Current Fluctuations of the Stationary ASEP and Six-Vertex Model

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Outline

1. ASEP
   - Definition of model
   - Statement of the result

2. Stochastic six-vertex model
   - Definition of model
   - Statement of the result

3. Ferroelectric Symmetric Six-Vertex Model
   - Definition of model
   - Informal statement of the result
   - Context and previous predictions
   - More formal statement of the result

4. On the proofs
The Asymmetric Simple Exclusion Process (ASEP)

Place particles on $\mathbb{Z}$ such that at most one particle occupies any site.

- jump to the left with exponential rate $L$,
- jump to the right with exponential rate $R$,
- so that jumps to occupied locations are suppressed.

Assume $R > L \geq 0$ to force a drift to the right.
Current of the ASEP

We would like to analyze the current $J_t(x)$ of the ASEP. To define it,

- color all particles to weakly the left of 0 blue;
- color all particles to strictly the right of 0 blue.

Run the ASEP for time $t$.

Define $J_t(x) = (\text{Number of Blue Particles strictly to the right of } x) - (\text{Number of red particles weakly to the left of } x)$.

Above, $J_t(4) = 1$. 
Asymptotics of the Current

- We are interested in the asymptotic fluctuations of the current, meaning the behavior of \( J_t(x) - \mathbb{E}[J_t(x)] \) for large \( t \).
- Provable results are only available for certain classes of initial data, including step and step-Bernoulli (Tracy-Widom).
- Of widespread interest is stationary (equilibrium): particles occupy sites independently with fixed probability \( \rho \in (0, 1) \); this is invariant under the ASEP dynamics.
- Since the ASEP is a discretization of the Kardar-Parisi-Zhang (KPZ) equation, one expects the fluctuations of \( J_T(xT) \) to be of order \( T^{1/3} \) and to converge to the long-time height fluctuations of the stationary KPZ equation, for some value of \( x = (R - L)(1 - 2\rho) \) (called the characteristic velocity).
  - The latter fluctuations have been studied and are known to converge the Baik-Rains distribution (Borodin-Corwin-Ferrari-Vető).
  - We expect the same for the ASEP current \( J_T(xT) \).
New Results

Our first result is a **precise fluctuation theorem** for the current of the stationary ASEP along the characteristic line, that confirms this prediction on the level of exact statistics.

**Theorem (A., 2016)**

Assume that \( R > L \) and set \( \delta = R - L \). There exist explicit \( c, \chi \in \mathbb{R} \) so that

\[
\lim_{T \to \infty} \mathbb{P} \left[ \frac{J_T(\delta(1 - 2\rho)T) - cT}{\chi T^{1/3}} \geq s \right] = \Phi(s),
\]

for any \( s \in \mathbb{R} \), where \( \Phi(s) \) is the Baik-Rains distribution.

- Previous studies of the stationary ASEP by Ferrari-Spohn (2006) and Balász-Seppäläinen (2010), as well as many others.
- Proof of the above theorem uses an analysis of the *six-vertex model*. 
Asymmetric Six-Vertex Model

- Consider a two-dimensional lattice (torus, square, $\mathbb{Z}^2$, quadrant).
- Give each vertex one of the six following edge configurations, weighted as below.

<table>
<thead>
<tr>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$c_1$</th>
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- When $a_1 = a = a_2$; $b_1 = b = b_2$; and $c_1 = c = c_2$, this becomes the **symmetric six-vertex model**.
  - Introduced independently by Pauling (1935) and Slater (1941).
  - Studied by Lieb, Baxter, and many others (1967 to present).
- When $a_1 = 1 = a_2$; $b_1 + c_1 = 1$; and $b_2 + c_2 = 1$, this becomes the **stochastic six-vertex model**.
  - Studied by Borodin-Corwin-Gorin and Reshetikhin-Sridhar (2016).
Consider the stochastic six-vertex model on the positive quadrant with weights as below.

