Dimers and related models in statistical mechanics

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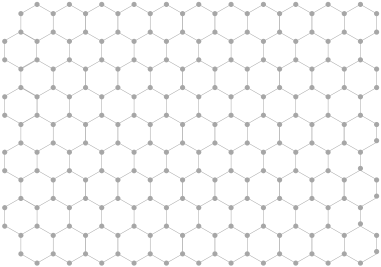
Statistical mechanics

Study the macroscopic properties of a physics system whose interactions are described on the microscopic level

The general framework is the following.

- The structure of the physics system is represented by a finite graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$.



Statistical mechanics

- Set of configurations on the graph G: $\mathcal{C}(\mathsf{G})$,
  - vertex configurations,
  - edge configurations,
  - vertex/edge configurations.
- Parameters representing:
  - the intensity of interactions between microscopic components,
  - the external temperature.

$\Rightarrow$ Positive weight function w on edges/vertices.

Statistical mechanics

- To a configuration C, one assigns an energy $\mathcal{E}_w(\mathsf{C})$.
- Boltzmann probability on configurations:

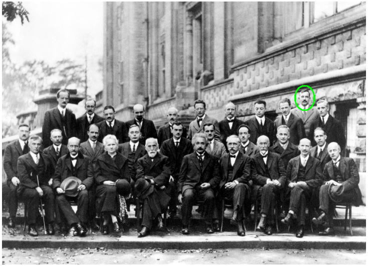
$$\forall\, \mathsf{C} \in \mathcal{C}(\mathsf{G}), \quad \mathbb{P}(\mathsf{C}) = \frac{e^{-\mathcal{E}_w(\mathsf{C})}}{Z(\mathsf{G}, w)},$$

where $Z(\mathsf{G}, w) = \sum_{\mathsf{C} \in \mathcal{C}(\mathsf{G})} e^{-\mathcal{E}_w(\mathsf{C})}$ is the partition function.

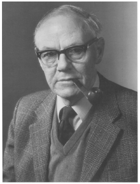
Understand the model
when the graph is large (infinite).

The dimer model

Adsorption of di-atomic molecules on the surface of a crystal



Sir Ralph H. Fowler (1889-1944)
Congrès Solvay 1927.



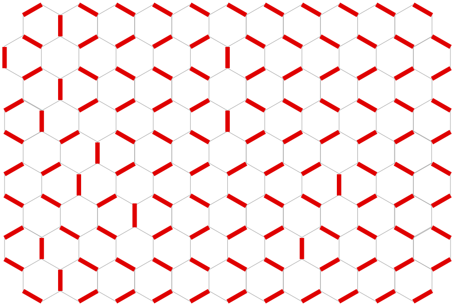
George S. Rushbrooke (1915-1995)

- Graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$.
- A dimer configuration or perfect matching: subset of edges such that every vertex is incident to exactly one edge.

$$\Rightarrow \mathcal{M}(\mathsf{G}) = \text{set of dimer configurations}.$$

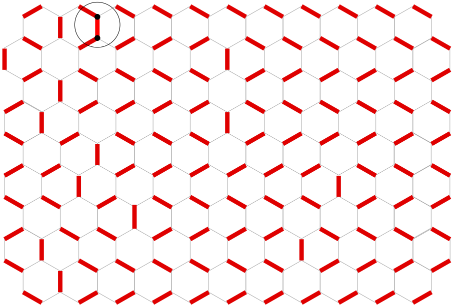
The dimer model

- A dimer configuration



The dimer model

- A dimer configuration



The dimer model

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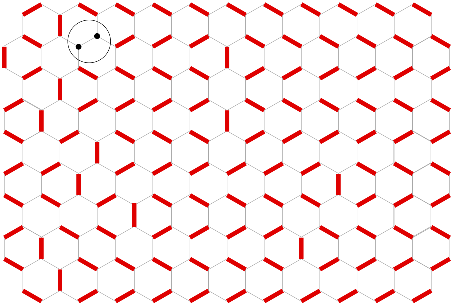
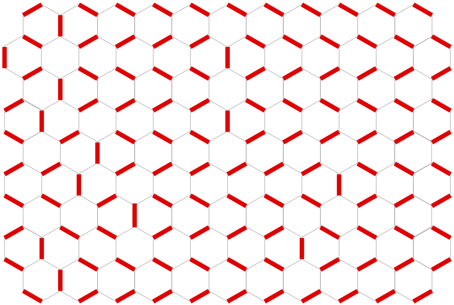


