Theorem (A., 2016)

Fix $x, t \in \mathbb{Z}_{>0}$ and $\delta_1, \delta_2, b_1, b_2 \in (0,1)$. Denote $q = \delta_1/\delta_2$, $\beta_1 = b_1/(1-b_1)$, $\beta_2 = b_2/(1-b_2)$, and $\kappa = (1-\delta_1)/(1-\delta_2)$. Assume that $\kappa\beta_2 < \beta_1$. Let $\zeta = -q^p < 0$ for some real number $p \in \mathbb{R}$. Then,

$$(\kappa\beta_2\beta_1^{-1}; q)_\infty \sum_{M=0}^{\infty} \frac{\left(\kappa\beta_2\beta_1^{-1}\right)^M}{(q;q)_M} \mathbb{E}\left[\frac{1}{(\zeta q^{H(x,t)-M}; q)_\infty}\right] = \det\left(\mathrm{Id} + V_\zeta\right)_{L^2(\mathcal{C}_V)},$$

for some explicit contour $\mathcal{C}_V$ and kernel K_ζ.

Observe the singularity of the left side of the equality as β_1 tends to $\kappa\beta_2$. This is what leads to the Baik-Rains $\Phi(s)$ fluctuations in the translation invariant case, rather than the Tracy-Widom fluctuations for step initial data.

Conclusion

- **Confirmed several predictions from the physics literature by proving KPZ growth exponents** in height fluctuations of the
  - Stationary ASEP
  - Translation-invariant stochastic six-vertex model
  - Ferroelectric symmetric six-vertex model at the conical singularity
- Established fluctuation theorems **on the level of exact statistics** (Baik-Rains distribution), in each of the above models.
- **Universality of these phenomena** under perturbations of the models (for example, asymmetric exclusion processes with longer jumps) **remains unknown**.
- At criticality ($-1 < \Delta < 1$), the six-vertex model is believed to be **conformally invariant**, but this **remains unproven**.