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<tr>
<th></th>
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<th>(\delta_1)</th>
<th>(\delta_2)</th>
<th>(1 - \delta_1)</th>
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<tr>
<td>1</td>
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<td>(\delta_1)</td>
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<td>(1 - \delta_1)</td>
<td>(1 - \delta_2)</td>
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- The stochastic six-vertex model can be sampled row by row.
- Under this sampling, the \(y\)-axis tracks time evolution.
Stochastic Six-Vertex Model as a Markov Process
Stochastic Six-Vertex Model as a Markov Process
Stochastic Six-Vertex Model as a Markov Process
Stochastic Six-Vertex Model as a Markov Process
Height Function of the Six-Vertex Model

We are interested in asymptotics of the height function $H(X, Y)$ of the six-vertex model. To define it,

- Color all paths emanating from the $x$-axis red.
- Color all paths emanating from the $y$-axis blue.

Let $H(X, Y) = \text{(Number of blue paths strictly to the right of } (X, Y)) - \text{(Number of red paths weakly to the left of } (X, Y))$. 
We are interested in asymptotics of the height function $H(X, Y)$ of the six-vertex model. To define it,

- Color all paths emanating from the $x$-axis blue.
- Color all paths emanating from the $y$-axis red.

Let $H(X, Y) = (\text{Number of blue paths strictly to the right of } (X, Y)) - (\text{Number of red paths weakly to the left of } (X, Y))$. 
Consider the stochastic six-vertex model on the positive quadrant.

There is a limit degeneration from the stochastic six-vertex model to the ASEP.

- Let \( b_1 = \varepsilon L \), \( b_2 = \varepsilon R \); scale time by \( \varepsilon^{-1} \); and observe on the diagonal.

This can be used to degenerate exact identities for the stochastic six-vertex model to the ASEP.

We will consider the stochastic six-vertex model with *double-sided* \((\rho_1, \rho_2)\)-Bernoulli initial data.

- This means that arrows enter through the \( y \)-axis with probability \( \rho_1 \) and \( x \)-axis with probability \( \rho_2 \).
New Results for the Stochastic Six-Vertex Model

Our second result is a precise fluctuation theorem for the height function of the stochastic six-vertex model with certain boundary data.

Theorem (A., 2016)

Assume that $\delta_2 > \delta_1$. Consider the stochastic six-vertex model with $(\rho_1, \rho_2)$-Bernoulli initial data, where

$$\frac{\rho_1}{1 - \rho_1} = \left( \frac{1 - \delta_1}{1 - \delta_2} \right) \frac{\rho_2}{1 - \rho_2}.$$  

Then, there exist (explicit) $c, v, \mathcal{F} \in \mathbb{R}$ such that

$$\lim_{X \to \infty} \mathbb{P} \left[ \frac{H(X, vX) - cX}{\mathcal{F}X^{1/3}} \geq s \right] = \Phi(s),$$  

for any $s \in \mathbb{R}$.

A similar statement holds for the ferroelectric symmetric six-vertex model.
Predictions for the Translation-Invariant Stochastic Six-Vertex Model

- The stochastic six-vertex model with double-sided \((\rho_1, \rho_2)\)-Bernoulli initial data was considered by Gwa and Spohn (1992) in the case

\[
\frac{\rho_1}{1 - \rho_1} = \left( \frac{1 - \delta_1}{1 - \delta_2} \right) \frac{\rho_2}{1 - \rho_2}.
\]

- In this case, they observed that the stochastic six-vertex model should be translation-invariant.

- Predicted that the height fluctuations of the stochastic six-vertex model should be of order \(T^{1/3}\) along an explicit characteristic line.