The dimer model

- A dimer configuration



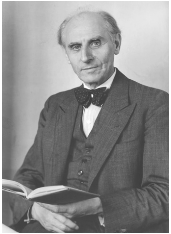
- Positive weight function on the edges: $\nu = (\nu_{\mathsf{e}})_{\mathsf{e} \in \mathsf{E}}$.
- Energy of a configuration M: $\mathcal{E}_\nu(\mathsf{M}) = -\sum_{\mathsf{e} \in \mathsf{M}} \log \nu_{\mathsf{e}}$.
- Dimer Boltzmann measure:

$$\forall\, \mathsf{M} \in \mathcal{M}(\mathsf{G}), \quad \mathbb{P}_{\mathrm{dimer}}(\mathsf{M}) = \frac{\prod_{\mathsf{e} \in \mathsf{M}} \nu_{\mathsf{e}}}{Z_{\mathrm{dimer}}(\mathsf{G}, \nu)}.$$

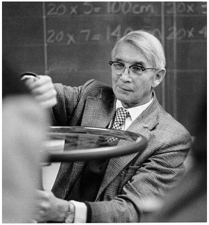
  - The highest the weight ν_{e}, the more likely is the edge e.

The Ising model

Model of ferromagnetism / mixture of two materials



Wilhelm Lenz (1888-1957)



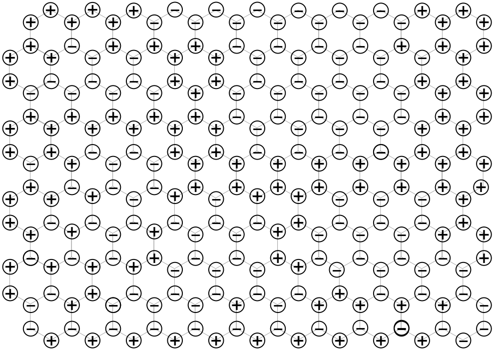
Ernst Ising (1900-1998)

- Graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$.
- A spin configuration σ assigns to every vertex x of the graph G a spin $\sigma_{\mathsf{x}} \in \{-1, 1\}$.

$$\Rightarrow \mathcal{C}(\mathsf{G}) = \{-1, 1\}^{\mathsf{V}} = \text{set of spin configurations.}$$

The Ising model

- A spin configuration

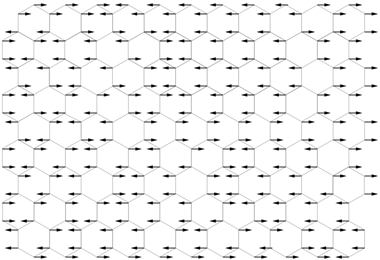


The Ising model

- A spin configuration / two interpretations.

Magnetic moments:

$+1/\rightarrow$, $-1/\leftarrow$



The Ising model

- A spin configuration / two interpretations.

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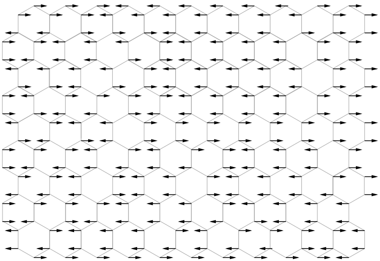
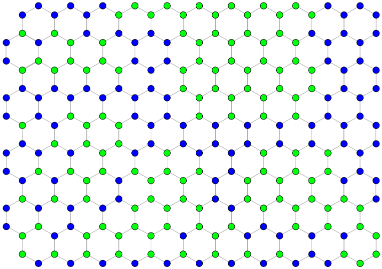


Mixture of two materials:
+1/●, -1/●.