- From KPZ universality, they predicted that the rescaled fluctuations converge to the long-time statistics of the stationary KPZ equation, which is the Baik-Rains distribution.
Theorem (A., 2016)

Assume that $\delta_2 > \delta_1$. Consider the stochastic six-vertex model with $(\rho_1, \rho_2)$-Bernoulli initial data, where

$$\frac{\rho_1}{1 - \rho_1} = \left( \frac{1 - \delta_1}{1 - \delta_2} \right) \frac{\rho_2}{1 - \rho_2}.$$

Then, there exist (explicit) $c, v, F \in \mathbb{R}$ such that

$$\lim_{X \to \infty} \mathbb{P} \left[ \frac{H(X, vX) - cX}{F X^{1/3}} \geq s \right] = \Phi(s),$$

for any $s \in \mathbb{R}$.

- Confirms the Gwa-Spohn prediction, on the level of exact statistics.
- Similar statement holds for the ferroelectric symmetric six-vertex model.
Phases of the Six-Vertex Model

The symmetric six-vertex model has the following weights.

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<tr>
<td>(c)</td>
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Properties of the six-vertex model are dependent on the parameter

\[
\Delta = \frac{a^2 + b^2 - c^2}{2ab}.
\]

- Ferroelectric phase \(\Delta > 1\).
- Disordered phase \(\Delta \in (-1, 1)\).
- Anti-ferroelectric phase \(\Delta < -1\).

We will be interested in the ferroelectric phase.
Theorem (A., 2016, Informal Version)

Let $a, b, c > 0$ satisfy $\Delta = (a^2 + b^2 - c^2)/2ab > 1$. There exists a one-parameter family of translation-invariant Gibbs measures for the symmetric, ferroelectric six-vertex model with weights $(a, b, c)$ such that the following holds. For any $s \in \mathbb{R}$, we have that

$$\lim_{X \to \infty} \mathbb{P} \left[ \frac{H(X, vX) - cX}{\mathcal{F}X^{1/3}} \geq s \right] = \Phi(s),$$

for some (explicit) $c, v, \mathcal{F} \in \mathbb{R}$.

Before explaining the above theorem more carefully, we begin with some context.
Analysis of the *free energy* of the symmetric six-vertex model dates back to Lieb and Sutherland-Yang-Yang (1967).

In 1995, Bukman and Shore analyzed the free energy profile $F(H, V)$ of the ferroelectric, symmetric six-vertex model in the presence of a magnetic field $(H, V)$.

\[
\begin{array}{cccccc}
  & \uparrow & \uparrow & \rightarrow & \rightarrow & \uparrow \\
  ae^{-H-V} & ae^{H+V} & be^{V-H} & be^{H-V} & c & c
\end{array}
\]
Free Energy Profile

Figure: Figure 3 of D. J. Bukman and J. D. Shore, The Conical Point in the Ferroelectric Six-Vertex Model, J. Stat. Phys. 78, 1277–1309, 1995.
Conical Singularity

Figure: Figure 2 of D. J. Bukman and J. D. Shore, The Conical Point in the Ferroelectric Six-Vertex Model, J. Stat. Phys. 78, 1277–1309, 1995.
They found that the free energy exhibits a singularity at the corners above, called *conical singularities* (or *tricritical points*).

These were missed in the original Sutherland-Yang-Yang analysis almost 30 years earlier.
The Conical Singularity in Terms of Slopes

- Associated with any translation-invariant six-vertex model is a slope \((x, y)\), where
  \[
  x = \mathbb{E}[H(X + 1, Y) - H(X, Y)]; \\
  y = \mathbb{E}[H(X, Y + 1) - H(X, Y)].
  \]

- What is the slope corresponding to the conical singularity?
- The slope \((x, y)\) can usually be recovered from the magnetic field \((H, V)\) by taking the Legendre dual.
  - This expresses \(x\) and \(y\) in terms of derivatives of \(F(H, V)\).
- However, \(F\) is not differentiable at the conical singularity, so this does not apply.
The Conical Singularity in Terms of Slopes

Instead, the conical singularity corresponds to a one-parameter family of slopes \((x, y) = (\rho_1, \rho_2)\), where \(\rho_1\) and \(\rho_2\) satisfy

\[
\frac{\rho_1}{1 - \rho_1} = \kappa \frac{\rho_2}{1 - \rho_2},
\]

for some \(\kappa\) that is explicit in terms of \(a, b,\) and \(c\).

**Figure:** Figure 5 of D. J. Bukman and J. D. Shore, The Conical Point in the Ferroelectric Six-Vertex Model, J. Stat. Phys. **78**, 1277–1309, 1995.
Bukman-Shore Predictions

- Bukman and Shore analyzed the free energy $F(x, y)$ of the six-vertex model (on the $N \times N$ torus) with slope $(x, y)$.
- They observed that a second derivative of $F(x, y)$ is singular when $(x, y) = (\rho_1, \rho_2)$ lies on the conical singularity, a second-order phase transition.
- After more careful analysis, they found that the second derivative of $F(x, y)$ is of order $\gamma^{-1/3}$, where $\gamma$ is the distance from $(x, y)$ to the conical singularity.
- Unfortunately, turning their physical heuristics into a mathematical proof at the moment seems inaccessible.
- Still, it suggests that the two-point function of the six-vertex model should decay as $N^{-2/3}$ at the tricritical point, leading to their prediction of KPZ fluctuations at the conical singularity.
- We will assess this prediction from the infinite-volume viewpoint.
Let $\Omega$ denote the set of all six-vertex configurations on $\mathbb{Z}^2$.

For any $\omega \in \Omega$, let $\omega|_{\Lambda}$ denote the restriction of $\omega$ to $\Lambda$.

**Definition**

A probability measure $\mu$ on $\Omega$ is said to have the *Gibbs property* if the following holds. For any finite subset $\Lambda \subset \mathbb{Z}^2$, the probability $\mu_\Lambda(\omega)$ of selecting $\omega \in \Omega$, conditioned on $\omega|_{\mathbb{Z}^2\setminus\Lambda}$, is proportional to $a_1^{N_1} a_2^{N_2} b_1^{N_3} b_2^{N_4} c_1^{N_5} c_2^{N_6}$.
Gibbs Measures and Translation Invariance

Lemma (A., 2016)

Let $a, b, c > 0$ such that $\Delta = (a^2 + b^2 - c^2)/2ab > 1$. Define

$$\delta_1 = \frac{b}{a} (\Delta - \sqrt{\Delta^2 - 1}); \quad \delta_2 = \frac{b}{a} (\Delta + \sqrt{\Delta^2 - 1});$$

and let $\rho_1, \rho_2 \in (0, 1)$ be any positive real numbers satisfying

$$\frac{\rho_1}{1 - \rho_1} = \left(\frac{1 - \delta_1}{1 - \delta_2}\right) \frac{\rho_2}{1 - \rho_2}.$$

Then, the stochastic six-vertex model with double-sided $(\rho_1, \rho_2)$-Bernoulli initial data is a translation-invariant Gibbs measure for the ferroelectric, symmetric six-vertex model with weights $(a, a, b, b, c, c)$. 
Theorem (A., 2016)

Let \( a, b, c > 0 \) satisfy \( \Delta = (a^2 + b^2 - c^2)/2ab > 1 \). Let \( \rho_1, \rho_2 \in (0, 1) \) satisfy

\[
\frac{\rho_1}{1 - \rho_1} = \kappa \frac{\rho_2}{1 - \rho_2}.
\]

Consider the infinite-volume ferroelectric six-vertex model with weights \((a, b, c)\) under the previously defined Gibbs measure with slope \((\rho_1, \rho_2)\).