The Ising model

- Positive weight function: coupling constants $\mathsf{J} = (\mathsf{J}_e)_{e \in \mathsf{E}}$.
- Energy of a spin configuration: $\mathcal{E}_{\mathsf{J}}(\sigma) = -\sum_{e=xy \in \mathsf{E}} \mathsf{J}_{xy} \sigma_x \sigma_y$.
- Ising Boltzmann measure:

$$\forall\, \sigma \in \{-1, 1\}^{\mathsf{V}}, \quad \mathbb{P}_{\mathrm{Ising}}(\sigma) = \frac{e^{-\mathcal{E}_{\mathsf{J}}(\sigma)}}{Z_{\mathrm{Ising}}(\mathsf{G}, \mathsf{J})}.$$

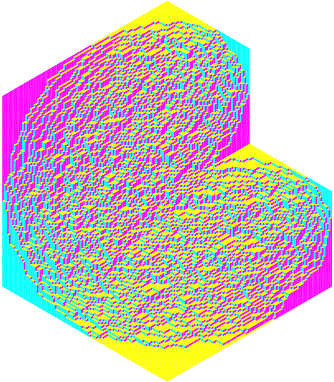
  - Two neighboring spins σ_x, σ_y tend to align.
  - The higher the coupling J_{xy}, the strongest is this tendency.

Other models

- Percolation: flow of a liquid through a porous material.
- Spanning trees/forests: related to electrical networks and random walks.
- Random cluster model:
  includes percolation, Ising, Potts, spanning trees.
- Vertex models (6-8-$\cdots$): 6-vertex is a model for ice.
- $O(n)$ loop models.
- ...

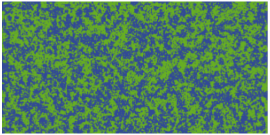
Macroscopic behavior

- Rhombus tilings $\leftrightarrow$ dimers on the honeycomb lattice (illustration by R. Kenyon)

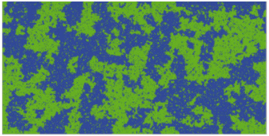


Macroscopic behavior

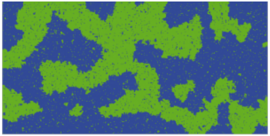
- Ising model on the square lattice (illustrations by R. Cerf)



J small



J critical



J large

Macroscopic behavior

- Identification of the phase transition.
- Understanding of the sub/super critical regimes.
- Understanding the critical model (at the phase transition):
  - Universality and conformal invariance.
  - Conjectures: Cardy, Duplantier, Nienhuis ...

    Proofs: Lawler, Schramm, Werner, D. Chelkak, S. Smirnov ...

Exactly solvable models

- One of the tools to study the macroscopic behavior is the partition function:

$$Z(\mathsf{G}, w) = \sum_{\mathsf{C} \in \mathcal{C}(\mathsf{G})} e^{-\mathcal{E}_w(\mathsf{C})},$$

  the normalizing constant of the Boltzmann measure

$$\forall\, \mathsf{C} \in \mathcal{C}(\mathsf{G}), \quad \mathbb{P}(\mathsf{C}) = \frac{e^{-\mathcal{E}_w(\mathsf{C})}}{Z(\mathsf{G}, w)}.$$

- The model is exactly solvable if there exists an exact, explicit formula, for the partition function
- The two models considered are exactly solvable in 2d:
  - Ising: Onsager (1944) - Kaufman - Kac & Ward (1952) - Fisher (1966).
  - Dimers: Kasteleyn - Temperley & Fisher (1961).

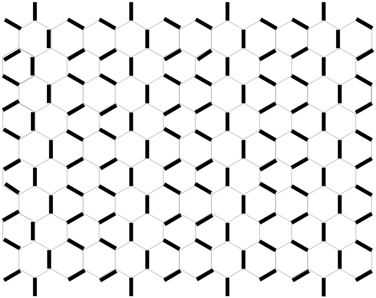
Partition function for the dimer model
Kasteleyn, Temperley & Fisher

- Dimer model on a finite, planar graph G with weights ν.
- Computation of the partition function,

$$Z_{\mathrm{dimer}}(\mathsf{G},\nu) = \sum_{\mathsf{M}\in\mathcal{M}(\mathsf{G})} \prod_{\mathsf{e}\in\mathsf{E}} \nu_{\mathsf{e}}.$$

Superimposition of two dimer configurations

- Let M_1, M_2 be two dimer configurations of G and let $M_1 \cup M_2$ be their superimposition.