Then, there exist (explicit) \( c, v, F \in \mathbb{R} \) such that

\[
\lim_{X \to \infty} \mathbb{P} \left[ \frac{H(X, vX) - cX}{FX^{1/3}} \geq s \right] = \Phi(s),
\]

for any \( s \in \mathbb{R} \).
New Results for the Ferroelectric Six-Vertex Model

- **Establishes KPZ growth** of the ferroelectric six-vertex model at any \((\rho_1, \rho_2)\) at the conical singularity.
  - This **confirms the Bukman-Shore prediction**.

- Proves that the **height fluctuations converge to the Baik-Rains distribution** (not predicted by Bukman-Shore).
  - Exact statistics for height fluctuations of the translation-invariant six-vertex model are **very rare**.
  - To the best of our knowledge, they have only **been proven in dimer-type specializations** of the six-vertex model (Kenyon, Johansson, Kenyon-Okounkov-Sheffield, Okounkov-Reshetikhin).
  - These can be mapped to free-fermionic degenerations \((\Delta \in \{0, \infty\})\) of the six-vertex model, where a wealth of determinantal methods (Kasteleyn matrix, Schur processes) exist.
  - Such methods **do not apply for the generic ferroelectric six-vertex model**.
New Results for the Ferroelectric Six-Vertex Model

- Proves that the fluctuations of the height function of the six-vertex model are in the KPZ universality class only along a single characteristic line (also not predicted by Bukman-Shore).
- Gaussian fluctuations of $H(X, v'X)$ of order $X^{1/2}$, for all $v' \neq v$.
- Implies that spin-spin correlations decay as an inverse power of the distance (exponent 2/3) along a single characteristic line and exponentially elsewhere (predicted by Reshetikhin-Sridhar).
- Power-law decay of correlations without rotational symmetry is an unusual phenomenon from the viewpoint of classical spin systems.
The Inhomogeneous Stochastic Higher Spin Vertex Model

Paths enter through the $x$-axis and $y$-axis and then move up and right according to the probabilities below. This produces a random ensemble of paths called the *inhomogeneous stochastic higher spin vertex model*.

- The $\{u_y\}$ are *spectral parameters*.
- The $\{s_x\}$ are *spin parameters*.
- The $\{\xi_x\}$ are *inhomogeneity parameters*.
Degeneration to the Stochastic Six-Vertex Model

Fix \( 0 < \delta_1 < \delta_2 < 1 \). Setting

\[
q = \frac{\delta_1}{\delta_2} < 1; \quad \kappa = \frac{1 - \delta_1}{1 - \delta_2} > 1; \quad s_1 = s_2 = \cdots = s = q^{-1/2};
\]

\[
\xi_1 = \xi_2 = \cdots = \xi = 1; \quad u_1 = u_2 = \cdots = u = \kappa s,
\]

the probabilities above become the stochastic six-vertex probabilities below.
Integral Identities

- What makes the inhomogeneous stochastic higher spin vertex model accessible is that its weights satisfy the **Yang-Baxter equation**.

- Using this fact, Borodin-Petrov established the contour integral identity for the height function of the inhomogeneous stochastic higher spin vertex model, run with **step initial data** (one path enters through each vertex of the $y$-axis, and no paths enter through the $x$-axis),

\[
\mathbb{E}[q^{kh_t}(x)] = \frac{q^2}{(2\pi i)^k} \oint \cdots \oint \prod_{i=1}^{k} \left( \prod_{j=1}^{x-1} \frac{s_j\xi_j - s_j^2 w_i}{s_j\xi_j - w_i} \prod_{j=1}^{t} \frac{1 - qu_j w_i}{1 - u_j w_i} \right) \times \prod_{1 \leq i < j \leq k} \frac{w_i - w_j}{w_i - qw_j} \prod_{i=1}^{k} \frac{dw_i}{w_i},
\]

for suitable contours for the $w_j$. 
Half-Stationary Initial Data

- Their identities **only hold for step initial data**.
- Recall that the stochastic six-vertex model was obtained from a *homogeneous degeneration* of the higher spin model, that is, by setting all \(\{u_y\}, \{s_x\}, \text{and} \{\xi_x\}\) equal.
- Instead, we can try altering the values of \(s_1\) and \(\xi_1\), while keeping the \(\{u_y\}_{y \geq 1}\) and other \(\{s_x\}_{x \geq 2}\) and \(\{\xi_x\}_{x \geq 2}\) as in the homogeneous six-vertex case.
  - In particular, let us set \(\xi_1 = \frac{b_1}{s_1 u (1 - b_1)}\), and then let \(s_1\) tend to 0.
  - The stochastic weights in the first column become the following.
Due to the step initial data, the red probabilities above are irrelevant.

The blue probabilities produce a “filter” at the first column, out of which paths exit randomly and independently, with probability $b_1$.

Shifting the diagram to the left yields the stochastic six-vertex model with \textit{step-Bernoulli}, or \textit{half-stationary}, initial data (analyzed by AA-Borodin).
Analytic Continuation and Horizontal Initial Data

- Altering the first spin and inhomogeneity parameters alters the boundary data at the $y$-axis. We also would like to alter the boundary data through the $x$-axis.
- To that end, we change the first $J$ spectral parameters. Specifically, set $u_1 = v$, $u_2 = qv$, $\ldots$, $u_J = q^{J-1}v$ (fusion), and $u_{J+1} = u_{J+2} = \cdots = u$, where $v = (b_2 - 1)/b_2s$.
- Shift the model up $J$ coordinates. Then, $J$ arrows enter through the $x$-axis.
- The entrance law of these arrows will be analytic in $q^{-J}$. Set $J = -\infty$ so that $q^{-J} = 0$.
- The result will be that arrows independently enter through the $x$-axis with probability $b_2$. 

![Diagram of analytic continuation and horizontal initial data]
A Fredholm Determinant Identity

A suitable combination of this analytic continuation in $q^{-J}$ with the contour integral identities of Borodin-Petrov yields the following Fredholm determinant identity.

**Theorem (A., 2016)**

Fix $x, t \in \mathbb{Z}_{>0}$ and $\delta_1, \delta_2, b_1, b_2 \in (0, 1)$. Denote $q = \delta_1/\delta_2$, $\beta_1 = b_1/(1 - b_1)$, $\beta_2 = b_2/(1 - b_2)$, and $\kappa = (1 - \delta_1)/(1 - \delta_2)$. Assume that $\kappa \beta_2 < \beta_1$. Let $\zeta = -q^p < 0$ for some real number $p \in \mathbb{R}$. Then,

$$
(k \beta_2 \beta_1^{-1}; q)_{\infty} \sum_{M=0}^{\infty} \frac{(k \beta_2 \beta_1^{-1})^M}{(q; q)_M} \mathbb{E} \left[ \frac{1}{(\zeta q^{H(x,t)-M}; q)_{\infty}} \right] = \det \left( \text{Id} + V_{\zeta} \right)_{L^2(C_V)},
$$

for some explicit contour $C_V$ and kernel $K_{\zeta}$.

Observe the singularity of the left side of the equality as $\beta_1$ tends to $\kappa \beta_2$. This is what leads to the Baik-Rains $\Phi(s)$ fluctuations in the translation invariant case, rather than the Tracy-Widom fluctuations for step initial data.
Conclusion

- **Confirmed several predictions from the physics literature by proving KPZ growth exponents** in height fluctuations of the
  - Stationary ASEP
  - Translation-invariant stochastic six-vertex model
  - Ferroelectric symmetric six-vertex model at the conical singularity

- Established fluctuation theorems **on the level of exact statistics** (Baik-Rains distribution), in each of the above models.

- **Universality of these phenomena** under perturbations of the models (for example, asymmetric exclusion processes with longer jumps) **remains unknown**.

- At criticality \((-1 < \Delta < 1)\), the six-vertex model is believed to be **conformally invariant**, but this **remains unproven**.