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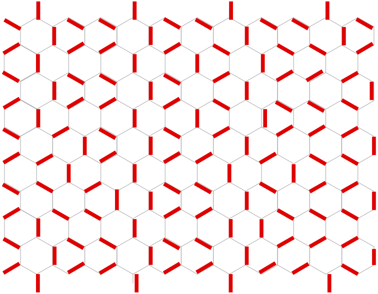

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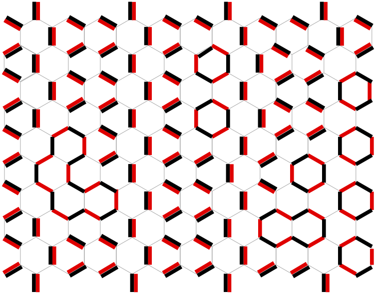

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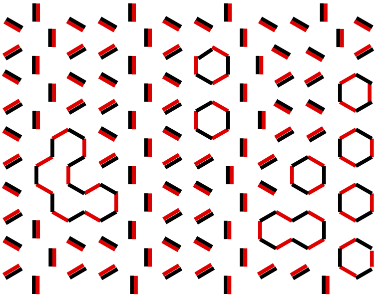

- $\mathsf{M}_1 \cup \mathsf{M}_2$ is a disjoint union of alternating cycles, where an alternating cycle has edges alternating between M_1 and M_2. An alternating cycle of length 2 is a doubled edge.

Isoradial graphs

- A graph is isoradial if it is planar and can be embedded in the plane in such a way that all inner faces are inscribable in a circle of radius 1 and all circumcenters are in the interior of the faces (Duffin-Mercat-Kenyon).

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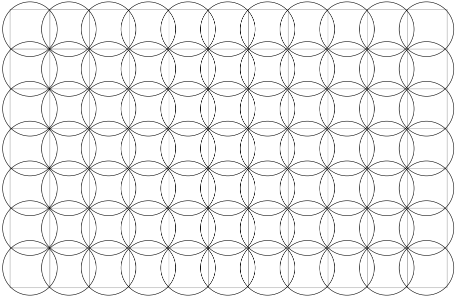

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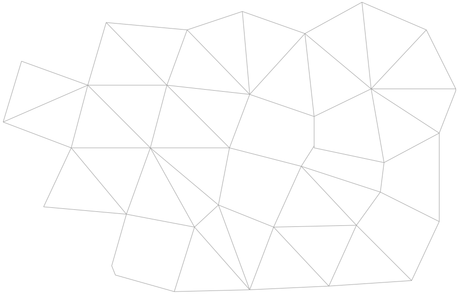

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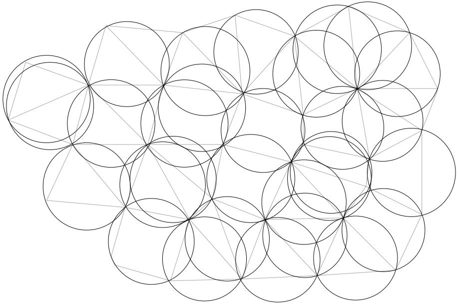
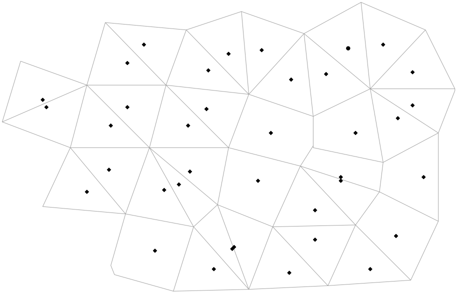


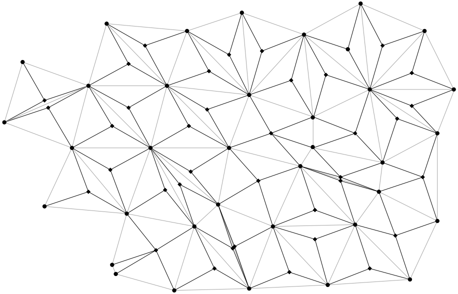
Associated diamond graph and rhombus angles

- Take the center of the circumcircles.



Associated diamond graph and rhombus angles

- Join each center to the “neighboring” vertices of G.
  $\Rightarrow$ Associated rhombus graph $\mathsf{G}^{\diamond}$.



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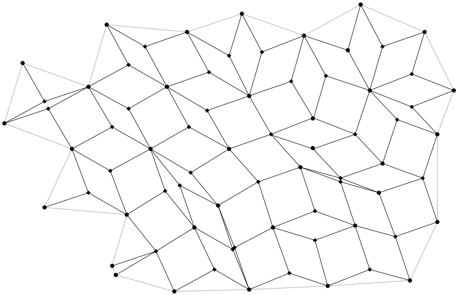

Associated diamond graph and rhombus angles

- To every edge e, one assigns the half-angle $\bar{\theta}_e$ of the corresponding rhombus.